Chapters 2.9 to 2.11
Homotopy Type Theory Seminar, Summer '17

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$\prod$-types and the function extensionality axiom

$\prod$-types

Given

$$A : \mathcal{U}$$

$$B : A \to \mathcal{U}$$

$$f, g : \prod_{x:A} B(x)$$

consider

$$(f = g) \simeq \left(\prod_{x:A} (f(x) =_{B(x)} g(x)) \right).$$

Not provable!

Only

$$\mathsf{happly} : (f = g) \to \prod_{x:A} (f(x) =_{B(x)} g(x))$$

definable by path induction.

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$$\mathsf{happly} : (f = g) \to \prod_{x:A} (f(x) =_{B(x)} g(x))$$

definable by path induction.

Axiom (Function extensionality)

For any A, B, f, and g, happly *is an equivalence.*

Implication: happly has a quasi-inverse:

$$\mathsf{funext} : \left(\prod_{x:A} (f(x) = g(x))\right) \to (f = g).$$

Rules for $f = g$

Introduction rule: $\mathsf{funext} : \left(\prod_{(x:A)}(f(x) = g(x))\right) \to (f = g)$

Elimination rule: $\mathsf{happly} : (f = g) \to \prod_{(x:A)}(f(x) =_{B(x)} g(x))$

Computation rule: $\mathsf{happly}(\mathsf{funext}(h), x) = h(x)$

Uniqueness rule: $p = \mathsf{funext}(x \mapsto \mathsf{happly}(p, x))$

Transporting functions

Given

$$X : \mathcal{U}$$

$$p : x_1 =_X x_2$$

$$A, B : X \to \mathcal{U}$$

$$f : A(x_1) \to B(x_1)$$

we define

$$\mathrm{transport}^{A \to B}(p, f) = a_2 \mapsto \mathrm{transport}^{B}(p, f(\mathrm{transport}^{A}(p^{-1}, a_2)))$$

where

$$(A \to B)(x) :\equiv (A(x) \to B(x)).$$

Transport dependent functions

Given

$$X : \mathcal{U}$$

$$p : x_1 =_X x_2$$

$$A : X \to \mathcal{U}$$

$$B : \prod_{x:X} A(x) \to \mathcal{U}$$

$$f : \prod_{a:A(x_1)} B(x_1, a)$$

we define

$$\mathsf{transport}^{\Pi_A(B)}(p, f)(a) =$$
$$\mathsf{transport}^{\widehat{B}}\Big((\mathsf{pair}^{=}(p^{-1}, \mathsf{refl}_{{p^{-1}}_*(a)}))^{-1},\ f(\mathsf{transport}^{A}(p^{-1}, a))\Big)$$

Dependent paths between functions

Given

$$A, B : X \to \mathcal{U}$$

$$p : x =_X y$$

$$f : A(x) \to B(x)$$

$$g : A(y) \to B(y)$$

we have an equivalence

$$\big(p_*(f) = g\big) \;\simeq\; \prod_{a:A(x)} (p_*(f(a)) = g(p_*(a))).$$

Dependent paths between functions

Moreover, if $q : p_*(f) = g$ corresponds under this equivalence to $\widehat{q}$, then for $a : A(x)$, the path

$$\mathsf{happly}(q, p_*(a)) : (p_*(f))(p_*(a)) = g(p_*(a))$$

is equal to the composite

$$\begin{aligned}(p_*(f))(p_*(a)) &= p_*\big(f({p^{-1}}_*(p_*(a)))\big) && \text{(transport)}\\ &= p_*(f(a))\\ &= g(p_*(a)).\end{aligned}$$

Dependent paths between dependent functions

Given

$$A : X \to \mathcal{U}$$

$$B : \prod_{x:X} A(x) \to \mathcal{U}$$

$$p : x =_X y$$

$$f : \prod_{a:A(x)} B(x, a)$$

$$g : \prod_{a:A(y)} B(y, a)$$

we have an equivalence

$$\big(p_*(f) = g\big) \;\simeq\; \Big(\prod_{a:A(x)} \mathsf{transport}^{\widehat{B}}(\mathsf{pair}^{=}(p, \mathsf{refl}_{p_*(a)}), f(a)) = g(p_*(a))\Big)$$

Universes and the univalence axiom

Identification of types

Given types $A, B : \mathcal{U}$, we can form the identity type $A =_{\mathcal{U}} B$ aswell as a function

$$\mathsf{idtoeqv} : (A =_{\mathcal{U}} B) \to (A \simeq B).$$

Identification of types

Given types $A, B : \mathcal{U}$, we can form the identity type $A =_{\mathcal{U}} B$ aswell as a function

