Introduction to Homotopy

Homotopy Type Theory Seminar 2017SS

Jonathan Krebs

Friedrich-Alexander-Universität Erlangen-Nürnberg

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What is this about?

Basic Topology

Homotopy

The Fundamental Group

Basic Topology

Topological Spaces

Definition (Topology, Topological space)

Let X be a set, $\mathcal{O} \subset \mathcal{P}(X)$ a set of subsets. The pair $(X, \mathcal{O})$ is a *topological space*, if

1. $\emptyset \in \mathcal{O}, X \in \mathcal{O}$
2. for any subset $\mathcal{A} \subset \mathcal{O}$ also the union $\bigcup_{A \in \mathcal{A}} A \in \mathcal{O}$
3. for $A, B \in \mathcal{O}$ the intersection $A \cap B \in \mathcal{O}$.

Then $\mathcal{O}$ is called *topology* of X, and $M \in \mathcal{O}$ an *open set*.

Example spaces

- $\emptyset$ and $\{pt\}$ are topological spaces with an unique topology.
- X some set. $\mathcal{P}(X)$ is called the *discrete topology*, $\{X, \emptyset\}$ is called *chaotic topology*
- for $\mathbb{R}$ there is a *standard topology*, c.f. blackboard.

More definitions

Let $(X, \mathcal{O})$ be a topological space.

Definition (closed)

A set $M \subset X$ is *closed*, if its complement is open: $X \backslash M \in \mathcal{O}$.

Remark

Swap $\cup$ *with* $\cap$ *and open with closed in the definition of a topology, and you have an alternative equivalent definition.*

Definition (Subspace topology)

Let $Y \subset X$ any subset. The topology given by $\mathcal{O}' = \{M \cap Y | M \in O\}$ is called *subspace topology*, $(Y, \mathcal{O}')$ a *subspace* of X.

Continuity

Let $(X, \mathcal{T}) and (Y, \mathcal{S})$ be topological spaces.

Definition (continuous)

A map $f : X \to Y$ is called *continuous* or a *morphism of topological spaces*, if the preimage of an open set is open: $f^{-1}(M) \in \mathcal{T}$ for all $M \in \mathcal{S}$.

Example

maps from a discrete space or to a chaotic space are continuous, so are constant maps or the identity.

Definition (homeomorphism)

$f : X \to Y$ is a *homeomorphism*, if it is continuous, invertible, and the inverse is continuous, too.

Definition

homeomorphic If there is a homeomorphism between X and Y, they are said to be *homeomorphic*.

Which Spaces are Homeomorphic?

Here the $\mathbb{R}^n$ have their standard topology, subsets the subspace topology.

- $\mathbb{R}^2$
- $\mathbb{R}^3$
- $D^2 := \left\{x \in \mathbb{R}^2 \middle| \, ||x|| \leq 1\right\}$
- $B^2 := \left\{x \in \mathbb{R}^2 \middle| \, ||x|| < 1\right\}$
- $S^1 := \left\{x \in \mathbb{R}^2 \middle| \, ||x|| = 1\right\}$
- $D^2 \setminus \frac{1}{2} B^2$
- $\mathbb{R}^2 \setminus \{0\}$

Remark

Homeomorphy is an equivalence relation (condition is symmetric in f or f^{-1}, id is invertable and continuous.) Write $X \approx Y$.

Only $\mathbb{R}^2 \approx B^2$: $f^{\pm 1} : x \mapsto \tanh^{\pm 1}(||x||) \cdot \frac{x}{||x||}$

Homeomorphic spaces are the same from topological perspective.
Next we look for a weaker relation for spaces that "look similar".

Constructions with Topological Spaces

Definition (Product space)

For $X \times Y$ chose the coarsest topology (fewest open sets), so that the maps $\pi_1 : (x, y) \mapsto x$ and π_2 are continuous. It also works for infinite products $\prod_{i \in I} A_i$. You can always project from a product to a component.

Example

$R^n = \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}, \quad S^1 \times S^1$ is a torus.

