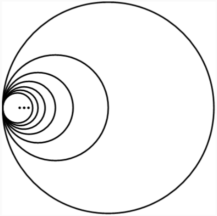
Nominal Topology for Data Languages



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Regular Languages: Algebraic Perspective

Regular languages $=$ **Languages recognizable by finite monoids**

A finite monoid M recognizes the language $L\colon \Sigma^* \to 2$

$$:\Longleftrightarrow$$

$\exists$ monoid morphism $e\colon \Sigma^* \to M$ and predicate $p\colon M \to 2$ with

$$\begin{tikzcd}
\Sigma^* \arrow[r, "L"] \arrow[d, "e"'] & 2 \\
M \arrow[ur, "p"'] &
\end{tikzcd}$$

Regular Languages: Topological Perspective

Topological space of profinite words:

$$\widehat{\Sigma^*} = \text{limit of all finite quotient monoids } e : \Sigma^* \twoheadrightarrow M.$$

Regular languages $\Sigma^* \to 2 \quad \cong \quad$ **continuous predicates** $\widehat{\Sigma^*} \to 2$

Stone Spaces

A topological space X is a Stone space if it is

- **compact:** every open cover has a finite subcover.
- **Hausdorff**: any $x \neq y$ in X can be separated by disjoint open sets.
- **zero-dimensional**: has a base of clopen sets .

↑ continuous predicates $X \to 2$

Stone: category of Stone spaces and continuous maps.

Example: Cantor space

$2^\omega = \{0,1\} \times \{0,1\} \times \cdots \quad \in$ **Stone**

Stone Spaces: Two Key Properties

- Stone spaces = Profinite spaces.

  cofiltered limits of finite discrete spaces

  Proof: uses Tychonoff's theorem (⇔ AC).

- Finite discrete spaces = Finitely copresentable objects in **Stone**.

continuous map

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "p_i"'] & D \\
X_i &
\end{tikzcd}$$

finite discrete space

cofiltered limit

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  Proof: uses Tychonoff's theorem ($\Leftrightarrow$ AC).

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continuous map

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p_i"'] & D \\ X_i \arrow[ur, dashed, "\exists f'"'] & \end{tikzcd}$$

finite discrete space

cofiltered limit

Stone Spaces: Universal Property

Stone is the free completion of $\mathbf{Set}_f$ under cofiltered limits:

$$\mathbf{Stone} = \mathrm{Pro}(\mathbf{Set}_f)$$

pres. cofiltered limits

has cofiltered limits

$$\begin{tikzcd}
\mathbf{Stone} \arrow[rr, dashed, "\exists ! F'"] & & \mathcal{C} \\
& \mathbf{Set}_f \arrow[ul, tail] \arrow[ur, "F"'] &
\end{tikzcd}$$

Regular Languages: Topological Perspective

Stone space of profinite words:

$$\widehat{\Sigma^*} = \text{limit of all finite quotient monoids } e : \Sigma^* \twoheadrightarrow M.$$

- Poset $\Sigma^* \downarrow \mathbf{Mon}_f$ of finite quotient monoids $e \colon \Sigma^* \twoheadrightarrow M$.
- Codirected diagram

$$D \colon \Sigma^* \downarrow \mathbf{Mon}_f \to \mathbf{Stone}, \quad (e \colon \Sigma^* \twoheadrightarrow M) \mapsto |M|\,.$$

- $\widehat{\Sigma^*} = \lim D$ with limit cone

$$\begin{tikzcd}
\widehat{\Sigma^*} \arrow[d, "p_e"] & \\
{|M|} & (e \colon \Sigma^* \twoheadrightarrow M)
\end{tikzcd}$$

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Regular languages $\Sigma^* \to 2$ $\cong$ **continuous predicates** $\widehat{\Sigma^*} \to 2$

$$\Sigma^* \xrightarrow{L} 2 \qquad \Longrightarrow \qquad \widehat{\Sigma^*} \xrightarrow{\widehat{L}} 2$$

