Mathematical Operational Semantics and Finitary System Behaviour

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Motivation

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Process algebra: SOS rules specify algebraic operations on system behaviour.

GSOS format → well-behaved operations (bisimilarity is a congruence)

B. Bloom, S. Istrail & A. Meyer: Bisimlation can‘t be traced. JACM 42, 1995.

Aceto‘s Theorem: The term model of a simple GSOS specification is regular.

Turi & Plotkin (Power et al., Bartels, Klin, …): Mathematical operational semantics
Interplay between syntax and semantics (sos rules) captured by distributive laws

Main question:

Can Aceto‘s Theorem be generalized to mathematical operational semantics?

Our results:

Generalization of Aceto‘s Theorem

Abstract rule format specifying operations on rational behaviour

Applications: concrete formats for: streams, (weighted) LTS‘s, (non-)determ. automata

Overview

- Abstract GSOS rules of Turi & Plotkin
- Simple GSOS and Aceto‘s Theorem
- Generalization of Aceto‘s Theorem
- Operations on rational behaviour
- Applications

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Operations on behaviour

SOS rules specify algebraic operations on system behaviour.

Example: Milner‘s CCS combinators

$$\frac{}{a.P \xrightarrow{a} P} \qquad \frac{P_1 \xrightarrow{a} P_1'}{P_1 + P_2 \xrightarrow{a} P_1'} \qquad \frac{P_2 \xrightarrow{a} P_2'}{P_1 + P_2 \xrightarrow{a} P_2'} \qquad \frac{P \xrightarrow{a} P'}{P \setminus L \xrightarrow{a} P' \setminus L}(a, \bar{a} \notin L)$$

$$\frac{P_1 \xrightarrow{a} P_1'}{P_1||P_2 \xrightarrow{a} P_1'||P_2} \qquad \frac{P_2 \xrightarrow{a} P_2'}{P_1||P_2 \xrightarrow{a} P_1||P_2'} \qquad \frac{P_1 \xrightarrow{a} P_1' \quad P_2 \xrightarrow{\bar{a}} P_2'}{P_1||P_2 \xrightarrow{\tau} P_1'||P_2'} \qquad \frac{P \xrightarrow{a} P'}{P[\rho] \xrightarrow{\rho(a)} P'[\rho]}$$

Example: the zip-operation on streams:

$$\frac{\sigma \xrightarrow{r_1} \sigma' \quad \tau \xrightarrow{r_2} \tau'}{\mathsf{zip}(\sigma, \tau) \xrightarrow{r_1} \mathsf{zip}(\tau, \sigma')}$$

e.g. $\mathsf{zip}((0,0,0,\ldots),(1,1,1,\ldots)) = (0,1,0,1,0,1,\ldots)$

Example: the shuffle-operation on languages:

$$\frac{s \xrightarrow{a} s'}{s \bowtie t \xrightarrow{a} s' \bowtie t} \qquad \frac{t \xrightarrow{a} t'}{s \bowtie t \xrightarrow{a} s \bowtie t'} \qquad \frac{s \downarrow \quad t \downarrow}{(s \bowtie t) \downarrow}$$

Abstract GSOS rules

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Turi's and Plotkin's abstract GSOS rules:

behaviour functor

$$\lambda : \Sigma(F \times Id) \Longrightarrow FT_\Sigma$$

free (term) monad

signature functor

captures GSOS format for:

