Sheet 5

Due 09.02.2016

Exercise 1 Derivations in BI (5 Points)

Derive the following sequents:

- $\phi \Rightarrow \phi * 1$
- $1 \mathrel{-\!\!*} \phi \Rightarrow \phi$
- $(\phi \mathrel{-\!\!*} \phi) * (\phi \mathrel{-\!\!*} \phi) \Leftrightarrow \phi \mathrel{-\!\!*} \phi$
- $\phi * (\phi \mathrel{-\!\!*} \phi) \Leftrightarrow \phi$

($\phi \Leftrightarrow \psi$, as in previous Blatts, means derivability of both $\phi \Rightarrow \psi$ and $\psi \Rightarrow \phi$)

Exercise 2 Generalized Excluded Middle

Prove that over discrete Kripke resource models, the rule of *generalized excluded middle*

$$(EM)\ \frac{\Gamma(\alpha \to \beta) \Rightarrow \gamma \qquad \Gamma(\alpha) \Rightarrow \gamma}{\Gamma(\emptyset_a) \Rightarrow \gamma}$$

is sound.

Exercise 3 To Bottom or not to Bottom (5 Points)

Show that over (total) Kripke resource models, the sequent

$$((\phi \mathrel{-\!\!*} \bot) \to \bot) \wedge ((\psi \mathrel{-\!\!*} \bot) \to \bot) \Rightarrow (\phi * \psi \mathrel{-\!\!*} \bot) \to \bot$$

is valid. *Hint:* First develop a characterization of what it means for a world to satisfy a formula of the form $(\alpha \mathrel{-\!\!*} \bot) \to \bot$.

Exercise 4 Multiplicative Weakening

a) Prove (using cut) that the addition of any of the following versions of the multiplicative weakening rule to the sequent system for BI has the same effect, i.e., renders all the remaining versions derivable:

(W1) $\emptyset_a \Rightarrow 1$	(W2) $\top \Rightarrow 1$	(W3) $\dfrac{\Gamma(\Delta) \Rightarrow \phi}{\Gamma(\Delta, \Delta') \Rightarrow \phi}$
(W4) $\phi, \psi \Rightarrow \phi$	(W5) $\phi * \psi \Rightarrow \phi$	(W6) $\dfrac{\phi; \psi \Rightarrow \chi}{\phi, \psi \Rightarrow \chi}$
(W7) $\phi, \psi \Rightarrow \phi \wedge \psi$	(W8) $\phi * \psi \Rightarrow \phi \wedge \psi$	(W9) $\dfrac{\Gamma(\Delta; \Delta') \Rightarrow \phi}{\Gamma(\Delta, \Delta') \Rightarrow \phi}$
(W10) $\phi \Rightarrow 1$	(W11) $\emptyset_a \Rightarrow \phi \to 1$	(W12) $\emptyset_m \Rightarrow \phi \mathrel{-\!\!*} 1$

Note: the list is by no means exhaustive; for starters, we could use more often reformulations where the left side of $\Rightarrow$ is just $\emptyset_a$ or $\emptyset_m$, the way we reformulated (W10) as either (W11) or (W12) respectively. Other rules where similar reformulations are immediately applicable are (W2), (W5) and (W8). Nevertheless, the default preference goes to rules which have as "structural" a shape as possible. In our context, the most "structural" ones seem to be either (W3) or (W9). As an exercise for yourself, you might be tempted to find as many reformulations of the rule (EM) from Exercise 2 as possible.

b) Given any PDRM satisfying *cancellativity* ($m \cdot n = m \cdot n'$ implies $n = n'$) and *indivisibility of units* ($m \cdot n = 1$ implies $m = 1$), show that the following definition

$$m \sqsubseteq n \quad \text{iff} \quad \exists x. m \cdot x = n$$

yields a PPRM (i.e., the relation is a partial order satisfying bifunctoriality). Furthermore, use whichever of the above formulations you prefer to show that in the resulting PPRM, multiplicative weakening is valid (i.e., holds under every valuation).

Aside. Here are some additional things you may enjoy proving for yourself (please don't submit it, no bonus points):

- Discrete and intuitionistic (partially ordered) versions of various concrete resource monoids defined in the lecture (resource allocation and deallocation, aliasing and update, trees and semistructured data ...) are related to each other in precisely this way: the PPRM is obtained from PDRM using the above procedure.
- The semantic relationship between these two variants of models can be lifted to a syntactic translation. That is, there is a translation—along the lines of the translation of the Gödel translation of intuitionistic logic into S4—embedding the set of formulas valid in the intuitionistic variant of a given cancellative monoid with indivisible unit (with $\sqsubseteq$ defined as above) into the set of theorems valid in the original PDRM. The crucial thing is to find a suitable counterpart of the $\Box$ modality of S4.

If you're going to try your hand in this for yourself, note further that the translation you are going to produce is not likely to be an embedding of base BI into minimal Boolean BI, i.e., BI extended with the scheme from Exercise 2. For good reasons, we have seen no proof to the effect that the original BI is complete wrt PPRM's obtained in such a special way. Nevertheless, there is in fact a syntactic translation from BI to BBI based on similar ideas, though both the translation itself and the proof of its correctness are definitely more complicated.