Exercise sheet 3

Submission deadline: 2016-12-08

Submission: by e-mail to christoph.rauch@fau.de. Feel free to ask questions!

Substitution Lemma (6 Points)

We have realised in the last tutorial that a proof for a lemma concerning substitution was still missing. The proof is via induction over the structure of the formula. Here is the lemma again:

Lemma. *Given a signature Σ, for any formula α of $FO(\Sigma)$, any model $\mathcal{A}$ and valuation κ,*

$$\mathcal{A}, \kappa \vDash \alpha[y/x] \quad \textit{iff} \quad \mathcal{A}, \kappa[x := y] \vDash \alpha.$$

To save you some work, here is the base case for α of the form $s \approx t$:

$$\mathcal{A}, \kappa \vDash (s \approx t)[y/x]$$
$$\Updownarrow$$
$$\mathcal{A}, \kappa \vDash s[y/x] \approx t[y/x]$$
$$\Updownarrow$$
$$\hat{\kappa}(s[y/x]) = \hat{\kappa}(t[y/x])$$

- $s = x$: $\hat{\kappa}(y) = \hat{\kappa}(t[y/x])$
  - $t = x$: $\hat{\kappa}(y) = \hat{\kappa}(y)$
  - $t \neq x$: $\hat{\kappa}(y) = \hat{\kappa}(t)$
- $s \neq x$: $\hat{\kappa}(s) = \hat{\kappa}(t[y/x])$
  - $t = x$: $\hat{\kappa}(s) = \hat{\kappa}(y)$
  - $t \neq x$: $\hat{\kappa}(s) = \hat{\kappa}(t)$

$$\hat{\kappa}(y) = \hat{\kappa}(t) \qquad \hat{\kappa}(s) = \hat{\kappa}(y)$$
$$\Updownarrow (\star)$$
$$\hat{\kappa}[x := y](s) = \hat{\kappa}[x := y](t)$$
$$\Updownarrow (\star\star)$$
$$\mathcal{A}, \kappa[x := y] \vDash \alpha$$

Finish the proof by

- first recalling how we recursively defined substitution as a function on formulas in the first tutorial
- describing in your own words why the equivalences $(\star)$ and $(\star\star)$ hold.
- providing the other base case, i.e. α of the form $P(t_1, \ldots, t_k)$; be careful to be general enough, i.e. assume that the elements of some subset of $\{t_1, \ldots, t_k\}$ are equal to x.
- providing the inductive cases for the boolean connectives and the universal quantifier; in the latter case, be careful again about substitution possibly capturing bound variables! That is, for $(\forall v. \alpha)[y/x]$, do consider the case where $x = v$.

Homomorphism Preservation Theorem (6 Points)

Recall the definition of a homomorphism:

Definition. *A homomorphism $f : \mathcal{A} \to \mathcal{B}$ is a mapping $f : A \to B$ with*

$$(a_1, \ldots, a_n) \in P^{\mathcal{A}} \implies (f(a_1), \ldots, f(a_n)) \in P^{\mathcal{B}} \text{ and } f(c^{\mathcal{A}}) = c^{\mathcal{B}}.$$

We say that a formula α is

- *atomic* if it is of the form $s \approx t$ or $P(t_1, \ldots, t_n)$ for s, t, and t_i variables or constants and P an n-ary predicate in the signature.
- *existential positive* if it is constructed from *atomic* formulas using $\wedge$, $\vee$, and $\exists$ only.

Let Σ be a signature consisting of the constants '`male`', '`female`' and binary predicates '*attends*' (as in "x attends course y"), '*gender*' ("x has gender y"), and '*mail*' ("x has mail address y").

1. Define a model $\mathcal{A}$ for this signature. Include at least one, preferably two pairs of elements in the interpretation of each binary predicate.
2. Recall that a *query* $\alpha_{\overline{v}}$ is a formula α over $FO(\Sigma)$ together with a subset $\overline{v}$ of the free variables of α. Invent two distinct queries whose formula is *existential positive*. Make $\overline{v}$ nonempty and let each formula contain at least one existential quantifier, conjunction and disjunction. Also ensure that the *answer sets*, $\alpha_{\overline{v}}(\mathcal{A})$, of the two queries, are distinct.
3. Compute the answer sets for the two queries explicitly.
4. Define another model $\mathcal{B}$ for Σ. Try to make the domains of $\mathcal{A}$ and $\mathcal{B}$ disjoint and relate a different number of elements using the binary predicates* than in $\mathcal{A}$.
5. Define a *homomorphism* between the models $\mathcal{A}$ and $\mathcal{B}$.
6. For each query $\alpha_{\overline{v}}$ of the two queries defined in 2., check the following:

   $$\mathcal{A}, \kappa \models \alpha \overset{?}{\implies} \mathcal{B}, f \circ \kappa \models \alpha$$

   Might this always be the case, i.e. do homomorphisms preserve existential positive formulas? How would one go about to prove it? Do not carry out the proof, but sketch how such a proof would look like.

*Remember that a model does not have to be total, i.e. it could lack gender information for some person, but it can also be nonsensical, e.g. give gender information to a course