$FX = \mathcal{P}_{\mathsf{f}}(A \times X)$ and $\Sigma =$ polynomial on Set

Abstract GSOS laws induce:

operational model

F-coalgebra on the initial Σ-algebra: $\mu\Sigma \to F(\mu\Sigma)$

Σ-algebra on the final F-coalgebra: $\Sigma(\nu F) \to \nu F$

initial

final

λ-bialgebra

denotational model

Example: labelled transition systems

GSOS format

B. Bloom, S. Istrail & A. Meyer: Bisimulation can‘t be traced. J. ACM 42, 1995.

Classical transition system specifications with rules of the form

$$\frac{\{x_{i_j} \xrightarrow{a_j} y_j\}_{j=1..m} \qquad \{x_{i_k} \stackrel{b_k}{\not\longrightarrow}\}_{k=1..l}}{f(x_1,\dots,x_n) \xrightarrow{c} t}$$

operation symbol from given signature Σ

arbitrary Σ-term on $\{x_1,\dots,x_n,y_1,\dots,y_m\}$

Example: Milner‘s CCS combinators

$$\frac{}{a.P \xrightarrow{a} P} \qquad \frac{P_1 \xrightarrow{a} P_1'}{P_1 + P_2 \xrightarrow{a} P_1'} \qquad \frac{P_2 \xrightarrow{a} P_2'}{P_1 + P_2 \xrightarrow{a} P_2'} \qquad \frac{P \xrightarrow{a} P'}{P \setminus L \xrightarrow{a} P' \setminus L}(a,\bar a \notin L)$$

$$\frac{P_1 \xrightarrow{a} P_1'}{P_1||P_2 \xrightarrow{a} P_1'||P_2} \qquad \frac{P_2 \xrightarrow{a} P_2'}{P_1||P_2 \xrightarrow{a} P_1||P_2'} \qquad \frac{P_1 \xrightarrow{a} P_1' \quad P_2 \xrightarrow{\bar a} P_2'}{P_1||P_2 \xrightarrow{\tau} P_1'||P_2'} \qquad \frac{P \xrightarrow{a} P'}{P[\rho] \xrightarrow{\rho(a)} P'[\rho]}$$

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Example: labelled transition systems

Aceto's Simple GSOS

L. Aceto: GSOS and Finite Labelled Transition Systems, TCS 131, 1994.

Classical transition system specifications with rules of the form

$$\frac{\{x_{i_j} \xrightarrow{a_j} y_j\}_{j=1..m} \qquad \{x_{i_k} \overset{b_k}{\not\longrightarrow}\}_{k=1..l}}{f(x_1,\ldots,x_n) \xrightarrow{c} t}$$

where $t = \begin{cases} z & \text{single variable} \\ g(z_1,\ldots,z_r) & \text{flat term} \end{cases}$

$$z, z_1, \ldots z_r \in \{\, x_{i_j}, y_j \mid i = 1..n, j = 1..m\}$$

Signature Σ of opns f can be infinite but:

Specification is bounded:

$\forall\ f$ and $\forall$ *positive trigger*: $\exists$ only finitely many rules with conclusion $f(x_1,\ldots,x_n) \xrightarrow{c} t$

Specification has finite dependency:

dependency $=$ transitive closure of $\begin{cases} \text{“}f\text{ depends on }g\text{” iff} \\ f(\vec{x}) \xrightarrow{c} g(\vec{z}) \text{ rule conclusion} \end{cases}$

Aceto's Theorem

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Examples: take constants $c_0, c_1, c_2, \ldots$ and infinitely many rules

$$\frac{}{c_n \xrightarrow{a} c_{n+1}}$$

infinite dependency

$$\frac{}{c_{n+1} \xrightarrow{a} c_n}$$

finite dependency

Theorem (L. Aceto).

L. Aceto: GSOS and finite labelled transition systems. TCS 131, 1994.

For a bounded transition system specification having finite dependency the operational model is *regular.*

How to generalize this to distributive laws?

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Bipointed Specifications

Given: $\mathcal{A}$ locally finitely presentable category

(e.g. Set, Pos, Graph, finitary variety, . . .)