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Axiom (Univalence)

For any $A, B : \mathcal{U}$, idtoeqv *is an equivalence.*

Therefore we have

$$(A =_{\mathcal{U}} B) \simeq (A \simeq B).$$

Rules for $(A =_{\mathcal{U}} B)$

Introduction: $\mathsf{ua} : (A \simeq B) \to (A =_{\mathcal{U}} B)$

Elimination: $\mathsf{idtoeqv} \equiv \mathsf{transport}^{X \mapsto X} : (A =_{\mathcal{U}} B) \to (A \simeq B)$

Computation: $\mathsf{idtoeqv}(\mathsf{ua}(f))(x) = f(x)$

Uniqueness: $p = \mathsf{ua}(\mathsf{idtoeqv}(p))$

Univalence and transport

For any

$$B : A \to \mathcal{U}$$

$$x, y : A$$

$$p : x = y$$

$$u : B(x)$$

we have

$$\begin{aligned} \mathsf{transport}^{B}(p, u) &= \mathsf{transport}^{X \mapsto X}(\mathsf{ap}_{B}(p), u) \\ &= \mathsf{idtoeqv}(\mathsf{ap}_{B}(p))(u). \end{aligned}$$

Identity type

Paths between paths

There is no simple characterization of the type $p =_{a =_A a'} q$ of paths between paths. However, identity types do respect equivalence:
If $f : A \to B$ is an equivalence, then for all $a, a' : A$, so is

$$\mathsf{ap}_f : (a =_A a') \to (f(a) =_B f(a')).$$

Given a characterization of $a =_A a'$, this can be used to describe $p =_{a =_A a'} q$.

Paths between paths of pairs

Recall that

$$(w =_{A\times B} w') \to (\mathrm{pr}_1(w) =_A \mathrm{pr}_1(w')) \times (\mathrm{pr}_2(w) =_B \mathrm{pr}_2(w'))$$

is an equivalence. Thus paths $p = q$, where $p, q : w =_{A\times B} w'$, are equivalent to pairs of paths

$$\mathrm{ap}_{\mathrm{pr}_1} p =_{\mathrm{pr}_1 w =_A \mathrm{pr}_1 w'} \mathrm{ap}_{\mathrm{pr}_1} q \quad \text{and} \quad \mathrm{ap}_{\mathrm{pr}_2} p =_{\mathrm{pr}_2 w =_B \mathrm{pr}_2 w'} \mathrm{ap}_{\mathrm{pr}_2} q.$$

Paths between paths of $\prod$-types

Paths $p = q$, where $p, q : f =_{\prod_{(x:A)} B(x)} g$, are equivalent to homotopies

$$\prod_{x:A} (\mathsf{happly}(p)(x) =_{f(x)=g(x)} \mathsf{happly}(q)(x)).$$

Transport in families of paths

For any A and $a : A$ with $p : x_1 = x_2$, we have

$$\begin{aligned}
\mathsf{transport}^{x \mapsto (a = x)}(p, q) &= q \cdot p && \text{for } q : a = x_1, \\
\mathsf{transport}^{x \mapsto (x = a)}(p, q) &= p^{-1} \cdot q && \text{for } q : x_1 = a, \\
\mathsf{transport}^{x \mapsto (x = x)}(p, q) &= p^{-1} \cdot q \cdot p && \text{for } q : x_1 = x_1.
\end{aligned}$$

Spot the hom-functors!

Transport in families of paths

For $f, g : A \to B$, with $p : a =_A a'$ and $q : f(a) =_B g(a)$, we have

$$\mathsf{transport}^{x \mapsto f(x) =_B g(x)}(p, q) =_{f(a') = g(a')} (\mathsf{ap}_f\, p)^{-1} \cdot q \cdot \mathsf{ap}_g\, p.$$

Transport in families of paths

Let $B : A \to \mathcal{U}$ and $f, g : \prod_{(x:A)} B(x)$, with $p : a =_A a'$ and $q : f(a) =_{B(a)} g(a)$. Then we have

$$\mathsf{transport}^{x \mapsto f(x) =_{B(x)} g(x)}(p, q) = (\mathsf{apd}_f(p))^{-1} \cdot \mathsf{ap}_{(\mathsf{transport}^B p)}(q) \cdot \mathsf{apd}_g(p).$$

Dependent paths for families of paths

For $p : a =_A a'$ with $q : a = a$ and $r : a' = a'$, we have

$$\left(\mathsf{transport}^{x \mapsto (x=x)}(p, q) = r\right) \;\simeq\; \left(q \cdot p = p \cdot r\right).$$