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$$\begin{tikzcd}
\Sigma^* \arrow[r, "L"] \arrow[d, two heads, "e"'] & 2 \\
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\end{tikzcd}
\qquad \Longrightarrow \qquad
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\widehat{\Sigma^*} \arrow[r, "\widehat{L}"] \arrow[d, "p_e"'] \arrow[dr, phantom, "\text{def.}"] & 2 \\
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$$\begin{tikzcd}
\Sigma^* \arrow[r, "L"] \arrow[d, two heads, "e"'] & 2 & & \widehat{\Sigma^*} \arrow[r, "\widehat{L}"] \arrow[d, "p_e"'] & 2 \\
M \arrow[ur, "f"', "\text{def.}"] & & \Longleftarrow & {|M|} \arrow[ur, "f"'] &
\end{tikzcd}$$

fin. copres.

Regular Languages: Topological Perspective

Stone duality

$$\begin{array}{ccc} \mathbf{Stone}^{\mathrm{op}} & \cong & \mathbf{BoolAlg} \\ X & \mapsto & \{\, \text{continuous predicates } X \to 2 \,\} \end{array}$$

$$\widehat{\Sigma^*} \quad \mapsto \quad \{\, \text{regular languages } \Sigma^* \to 2 \,\}$$

Pippenger 1997

Gehrke, Grigorieff, Pin 2008

Adámek, Chen, Milius, Myers, Urbat 2014–2017

Nominal Sets

$\mathbb{A}$ = fixed countably infinite set of names

- Nominal set: set X with a renaming operator, i.e. group action

  $$\mathrm{Perm}(\mathbb{A}) \times X \to X, \quad (\pi, x) \mapsto \pi \cdot x,$$

  such that every $x \in X$ has finite support

  $$\mathrm{supp}_X(x) \subseteq \mathbb{A}.$$
- Orbit-finite nominal set: finite up to renaming.
- **Nom**: category of nominal sets and equivariant maps.

Example: $X = \mathbb{A}^*$ (finite words over $\mathbb{A}$)

$(bc) \cdot abbc = accb$ and $\mathrm{supp}_{\mathbb{A}^*}(abbc) = \{a, b, c\}$

Data Languages

- Data language: equivariant map $L\colon \Sigma^* \to 2$ with $\Sigma \in \mathbf{Nom}$.
- Nominal monoid: nominal set M with equivariant monoid structure.
- Regular data language: recognizable by an orbit-finite nom. monoid.

$$\begin{tikzcd}
\Sigma^* \arrow[r, "L"] \arrow[d, "e"'] & 2 \\
M \arrow[ur, "p"'] &
\end{tikzcd}$$

Equivalent descriptions: rigid MSO and single-use register automata.

Goal: Topological perspective

Data Languages: Topological Perspective

For classical regular languages, topological view based on

$$\mathrm{Pro}(\mathbf{Set}_f) \quad = \quad \mathbf{Stone}.$$

For regular data languages: try to prove

$$\mathrm{Pro}(\mathbf{Nom}_{\mathrm{of}}) \quad = \quad \text{Nominal Stone spaces (?)}.$$

pres. cofiltered limits

$$\begin{tikzcd}
\mathrm{Pro}(\mathbf{Nom}_{\mathrm{of}}) \arrow[rr, dashed, "U"] & & \mathbf{Nom} \\
& \mathbf{Nom}_{\mathrm{of}} \arrow[ul, tail] \arrow[ur, tail] &
\end{tikzcd}$$

☹ U not faithful!

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☹ U not faithful!

The Issue With $\mathrm{Pro}(\mathbf{Nom}_{\mathrm{of}})$

Consider ω^{op}-cochain in $\mathbf{Nom}_{\mathrm{of}}$:

$$\mathbb{A}^{\#0} \leftarrow \mathbb{A}^{\#1} \leftarrow \mathbb{A}^{\#2} \leftarrow \cdots\cdots \mathbb{A}^{\#n} \leftarrow \cdots$$

where

$$\mathbb{A}^{\#n} \;=\; \{\,(a_1,\ldots,a_n) \in \mathbb{A}^n \;:\; a_i \neq a_j \text{ for } i \neq j\,\}.$$

Limit in **Nom** is $\emptyset$ – but limit in $\mathrm{Pro}(\mathbf{Nom}_{\mathrm{of}})$ is nontrivial!

