Assignment 2

Deadline for solutions: 30.05.2015, **12.15 a.m.**

Exercise 1 By far non-Pythagorean numbers (8 Points)

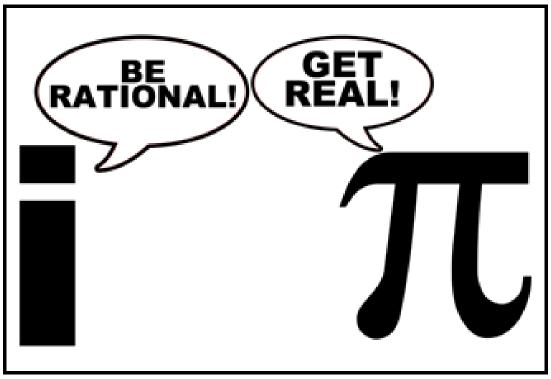
Consider a notion of number which includes all natural numbers and supports the operations of summation and multiplication. Let us denote by S the set of such numbers. We can extend S to the numbers of the form

$$a + \sqrt{2} \cdot b \qquad (*)$$

with $a, b \in S$ and denote the extended numbers as $S[\sqrt{2}]$. Note that depending on S, S may be equi-expressive with $S[\sqrt{2}]$ (e.g. if S are all real numbers) or properly less expressive (e.g. if S are all rational numbers).

Implement the numbers ($*$) in Haskell as an algebraic data type

```
Sq2Num a
```



where `a` is the type capturing the elements of S. Ensure that `Sq2Num a` (under suitable assumptions) is an instance of the following type classes: `Eq`, `Ord`, `Show`, `Num`, `Fractional`, e.g. by completing the following declarations:

```
instance (Num a, Eq a) => Eq  (Sq2Num a)
instance (Num a, Eq a) => Ord (Sq2Num a)
instance (Num a, Eq a) => Num (Sq2Num a)
instance (Fractional a, Eq a) => Fractional (Sq2Num a)
```

Additionally, provide a conversion function

```
getReal :: Floating a => Sq2Num a -> a
```

reducing from $S[\sqrt{2}]$ to S in such a way that real numbers are converted to themselves.

Hint: For inspiration, use the standard implementation of complex numbers in Haskell [1]. Like in the case of complex numbers, you need to prove (!) and implement the mathematical fact that the numbers ($*$) are closed under summation and multiplication and additionally under division, provided that so are the numbers from S.

Exercise 2 Unfolding Y (8 Points)

Recall the call-by-value small-step rule

$$Y f \to f(\lambda x.\,(Y f)x)$$

where $f : (A \to B) \to (A \to B)$. The soundness theorem implies that

$$[\![- \vdash (Y f) : A \to B]\!] = [\![- \vdash f(\lambda x.\,(Y f)x) : A \to B]\!]$$

where "$-$" denotes the empty variable context.

(a) Prove that moreover

$$[\![\Gamma \vdash (Y f) : A \to B]\!](\rho) = [\![\Gamma \vdash f(\lambda x.\,(Y f)x) : A \to B]\!](\rho) \tag{1}$$

for every $\rho \in [\![\Gamma]\!]$, directly using the rules of denotational call-by-value semantics of PCF.

(b) Implement the factorial function $fac : Nat \to Nat$ in PCF as a function of the form $Y p$ for a suitable term p. Prove formally that $[\![n : Nat \vdash fac(n) : Nat]\!](3) = 6$ under the call-by-value semantics.

Hint: use (1).

Exercise 3 `curry` and `uncurry` (4 Points)

Chose one of the functions `curry` $: A^{B\times C} \to (A^B)^C$ and `uncurry` $: (A^B)^C \to A^{B\times C}$, defined at the lecture, as you please, and prove it to be continuous (in particular monotone).

References

[1] https://www.haskell.org/onlinereport/complex.html.