Sheet 3

Due 15.12.2015

Exercise 1 A Farewell to Classics (5 Points)

Show that the following sequents (derivable classically) fail to be derivable in the intuitionistic sequent calculus.

$$\begin{aligned} &\Rightarrow a, a \to (b \vee \neg b) \\ (a \to b) \to a &\Rightarrow a \\ \neg a \to (b \vee c) &\Rightarrow \neg a \to b, \neg a \to c. \end{aligned}$$

In all cases, extract a Kripke countermodel from the failed proof search.

Exercise 2 Intuitionistic Sequent Proofs (5 Points)

Derive the following sequents in the intuitionistic sequent calculus.

$$\begin{aligned} a \to (b \to c) &\Rightarrow (a \to b) \to (a \to c) \\ a \vee b &\Rightarrow (\neg a \wedge \neg b) \to c \\ \neg\neg\neg a &\Rightarrow \neg a. \end{aligned}$$

Exercise 3 Contraction, Intuitionistically (5 Points)

Prove by induction on *proof height* (i.e. the maximal length of a branch in a proof) that the intuitionistic sequent calculus (in the form that absorbs the structural rules as given in the lecture) admits the contraction rules.

Hints: As in the classical case, the proof relies on the fact that the inverted rules are *height-preserving admissible*, that is, not only are the conclusions of the inverted rules provable if the premise is provable, but the proof of the conclusion also has at most the same height as that of the premise. Moreover, some cases require applying the induction hypothesis several times, so one actually needs to show the stronger statement that the contraction rules are height-preserving admissible.

Exercise 4 Skeletons in the Closet (5 Points)

As we have learned, Int can be identified with a fragment of $\mathsf{S4}$. However, it is a somewhat restricted fragment, where boxes appear everywhere; in particular, atoms can only appear prefixed by box. Consequently, satisfaction of formulas from this fragment is invariant under some operations that would not be safe for arbitrary modal formulas.

Given an $\mathsf{S4}$ frame (a reflexive and transitive structure) (W, R) and any $w \in W$, define

$$\begin{aligned} [w]_\equiv &:= \{w' \mid wRw' \,\&\, w'Rw\} \\ W_\equiv &:= \{[w]_\equiv \mid w \in W\} \\ R_\equiv &:= \{([w]_\equiv, [w']_\equiv) \mid (w, w') \in R\}. \end{aligned}$$

Show that

- $W_\equiv$ is a *partition* of W, i.e., for any $w \in W$, $w \in [w]_\equiv$ and for any $v \in W$, either $[w]_\equiv \cap [v]_\equiv = \emptyset$ or $[w]_\equiv = [v]_\equiv$.
- $R_\equiv$ is well-defined, i.e., the definition does not depend on the choice of representatives. In other words, for any w, w', v, v'

$$w' \in [w]_\equiv, v' \in [v]_\equiv \text{ implies } (wRv \text{ iff } w'Rv').$$

- $R_\equiv$ is a partial ordering.

$(W_\equiv, R_\equiv)$ is called the *skeleton* of (W, R).

Now assume π is a valuation s.t.

$$\textbf{(*)} \qquad \text{for any } a \in \mathsf{Atoms}, \quad (W, R, \pi) \vDash a \to \Box a.$$

- Show that $\pi_\equiv(a) = \{[w]_\equiv \mid w \in \pi(a)\}$ is well-defined, i.e., $w' \in [w]_\equiv$ implies $w' \in \pi(a)$ iff $w \in \pi(a)$.
- Show that $\{(w, [w]_\equiv) \mid w \in W\}$ is a bisimulation between (W, R, π) and $(W_\equiv, R_\equiv, \pi_\equiv)$.
- Make precise the statement that *intuitionistic formulas are invariant under passing to the skeleton* and prove it using some of the previous results in this exercise and in the lecture.
- Show that without **(*)**, the previous claim may fail, and in fact there is even an example of a modal formula ϕ, an $\mathsf{S4}$ model (W, R, π) and $w \in W$ s.t.
  - $(W, R, \pi), w \vDash \phi$
  - $(W_\equiv, R_\equiv) \vDash \neg\phi$.

Aside. Note we can use now the finite model property of $\mathsf{S4}$, the Gödel translation and the results proved in this exercise to show that Int, unlike $\mathsf{S4}$, has the finite poset property. It would be possible to go further and give a semantic proof of the finite tree property.

Exercise 5 Intuitionistic Frame Conditions (5 Points)

Show that the following formulas in intuitionistic propositional logic correspond to the listed properties of partial-order Kripke frames:

1. $\neg a \vee \neg\neg a$: Directedness $(\forall x, y, z.\, x \leq y \wedge x \leq z \to \exists w.\, y \leq w \wedge z \leq w)$.

   Aside. For rooted finite frames, directedness is equivalent to boundedness $(\exists x.\, \forall y.\, y \leq x)$. You are, however, asked for a proof which works for arbitrary (possibly infinite) frames.

2. $(a \to b) \vee (b \to a)$: Linearity (on *rooted* frames, i.e., frames with a smallest element), i.e., $\forall x, y.\, x \leq y \vee y \leq x$.