Sheet 2

Due 24.11.2015

Exercise 1 Frame Correspondence (5 Points)

Consider the following table:

reflexivity ($\forall x.xRx$)	$\Box p \to p$
transitivity ($\forall xyz.xRy \& yRz \Rightarrow xRz$)	$\Box p \to \Box\Box p$
symmetry ($\forall xy.xRy \Rightarrow yRx$)	$p \to \Box\Diamond p$

For each of these rows, show that a frame satisfies the condition in the left column **iff** it satisfies the formula in the right column.

Aside 1. Note that this does not automatically mean that logics obtained by, e.g, adding these formulas to the minimal sequent system for K as axioms would be **complete** wrt frames satisfying corresponding conditions. We would still need a generic way to produce countermodels with underlying frames satisfying the corresponding property. In each of these particular three cases, such a procedure exists (indeed, for the combination of reflexivity and transitivity, this is exactly the procedure for creating countermodels from failed attempts at S4 proofs), but there are many pathological examples where things go wrong.

Aside 2. A related question is whether there is a general result on correspondence (and perhaps also completeness) which yields the three cases above (and perhaps also, e.g., the density example we did in the class). Still further, one would like to know if this generic procedure always produces *elementary*, i.e., first-order conditions (there are many conditions on frames which cannot be defined in first-order predicate logic as known from GLoIn, finiteness being one example). The answer is: such algorithms exists if the formula in question is in a "good enough" shape. This is a famous result proved first by Hendrik Sahlqvist in mid-1970's. Exercise 3 and Exercise 4 below use a formula which is not covered by the Sahqlvist Theorem. Sadly, the theorem is beyond the scope of this lecture.

Exercise 2 Some Sequent Proofs (5 Points)

Prove syntactically that the following formulas are theorems of K (i.e., search for their Gentzen-style proofs)

- $\Box p \wedge \Diamond q \to \Diamond(p \wedge q)$
- $\neg(\Box\Diamond\top \wedge \Diamond(q \vee r) \wedge \Box(q \to \Box p) \wedge \Box(\Diamond\neg p \to \neg r) \wedge \Box\Box\neg p)$

Update 17 Nov. The second formula in the originally posted sheet did not contain the subformula marked in blue, i.e., $\Box\Diamond\top$. This oversight would make the whole subformula under negation satisfiable. Apologies for this oversight. Note that the original formula would be valid over frames satisfying $\Diamond\top$—as a bonus question to think of on your own, do you see what condition it corresponds to? Is it more general than, e.g., reflexivity? And can you see how to prove the original formula if the sequent system contains the rule for (T)?

Exercise 3 Failure of the Finite Tree Property for S4 (5 Points)

Our model-theoretical investigations established that K has both

- the finite model property (via filtrations)
- the tree model property (via unravelling)

Furthermore, our analysis of proof search in sequent calculus showed that these two results can be strengthened to one: K has *the finite tree property*, i.e., every non-theorem can be refuted on a **finite tree**.

However, one has to be careful about combining such results in general. S4 also has these two properties (i.e., the finite model property—see Exercise 5—and the tree model property, via the transitive variant of the unravelling construction). But S4 does not have the finite tree property—or, indeed, even a finite *poset* property. The proof is spread over two exercises, this and the next one:

Show that the formula $\Box\Diamond p \to \Diamond\Box p$ is valid on every **finite** *partially ordered set*, i.e., a finite structure which is reflexive, transitive and *antisymmetric*: $\forall xy.xRy \& yRx \Rightarrow x = y$.

Hint. Think of some obvious property distinguishing finite posets from, say, natural numbers with the standard ordering.

Exercise 4 Failing S4 Proof Search (5 Points)

Show that the above formula $\Box\Diamond p \to \Diamond\Box p$ is **not** valid in S4 by conducting an exhaustive (hopefully, failing) proof search. Draw the obtained countermodel.

Aside. The formula in question has a corresponding condition on frames which even ensures completeness, but for the purpose of this exercise it does not matter how this condition looks like. Furthermore, on arbitrary frames this condition is not elementary; thus, the formula cannot be covered by the Sahlqvist Theorem mentioned in Exercise 1. Over transitive frames (for example, taken together with S4 axioms), the corresponding condition becomes first-order definable (you may guess then that even over posets, this condition is something different than finiteness); if you're interested, you can try to find out exactly which FO formula it is, but this is not a part of this exercise.

Exercise 5 Transitive Filtrations (5 Points)

- Show that the *finest* (*least*) $\overline{R}_{min}$ filtration of a S4 model does not have to be transitive
- Show that the *coarsest* (*greatest*) $\overline{R}_{max}$ filtration does not necessarily do the job either. **Hint.** If you do not insist on your model being connected/having a root, just three points with suitably chosen order, formula and valuation will do. Otherwise, you will need more points (do you see why?)
- Show that a modest modification of the coarsest filtration would do. To wit: set

  $$[x]_\alpha \overline{R}_{tr} [y]_\alpha \text{ iff for any } \Box\beta \in cl(\alpha),\ \mathcal{M}, x \vDash \Box\beta \text{ implies } \mathcal{M}, y \vDash \beta \textbf{ and } \mathcal{M}, y \vDash \Box\beta$$

  Prove that $\overline{R}_{tr}$ is a transitive relation which moreover is a filtration **whenever $\mathcal{M}$ is transitive** (a spare question to think of in your free time: can you run into problems without such an assumption? if so, where exactly?)