$\Sigma,\ F : \mathcal{A} \to \mathcal{A}$ finitary endofunctors

Definition: Bipointed specifications are natural transformations

$$\Sigma(F \times Id) \Longrightarrow F(\Sigma + Id)$$

(cf. abstract GSOS rule: $\Sigma(F \times Id) \Longrightarrow FT_\Sigma$)

Example: $\mathcal{A} = \text{Set}$, $F = \mathcal{P}_{\text{fin}}(A \times -)$, $\Sigma =$ polynomial endofunctor

bipointed specifications = simple GSOS specification with bounded opns

What about finite dependency?

Finite dependency

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Definition:

Preserving finitely presentable objects

bipointed spec. $\lambda : \Sigma(F \times Id) \Rightarrow F(\Sigma + Id)$ has finite dependency if

1. there is filtered diagram of strongly finitary functors $\Sigma_i : \mathcal{A} \to \mathcal{A}$
2. the are bipointed specifications $\lambda_i : \Sigma_i(F \times Id) \to F(\Sigma_i + Id)$
3. λ is a filtered colimit of the λ_i:

$$\begin{array}{ccc} \Sigma_i(F \times Id) & \xrightarrow{\lambda_i} & F(\Sigma_i + Id) \\ {\scriptstyle \mathsf{in}_i(F\times Id)}\downarrow & & \downarrow{\scriptstyle F(\mathsf{in}_i + Id)} \\ \Sigma(F \times Id) & \xrightarrow[\lambda]{} & F(\Sigma + Id) \end{array}$$

Example:

$\mathcal{A} = \mathsf{Set}$

$\Sigma X = \{c_0, c_1, c_2, \ldots\}$ infinitely many constants

$\Sigma_n X = \{c_0, c_1, \ldots, c_n\}$

$$\Sigma = \bigcup_{n<\omega} \Sigma_n \qquad\qquad \frac{}{c_{n+1} \xrightarrow{a} c_n}$$

Aceto's Theorem generalized

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Operational model of a bipointed specification $\lambda : \Sigma(F \times Id) \to F(\Sigma + Id)$

$$\begin{tikzcd}
\Sigma(\mu\Sigma) \arrow[rr, "\iota"] \arrow[d, "\langle c, id \rangle"'] & & \mu\Sigma \arrow[d, "c"] \\
\Sigma(F(\mu\Sigma) \times \mu\Sigma) \arrow[r, "\lambda"'] & F(\Sigma(\mu\Sigma) + \mu\Sigma) \arrow[r, "{F[\iota, id]}"'] & F(\mu\Sigma)
\end{tikzcd}$$

unique

Theorem. Let $\lambda : \Sigma(F \times Id) \to F(\Sigma + Id)$ have finite dependency.

Then the operational model $(\mu\Sigma, c)$ is a locally finite F-coalgebra.

Example. $\mathcal{A} = \mathsf{Set}$

$C \xrightarrow{c} FC$ locally finite iff $\begin{cases} \forall x \in C \\ \exists \text{ finite } (C', c) \leq (C, c) \text{ with } x \in C' \end{cases}$

Definition. $C \xrightarrow{c} FC$ is locally finite if

finitely presentable objects

$$(C \xrightarrow{c} FC) = \text{filtered colimit of } C_i \xrightarrow{c_i} FC_i$$

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(Rational) Denotational Model

Denotational model of a bipointed specification $\lambda : \Sigma(F \times Id) \to F(\Sigma + Id)$

$$\begin{tikzcd}
\Sigma(\nu F) \arrow[r, "\Sigma\langle r, id\rangle"] \arrow[d, "\alpha"] & \Sigma(F(\nu F) \times \nu F) \arrow[r, "\lambda_{\nu F}"] & F(\Sigma(\nu F) + \nu F) \arrow[d, "{F[\alpha, id]}"] \\
\nu F \arrow[rr, "r"] & & F(\nu F)
\end{tikzcd}$$

unique

Now consider: final locally finite F-coalgebra

finitely presentable objects

$$(\varrho F \xrightarrow{r} F(\varrho F)) = \text{filtered colimit of all } C \to FC$$

Proposition. ϱF is a fixpoint of F (i.e. r isomorphism)