☹ Issue: Elements of $\mathbb{A}^{\#0}, \mathbb{A}^{\#1}, \mathbb{A}^{\#2}, \ldots$ have unbounded support.

☺ Remedy: Restrict to bounded cofiltered diagrams!

Bounded Diagrams

A nominal set X is *n*-bounded ($n \in \mathbb{N}$) if

$$\forall x \in X.\, |\mathrm{supp}_X(x)| \leq n.$$

A diagram $D\colon I \to \mathbf{Nom}$ is *n*-bounded if each D_i ($i \in I$) is n-bounded.

Observation

D n-bounded cofiltered $\implies$ $\lim D$ is n-bounded and formed in **Set**.

We will show the following result:

$$\underbrace{\mathrm{Pro}(\mathbf{Nom}_{\mathrm{of,n}})}_{} \quad = \quad n\text{-bounded nominal Stone spaces.}$$

n-bounded orbit-finite sets

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continuous predicates $X \to 2$

Now: Introduce nominal analogues of these notions.

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Nominal Topological Spaces

Nominal topological space: nominal set X with a family

$$\tau \;\subseteq_{\mathsf{eq}}\; \mathcal{P}_{\mathsf{fs}}X$$

of open sets closed under finite intersection and finite-supported union:

$$U_1, \ldots, U_n \in \tau \quad \Rightarrow \quad \bigcap_{i=1}^{n} U_i \in \tau \qquad \text{and} \qquad \tau' \subseteq \tau \text{ f.s.} \quad \Rightarrow \quad \bigcup \tau' \in \tau.$$

An equivariant map $f\colon X \to Y$ is continuous if

$$U \subseteq Y \text{ open} \quad \Rightarrow \quad f^{-1}[U] \subseteq X \text{ open}.$$

NomTop: Nominal topological spaces and equivariant continuous maps.

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Nominal Topological Spaces: Compactness

X : Nominal topological space with topology $\tau \subseteq_{\mathsf{eq}} \mathcal{P}_{\mathsf{fs}} X$.

- Open cover: finitely supported set $\mathcal{C} \subseteq \tau$ of open sets with

$$\bigcup \mathcal{C} = X.$$

- Subcover of $\mathcal{C}$: finitely supported set $\mathcal{C}' \subseteq \mathcal{C}$ with

$$\bigcup \mathcal{C}' = X.$$

- Orbit-finite cover: meets only finitely many orbits of τ.

The space X is compact if every open cover has an orbit-finite subcover.

Nominal Topological Spaces: Compactness

Example

X orbit-finite $\quad\Longrightarrow\quad$ X compact.

Example: Tychonoff fails!

$\mathbb{A}^\omega$ is not compact: the open sets

$$U_{i,j} = \{\, (a_n)_{n\in\mathbb{N}} \in \mathbb{A}^\omega \;:\; a_i = a_j \,\} \qquad (i \neq j)$$

cover $\mathbb{A}^\omega$, but there is no (orbit-)finite subcover.

Nominal Compactness vs. Classical Compactness

X : Nominal topological space with topology $\tau \subseteq_{\mathsf{eq}} \mathcal{P}_{\mathsf{fs}} X$.

For each $S \subseteq_{\mathsf{fin}} \mathbb{A}$, get ordinary topological space $|X|_S$ with

- same underlying set as X;
- open sets of $|X|_S =$ open sets of X with support S.

This gives forgetful functors

$$|{-}|_S : \mathbf{NomTop} \to \mathbf{Top} \qquad (S \subseteq_{\mathsf{fin}} \mathbb{A}).$$

Lemma

X compact $\iff$ $|X|_S$ compact for each $S \subseteq_{\mathsf{fin}} \mathbb{A}$.

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continuous predicates $X \to 2$

Now: Introduce nominal analogues of these notions.