Definition (Sum Space $X + Y$)

The disjoint union of topological spaces can be equipped with the disjoint union of the topologies. You embed every component into the sum: $\iota_1(x) := (1, x), \iota_2(x) := (2, x)$,

Constructions with Topological Spaces

Definition (Quotient Space)

Let X be a top. space,

- $\sim$ an equivalence relation. $X/ \ = \{[a]_\sim | a \in X\}$ is called *quotient space*, the topology is generated by taking equivalence classes.
- U be a subspace. $X/U := X/\sim$ with $a, b \in U \implies a \sim b$.

Remark

A map $f : A \to B$ with $f(a_1) = f(a_2)$ for $a_1 \sim a_2$ corresponds to a map $\tilde{f} : A/\sim \to B, \ \tilde{f}([a]) = f(a)$.

Example

- $[0,1]^n / \partial [0,1]^n \approx S^n$
- Möbius strip and Klein bottle can be constructed as $[0,1]^2/\sim$, $\to$ blackboard.

Homotopy

Homotopy

A *homotopy* is a continuous transformation between maps:

Definition

Homotopy Let X, Y be topological spaces, $A \subset X$ a subspace, $f, g : X \to Y$ continuous with $f_{|A} = g_{|A}$. A cont. map $h : [0,1] \times X \to Y$ is called a *homotopy* relative to A, if

- $h(0,x) = f(x)$ for all $x \in X$
- $h(1,x) = g(x)$ for all $x \in X$
- $h(t,a) = f(a) = g(a)$ for all $a \in A, t \in [0,1]$.

Then f and g are said to be *homotopic*. This is a equivalence relation, write $f \sim_A g$ or $f \sim g \iff f \sim_\emptyset g$.

Example

- Let $X = [0, 2\pi]$, $Y = S^1$, $f(x) = \binom{\cos(x)}{\sin(x)}$, $c(x) = (1,0)$. Are f and c homotopic relative to some set? Which sets are allowed for relative homotopy?
- Let $X = Y = \mathbb{R} \backslash 0$, $f = id$, $g(x) = -x$. What about them?

Homotopy Equivalence

Definition

Let X, Y be topological spaces. If there are morphisms $f : X \to Y, g : Y \to X$ with $f \circ g \sim \mathrm{id}_Y$ and $g \to f \sim \mathrm{id}_X$, then X and Y are called *homotopy equivalent*.

Example (Which are homotopy equivalent?)

- $\mathbb{R}^2$
- $\mathbb{R}^3$
- $D^2 := \left\{x \in \mathbb{R}^2 \middle| \; ||x|| \leq 1\right\}$
- $B^2 := \left\{x \in \mathbb{R}^2 \middle| \; ||x|| < 1\right\}$
- $S^1 := \left\{x \in \mathbb{R}^2 \middle| \; ||x|| = 1\right\}$
- $D^2 \backslash \frac{1}{2} B^2$
- $\mathbb{R}^2 \backslash \{0\}$
- $\{pt\}$

- $\mathbb{R}^2 \sim \mathbb{R}^3 \sim D^2 \sim B^2 \sim \{pt\}$
- $S^1 \sim D^2 \backslash \frac{1}{2} B^2 \sim \mathbb{R}^2 \backslash \{0\}$

Some explicit equivalences

Prove $\mathbb{R}^2 \sim \mathbb{R}^3$

- $f : \mathbb{R}^2 \to \mathbb{R}^3 : (x_1, x_2) \mapsto (x_1, x_2, 0)$
- $g : \mathbb{R}^3 \to \mathbb{R}^2 : (x_1, x_2, x_3) \mapsto (x_1, x_2)$
- $g \circ f = id_{\mathbb{R}^2}$, so $h = \pi_2 : (t, x) \mapsto x$
- $f \circ g = (x_1, x_2, x_3) \mapsto (x_1, x_2, 0) \sim \mathrm{id}_{\mathbb{R}^3}$ with $h(t, (x_1, x_2, x_3)) = (x_1, x_2, t \cdot x_3)$