J. Adamek, S. Milius, J. Velebil: Iterative Algebras at Work, MSCS 2006

ϱF is called rational fixpoint of F

Examples

FX	coalgebras	νF	ϱF
$\{0,1\} \times X^A$	deterministic automata	$\mathcal{P}(A^*)$	regular languages
$\{0,1\} \times \mathcal{P}_{\mathsf{f}}(X)^A$	non-determ. automata	branching behaviour (up to bisimilarity)	finite state branching behaviours
$k \times X$	stream automata	k^ω	eventually periodic streams

$\mathcal{A} = \mathsf{Vec}_k$

$FX = k \times X$	k^ω	rational streams

More examples:

rational formal power series, rational Σ-trees, rational λ-trees, …

Operations on the rational fixpoint

Theorem.

Extends: M. Bonsangue, S. Milius, J. Rot: On the specification of operations on the rational behaviour of systems, EXPRESS/SOS 2012.

Let $\lambda : \Sigma(F \times Id) \to F(\Sigma + Id)$ be a bipointed spec. with finite dependency.

Then there exist a unique $\beta : \Sigma(\varrho F) \to \varrho F$ such that

$$\begin{tikzcd} \Sigma(\varrho F) \arrow[r, "\Sigma\langle r, id\rangle"] \arrow[d, "\beta"'] & \Sigma(F(\varrho F) \times \varrho F) \arrow[r, "\lambda_{\varrho F}"] & F(\Sigma(\varrho F) + \varrho F) \arrow[d, "{F[\beta, id]}"] \\ \varrho F \arrow[rr, "r"] & & F(\varrho F) \end{tikzcd}$$

„rational denotational model“

And this “restricts” the denotational model:

$$\begin{tikzcd} \Sigma(\varrho F) \arrow[r, "\Sigma h"] \arrow[d, "\beta"'] & \Sigma(\nu F) \arrow[d, "\alpha"] \\ \varrho F \arrow[r, "h"'] & \nu F \end{tikzcd}$$

unique F-coalgeba homomorphism

What about more general rule formats?

Counterexample: rational behaviour is not closed under operations specified by abstract GSOS rules

Consider streams: $FX = k \times X$ and $\Sigma X = C_{\mathbb{N}}$

$$\frac{}{c_0 \xrightarrow{0} c_1} \qquad \frac{}{c_1 \xrightarrow{1} c_2} \qquad \frac{}{c_2 \xrightarrow{2} c_3} \qquad \dots$$

defines $c_0 = (0, 1, 2, 3, \dots) \in \nu F$

For $\Sigma X = X$

$$\frac{x \xrightarrow{n} x'}{f(x) \xrightarrow{n+1} f(f(x))}$$

defines $f : \nu F \to \nu F$ with $f(0, 0, \dots) = (1, 2, 4, 8, \dots)$

Conjecture: all results still hold true for Klin‘s „coGSOS“ laws:

$$\Sigma \hat{F} \Longrightarrow F(\Sigma + Id)$$

cofree comonad on F

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Application 1: Labelled Transition Systems

$$FX = \mathcal{P}_{\mathsf{f}}(A \times X)$$ and $\Sigma =$ polynomial for a finite signature on Set

Bipointed specifications are equivalent to Aceto's simple GSOS rules:

$$\frac{\{x_{i_j} \xrightarrow{a_j} y_j\}_{j=1..m} \qquad \{x_{i_k} \not\xrightarrow{b_k}\}_{k=1..l}}{f(x_1, \ldots, x_n) \xrightarrow{c} t}$$

$$t = \left\{ \begin{array}{l} \text{variable} \\ \text{flat term} \end{array} \right\} \text{ in } \{x_{i_j}, y_j \mid i = 1..n, j = 1..m\}$$

Corollary. Aceto's Theorem.