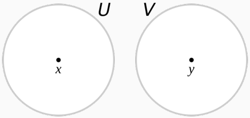
Nominal Topological Spaces: Hausdorff Property

$$x \equiv_S y \quad \text{iff} \quad \exists \pi \in \mathrm{Perm}_S(\mathbb{A}).\ x = \pi \cdot y.$$

permutations fixing S

A nominal topological space X is Hausdorff if, for $S \subseteq_{\mathrm{fin}} \mathbb{A}$ and $x, y \in X$,

$x \not\equiv_S y \quad \implies \quad \exists$ disjoint open sets U, V of support S with $x \in U$, $y \in V$.



Example

X discrete (i.e. $\tau = \mathcal{P}_{\mathrm{fs}} X$) $\quad \implies \quad$ X Hausdorff.

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continuous predicates $X \to 2$

Now: Introduce nominal analogues of these notions.

Nominal Topological Spaces: Representable Sets

X : Nominal topological space with topology $\tau \subseteq_{\mathsf{eq}} \mathcal{P}_{\mathsf{fs}} X$.

A representable set $C \subseteq X$ is one of the form

$$C = f^{-1}[d]$$

where

$$f : X \to D \text{ continuous}, \quad D \text{ discrete \& orbit-finite}, \quad d \in D.$$

Note

$$C \text{ representable} \quad \begin{array}{c} \Longrightarrow \\ \not\Longleftarrow \end{array} \quad C \text{ clopen}$$

X is zero-dimensional if it has a base of representable sets.

Nominal Stone Spaces

A nominal topological space X is a nominal Stone space if it is

- **compact:** every open cover has a orbit-finite subcover.
- **Hausdorff**: $x \not\equiv_S y \Rightarrow \exists$ disj. S-supported open sets separating x, y.
- **zero-dimensional**: has a base of representable sets.

Example

X orbit-finite & discrete $\quad\Longrightarrow\quad$ X Stone.

$\neq$ **Gabbay, Litak, Petrișan 2011**

Nominal Stone Spaces: Universal Property

Theorem

$$\mathrm{Pro}(\mathbf{Nom}_{\mathrm{of,n}}) \quad = \quad \mathbf{NomStone}_n$$

n-bounded orbit-finite sets

n-bounded nominal Stone spaces

pres. cofiltered limits

has cofiltered limits

$$\begin{tikzcd}
\mathbf{NomStone}_n \arrow[rr, dashed, "\exists! F'"] & & \mathcal{C} \\
& \mathbf{Nom}_{\mathrm{of,n}} \arrow[ul, tail] \arrow[ur, "F"'] &
\end{tikzcd}$$

Nominal Stone Spaces: Two Key Properties

- n-bounded nominal Stone spaces = n-bounded pro-orbit-finite spaces.

  cofiltered limits of n-bounded orbit-finite discrete spaces

  Proof: uses classical (!) Tychonoff theorem.

- n-bounded orbit-finite discrete spaces

  = finitely copresentable objects in $\mathbf{NomStone}_n$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "p_i"'] & D \\
X_i \arrow[ur, dashed, "\exists f'"'] &
\end{tikzcd}$$

continuous map

orbit-finite discrete space

n-bounded cofiltered limit

Data Languages: Topological Perspective

Nominal Stone space of *n*-bounded pro-orbit-finite words over $\Sigma \in \mathbf{Nom}$:

$$\widehat{\Sigma_n^*} = \text{limit of all } n\text{-bounded orbit-finite quotient monoids } e : \Sigma^* \twoheadrightarrow M.$$

Theorem

n-**regular data languages** $\Sigma^* \to 2 \;\cong\;$ **continuous predicates** $\widehat{\Sigma_n^*} \to 2$

recognizable by n**-bounded orbit-finite monoids**

Proof: As in classical case, use that 2 is finitely copresentable.

Data Languages: Dual Perspective

For classical regular languages: duality theory based on

$$\mathbf{Stone}^{\mathsf{op}} \quad \cong \quad \mathbf{BoolAlg}.$$

Candidate for nominal Stone duality:

$$\mathbf{NomStone}_n^{\mathsf{op}} \quad \cong \quad n\text{-}\mathbf{atomic\ nominal\ boolean\ algebras}.$$

$\neq$ **Gabbay, Litak, Petrișan 2011**