Prove $S^1 \sim \mathbb{R}^2 \setminus \{0\}$

- $f : S^1 \to \mathbb{R}^2, x \mapsto x$
- $g : \mathbb{R}^2 \to S^1 : x \mapsto \frac{x}{||x||}$
- $g \circ f = id_{S^1}$, so $h = \pi_2$
- $f \circ g = x \mapsto \frac{x}{||x||} \sim \mathrm{id}_{\mathbb{R}^2}$ with $h(t, x) = (1 - t) \cdot \frac{x}{||x||} + t \cdot x$

The Fundamental Group

Repetition - Groups

Definition (Group)

A *group* is a triplet $(G, *, e)$ of a set G, a function $* : G \times G \to G$ and an Element $e \in G$, where

- associativity: $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$
- e is neutral: $a * e = e * a = a$ for all $a \in G$
- inverse: for $a \in G$ there is an a^{-1} with $a^{-1} * a = a * a^{-1} = e$

Definition (Homomorphism)

Let $(G, *_G, e_G)$ and $(H, *_H, e_H)$ be groups. A map $f : G \to H$ is called *homomorphism (of groups)*, if $f(a *_G b) = f(a) *_H f(b)$ for all $a, b \in G$.
If it is bijective, it is called an *isomorphism*.

Remark

- *Choosing $*$ is a structure on G, the neutral e and an element's inverse are uniquely determined by it.*
- *It can be shown that $f(e_G) = e_H$ and $f(a^{-1}) = f(a)^{-1}$ for any homomorphism.*

Example Groups

Example (Bijective Functions)

Let X be a set. $S_X := \{f : X \to X | f \text{ bijective}\}$ is a group with function composition as operation. It's called *symmetric* Group. For a natural number n the group $S_n := S_{\{1,2,\ldots,n\}}$ has $n!$ elements.

Example (Free Group)

Let Σ be a set, the "alphabet". The *free* group $F_\Sigma = \langle \Sigma \rangle = \langle \sigma_1, \sigma_2, \ldots \rangle$ contains the terms generated by applying group operations $*$ and $\cdot^{-1}$ on the elements of Σ. Two terms are considered as equal, if they can by transformed into each other by group axioms.
$\langle a \rangle$ is also known as $\mathbb{Z}$.

A Group of Paths?

Definition (Path)

- A *path* is a continuous map $\gamma : [0,1] \to X$ for some top. space.
- A *loop* or *closed path* is a path γ with $\gamma(0) = \gamma(1)$.
- If two paths α, β fit together, i.e. $\alpha(1) = \beta(0)$, then

$$\alpha \bullet \beta(t) := \begin{cases} \alpha(2t) & 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} < t \leq 1 \end{cases}$$

- $\bar{\gamma} := \gamma(1-t)$

- If we chose a fixed $x_0 \in X$, all paths fit together.
- But $\bullet$ does not form a group!

A Group of Paths!

Definition (Fundamental Group)

- For a top. space X with $x_0 \in X$ let $\pi_1(X, x_0)$ be the homotopy classes of loops starting at x_0.
- $[\alpha] * [\beta] := [\alpha \bullet \beta]$, $[\alpha]^{-1} := [\bar{\alpha}]$
- $(\pi_1(X, x_0), *, \text{const}x_0)$ is called *Fundamental Group*

Theorem

- *$(\pi_1(X, x_0), *, const x_0)$ is a group.*
- *In a path-connected space X is $\pi_1(X, x) = \pi_1(X, y)$ for all $x, y \in X$. Write $\pi_1(X)$.*

Example

- $\{pt\}$ has fundamental group $\{e\}$
- $\pi_1(S_1) = \langle loop \rangle \approx \mathbb{Z}$

continued next week

References

- Lecture notes by Prof. Catherine Meusburger, https://www.algeo.math.fau.de/fileadmin/algeo/users/meusburger/Teaching/algebraic_topology/algebraische_topologie2.pdf
- Lecture notes by Prof. Jan Möllers, 15SS
- Möbius Strip fundamental group calculation: http://www3.nd.edu/~stolz/2015F_Math60330/Klein_bottle.pdf

Thanks

- Thanks for listening, I hope it was useful.