Corollary. Operations defined by simple GSOS rules restrict to the *rational fixpoint of F*.

coproduct of all *finite* labelled transition systems modulo bisimilarity

Examples. All CCS combinators, e.g.

$$\frac{s \xrightarrow{a} s'}{s||t \xrightarrow{a} s'||t} \qquad \frac{t \xrightarrow{a} t'}{s||t \xrightarrow{a} s||t'}$$

Application 2: Streams

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$FX = k \times X$ and $\Sigma =$ polynomial for a finite signature on ~~Set~~ Vec_k

Bipointed specifications ~~are equivalent to~~ can be given by bipointed stream SOS rules:

$$\frac{x_1 \xrightarrow{r_1} x_1' \quad \cdots \quad x_n \xrightarrow{r_n} x_n'}{f(x_1, \ldots, x_n) \xrightarrow{r} t}$$

$$t = \begin{Bmatrix} \text{variable} \\ \text{flat term} \end{Bmatrix} \text{ in } \{x_1, \ldots, x_n, x_1', \ldots, x_n'\}$$

Corollary. Operations defined by bipointed stream SOS rules restrict to ~~eventually periodic~~ rational streams.

Examples.

$$\frac{\sigma \xrightarrow{r_1} \sigma' \quad \tau \xrightarrow{r_2} \tau'}{\mathsf{zip}(\sigma, \tau) \xrightarrow{r_1} \mathsf{zip}(\tau, \sigma')}$$

coGSOS rule

$$\frac{\sigma \xrightarrow{r_1} \sigma' \quad \sigma' \xrightarrow{r_2} \sigma''}{\mathsf{tail}(\sigma) \xrightarrow{r_2} \sigma''}$$

Application 3: Non-deterministic automata

$FX = 2 \times (\mathcal{P}_f X)^A$ and $\Sigma =$ polynomial for a finite signature on Set

Bipointed NDA SOS specifications

$$\frac{\{x_{i_j} \xrightarrow{a_j} y_j\}_{j=1..m} \qquad \{x_{i_k} \overset{b_k}{\not\longrightarrow}\}_{k=1..l}}{f(x_1,\ldots,x_n) \xrightarrow{c} t} \qquad\qquad \frac{\{x_{i_j}\downarrow\}_{j=1..k}}{f(x_1,\ldots,x_n)\downarrow}$$

accepting states

$t = \left\{ \begin{array}{l} \text{variable} \\ \text{flat term} \end{array} \right\}$ in $\{x_1,\ldots,x_n,y_1,\ldots,y_m\}$

Remark. This format is not complete w.r.t. to bipointed specifications.

Corollary. Operations defined by NDA SOS specifications on νF restrict to the rational fixpoint ρF.

Example. Shuffle operator

$$\frac{s \xrightarrow{a} s'}{s \bowtie t \xrightarrow{a} s' \bowtie t} \qquad \frac{t \xrightarrow{a} t'}{s \bowtie t \xrightarrow{a} s \bowtie t'} \qquad \frac{s\downarrow \quad t\downarrow}{(s \bowtie t)\downarrow}$$

Application 4: Deterministic Automata

$FX = 2 \times X^A$ on $\mathsf{Jsl}_\perp$ ← join-semilattices with bottom

$$\Sigma = (\,\mathsf{Jsl}_\perp \xrightarrow{\text{forget}} \mathsf{Set} \xrightarrow{P} \mathsf{Set} \xrightarrow{\text{free}} \mathsf{Jsl}_\perp\,)$$

polynomial for a finite signature Σ

Bipointed DA SOS specifications

$$\frac{s_1 \xrightarrow{a_1} s_1' \quad \cdots \quad s_n \xrightarrow{a_n} s_n' \qquad s_{i_1}\downarrow \quad \cdots \quad s_{i_k}\downarrow}{f(s_1, \ldots, s_n) \xrightarrow{a} t} \qquad \frac{s_{i_1}\downarrow \quad \cdots \quad s_{i_k}\downarrow}{f(s_1, \ldots, s_n)\downarrow},$$

$t =$ { term over $\Sigma + \{\oplus, \perp\}$ with every variable guarded by at most one Σ-symbol

Remark. Not complete w.r.t. to bipointed specifications.

