Nonclassical Logics in Computer Science WiSe 2015/16

# Sheet 4

## Due 19.01.2016

The parts in green have been edited. So were the parts in blue.

## Exercise 1 Additive Cut (5 Points)

The *additive cut rule* is

$$\frac{\Gamma \Rightarrow \phi, \Delta \qquad \Gamma, \phi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}.$$

Show that given the existing sequent system for linear logic, the additive cut rule is mutually interderivable with the axiom $\phi \sqcup \phi^{\perp}$ (where axioms may be inserted to the left of $\Rightarrow$ in derivations). Conclude that additive cut is not admissible. *Hint:* You may (and will need to) use that multiplicative cut is admissible.

## Exercise 2 Axiomatizing the Structural Rules (5 Points)

Show (using cut) that the rules of !-weakening, !-contraction and !-dereliction are mutually interderivable with the axiom scheme

$$!\phi \Rightarrow 1 \sqcap \phi \sqcap \phi \cdot \phi.$$

Note: we keep !-dereliction. Use it smartly. If you don't see how, replace the above axiom with $!\phi \Rightarrow 1 \sqcap \phi \sqcap !\phi \cdot !\phi$. If there are no other errors, we won't subtract more than a point for this.

## Exercise 3 From Classical to Linear (5 or more Points)

There are many ways of translating both classical and intuitionistic logic into linear logic. In this exercise, we focus on a specific translation of cut-free classical proofs proposed already in Girard's original paper (though his translation of intuitionistic logic had more influence on subsequent literature; rather unsurprisingly so, as classical logic can be translated into intuitionistic logic and the fact that one can embed intuitionistic logic into classical linear logic seems more counterintuitive).

An important point: I am working with structural-rules-absorbing formulation of sequent calculus for classical logic exactly as it is found in Troelstra and Schwichtenberg "Basic Proof Theory", notably with calculus G3c as defined on page 77. This calculus is somewhat different to the one defined in the lecture, as I recalled belatedly. Namely, it includes left and right disjunction and implication rules as primitives, but does not include negation rules as primitive. I have sent around a screenshot with this formulation.

We can allow a further extension of the classical calculus G3c with the following rules for constants:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \bot, \Delta} \qquad \Gamma \Rightarrow \top, \Delta \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma, \top \Rightarrow \Delta}$$

This is the formulation we will work with. If you want additional bonus points, you can extend the classical calculus further with left and right rules for negation, which are now derived rather than primitive. Show their admissibility.

a) As a preparatory exercise, let us do the proof that we have not managed during tutorials, i.e.

$$\vdash\ !\phi \cdot !\psi \ \Leftrightarrow\ !(\phi \sqcap \psi)$$
$$\vdash\ ?\phi + ?\psi \ \Leftrightarrow\ ?(\phi \sqcup \psi)$$

where $\vdash \phi \Leftrightarrow \psi$ means derivability of $\phi \Rightarrow \psi$ and $\psi \Rightarrow \phi$.

b) Derive that in presence of cut, the following rules are admissible:

$$\frac{\Gamma, !\phi \cdot !\psi \Rightarrow \Delta}{\Gamma, !(\phi \sqcap \psi) \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow ?\phi + ?\psi, \Delta}{\Gamma \Rightarrow ?(\phi \sqcup \psi), \Delta} \quad \frac{\Gamma, !\phi \Rightarrow ?\psi, \Delta}{\Gamma \Rightarrow ?(\phi^{\perp} \sqcup \psi), \Delta} \tag{1}$$

If you want additional points for handling negation, you may need to find additional admissible rules for that and prove their admissibility.

c) Now let us define by mutual recursion two translations of classical formulas into linear formulas (a positive and a negative one):

$$\begin{array}{llll}
(\bot)^+ & = 0 & (\bot)^- & = \bot \\
(\top)^+ & = \top & (\top)^- & = 1 \\
(a)^+ & = a & (a)^- & = a \\
(\neg\phi)^+ & = (\phi)^{-\perp} & (\neg\phi)^- & = (\phi)^{+\perp} \\
(\phi \wedge \psi)^+ & = ?(\phi)^+ \cdot ?(\psi)^+ & (\phi \wedge \psi)^- & = (\phi)^- \sqcap (\psi)^- \\
(\phi \vee \psi)^+ & = (\phi)^+ \sqcup (\psi)^+ & (\phi \vee \psi)^- & = !(\phi)^- + !(\psi)^- \\
(\phi \to \psi)^+ & = (\phi)^{-\perp} \sqcup (\psi)^+ & (\phi \to \psi)^- & = ?(\phi)^+ \multimap !(\psi)^-.
\end{array}$$

If you're not looking for negation bonus points, remove the negation clause; it is pointless then.

Show that for any classical sequent $\Gamma \Rightarrow \Delta$,

$$\vdash_{G3} \Gamma \Rightarrow \Delta \qquad \text{iff} \qquad \vdash_{lin} !(\Gamma)^- \Rightarrow ?(\Delta)^+.$$

where $\vdash_{G3}$ means **cut-free** derivability in the standard sequent system for classical logic (absorbing the structural rules) that we presented in October (the subscript G3 is for consistency with the notation in Troelstra and Schwichtenberg's book), whereas $\vdash_{lin+}$ stands for **cut-free** derivability in the sequent system for linear logic extended with rules (1).

For the "if" direction, one needs to have a convention how linear sequents are interpreted as classical ones. This, however, is very obvious: both the multiplicative and the additive variant of any connective are translated as its unique classical counterpart (i.e., $+$ and $\sqcup$ as $\vee$, $\cdot$ and $\sqcap$ as $\wedge$ etc.), and ! and ? are simply removed.

## Exercise 4 Orthogonally Orthogonal a la Galois (5 Points)

Let $(M, \perp)$ be a phase space. Show that for all $A, B \subseteq M$,

$$A \subseteq B^{\perp} \text{ iff } B \subseteq A^{\perp}.$$

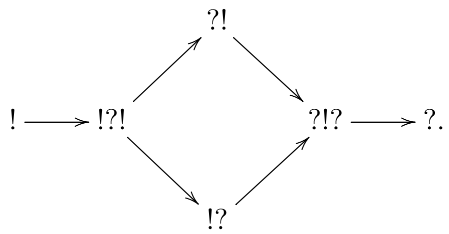
Derive from this the following conclusions for any $A, B$ (you can use each of derived inclusions in proofs of subsequent points):

- $A \subseteq A^{\perp\perp}$;
- $A \subseteq B$ then $B^{\perp} \subseteq A^{\perp}$;
- $A^{\perp\perp\perp} = A^{\perp}$.

Moreover, show that more generally, $(BA^{\perp\perp})^{\perp} = (BA)^{\perp}$ where $BA$ denotes complex multiplication ($BA = \{xy \mid x \in B, y \in A\}$).

## Exercise 5 The Six Modalities (5 Points)

Define a *compound modality* to be a non-empty sequence of ? and !, i.e., an element $\{?, !\}^*$ of length at least one. Show that there are at most six non-equivalent compound modalities, which are moreover ordered as follows:




Arrow such as $! \longrightarrow !?!$ should of course be read as derivability of $!a \Rightarrow !?!a$ in the Gentzen system for linear logic. You can obviously use basic facts we already know about it to reduce the number of derivations you have to do.

**Aside.** It would be possible to show that there are **exactly** six such modalities, but showing that these six are distinct is beyond the scope of this exercise. You just have to show all possible composite modalities collapse to one of those six, and that the relationship between them is as depicted.