Corollary. Regular languages are closed under operations defined by DA SOS specifications.

Application 4: Deterministic Automata

Shuffle operator

$$\frac{s \xrightarrow{a} s' \qquad t \xrightarrow{a} t' \qquad S}{s \bowtie t \xrightarrow{a} (s' \bowtie t) \oplus (s \bowtie t')} (S \subseteq \{s\downarrow, t\downarrow\}) \qquad\qquad \frac{s\downarrow \quad t\downarrow}{(s \bowtie t)\downarrow}$$

Sequential composition

$$\frac{s \xrightarrow{a} s' \qquad t \xrightarrow{a} t' \qquad S}{s;t \xrightarrow{a} s';t} (s\downarrow \notin S) \qquad\qquad \frac{s\downarrow \quad t\downarrow}{(s;t)\downarrow}$$

$$\frac{s \xrightarrow{a} s' \qquad t \xrightarrow{a} t' \qquad S}{s;t \xrightarrow{a} (s';t) \oplus t'} (s\downarrow \in S) \qquad\qquad \frac{t\downarrow}{(s;t)\downarrow}$$

Other examples: regular expression opns incl. Kleene star, …

Corollary. (to generalization of Aceto‘s theorem)

operational model $\mu\Sigma$ of regular expressions is locally finite

$\Longrightarrow$ regular expressions have finitely many derivatives

(modulo join-semilattice equations)

Application 5: weighted transition systems

$FX = (\mathcal{F}_{\mathbb{M}} X)^A$ $\Sigma =$ polynomial for a finite signature on Set

Bipointed WTS SOS specifications

$$\frac{\{x_{i_j} \xrightarrow{a_j, u_j} y_j\}_{j=1..m} \qquad \{x_i \overset{a}{\Rightarrow} w_{a,i}\}_{a \in D_i, i=1..n}}{f(x_1, \ldots, x_n) \xrightarrow{c,\ \beta(u_1, \ldots, u_k)} t}$$

$$t = \left\{ \begin{array}{l} \text{variable} \\ \text{flat term} \end{array} \right\} \text{ in } \{x_1, \ldots, x_n, y_1, \ldots, y_m\}$$

Remark. This format is not complete w.r.t. to bipointed specifications.

Corollary. Operations defined by WTS SOS specifications on νF restrict to the rational fixpoint ρF.

Example. Priority operator $\mathbb{M} = (\mathbb{R}^{\infty}, \min, \infty)$

$$\frac{x \overset{a}{\Rightarrow} w \quad x \overset{b}{\Rightarrow} v \quad x \xrightarrow{a,u} x'}{\partial_{ab}(x) \xrightarrow{a,u} \partial_{ab}(x')} \qquad \frac{x \overset{a}{\Rightarrow} v \quad x \overset{b}{\Rightarrow} w \quad x \xrightarrow{b,u} x'}{\partial_{ab}(x) \xrightarrow{b,u} \partial_{ab}(x')} \qquad \text{for all } w \leq v \in \mathbb{R}^{\infty}$$

Conclusions

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- Mathematical operational semantics meets finiteness:
  - bipointed specifications capture Aceto's simple GSOS
  - Generalization of Aceto's result that the operational model is regular
  - rational fixpoint is closed under operations specified by bipointed specifications
- Many interesting applications:
  - labelled transition systems, streams, (non-)deterministic automata, weighted transition systems, deterministic automata on join-semilattices, etc.

Future work

- More on bipointed specifications in algebraic categories (e.g. complete formats, other categories: locally finite varieties, …)
- Rational and context free power series
- Local finiteness of operational models and rational fixpoints: → decidability of bisimilarity, algorithms, tool development