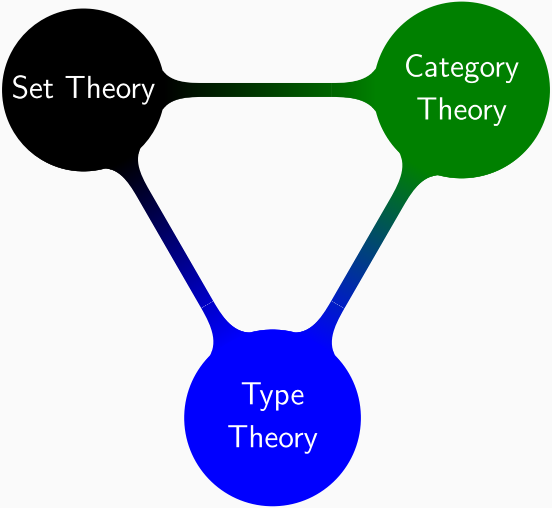
Elementary Theory of the Category of Sets

(based on Tom Leinster's *"Rethinking set theory"*)

Sergey Goncharov

Exclusively for HoTT Seminar, FAU TCS


Set Theory
Category Theory
Type Theory

Set theory

Type theory

Category theory

Introduction

We've considered two approaches to the foundations:

- **Classical:** (Zermelo-Fraenkel) Set Theory (briefly)
- **Modern:** (Homotopy) Type theory (whole course)

Here:

- **Mediate:** Elementary Theory of the Category of Sets
  - Unlike ZF(C): postulates functions (morphisms) and not sets (objects)
  - Unlike HoTT: postulates impredicative ($\approx$ nonconstructive) subobject classifier

However, ETCS is compatible both with set theory and with type theory, and additionally incorporates foundations into the spacious realm of category theory

*AN ELEMENTARY THEORY OF THE CATEGORY OF SETS**

BY F. WILLIAM LAWVERE

UNIVERSITY OF CHICAGO AND EIDG. TECHNISCHE HOCHSCHULE, ZÜRICH

Communicated by Saunders Mac Lane, October 26, 1964

We adjoin eight first-order axioms to the usual first-order theory of an abstract Eilenberg-Mac Lane category[1] to obtain an elementary theory with the following properties: (*a*) There is essentially only one category which satisfies these eight axioms together with the additional (nonelementary) axiom of completeness, namely, the category $\mathcal{S}$ of sets and mappings. Thus our theory distinguishes $\mathcal{S}$ structurally from other complete categories, such as those of topological spaces, groups, rings, partially ordered sets, etc. (*b*) The theory provides a foundation for number theory, analysis, and much of algebra and topology even though no relation $\in$ with the traditional properties can be defined. Thus we seem to have partially demonstrated that even in foundations, not Substance but invariant Form is the carrier of the relevant mathematical information.

As in the general theory of categories, our undefined terms are *mapping*, *domain*, *codomain*, and *composition*, the first being simply a name for the elements of the universe of discourse. Each mapping has a unique domain and a unique codomain,

Rethinking Set Theory

*AN ELEMENTARY THEORY OF THE CATEGORY OF SETS**

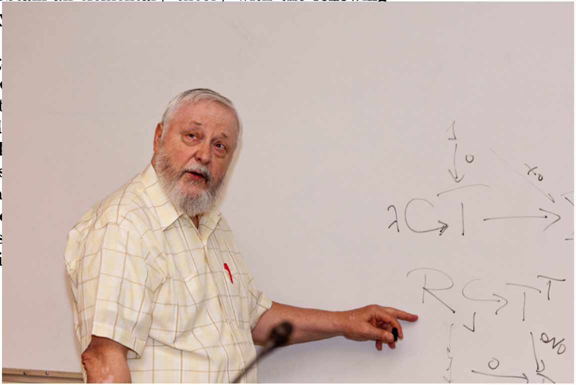
By F. William Lawvere

UNIVERSITY OF CHICAGO AND EIDG. TECHNISCHE HOCHSCHULE, ZÜRICH

Communicated by Saunders Mac Lane, October 26, 1964

We adjoin eight first-order axioms to the usual first-order theory of an abstract Eilenberg-Mac Lane category[1] to obtain an elementary theory with the following
properties: (*a*) There is essentially
axioms together with the additional
the category $\mathcal{S}$ of sets and mapping
from other complete categories, su
partially ordered sets, etc. (*b*) The t
analysis, and much of algebra and
traditional properties can be defined
that even in foundations, not Subs
relevant mathematical information

As in the general theory of categ
codomain, and *composition*, the firs
universe of discourse. Each mappi



Why Rethinking Set Theory?

The transition ZF(C) $\to$ ETCS is not as radical as ZF(C) $\to$ HoTT. Essentially, it is by rethinking Set Theory

But why do we need to rethink it?

- In classical set theory everything must be a set, e.g. 3 is the set $\{2, \emptyset\} = \{\{1, \emptyset\}, \emptyset\} = \{\{\{\emptyset\}, \emptyset\}, \emptyset\}$ hence $2 \in 3$ and $0 = \emptyset$
- Axioms heavily use this "feature", e.g. one says that every nonempty set X contains an element $x \in X$ such that $x \cap X = \emptyset$.
  Think of $\pi \cap \mathbb{R}$
- Because of this lack of structure it is difficult to remember/understand/analyze/modify the axioms of set theory. Roughly, the axioms are not worked principles, but only technical tricks implying such principles (think of induction)

Categories

A (locally small) category $\mathbf{C}$ consists of

- a class (generally $\geqslant$ set) of objects $|\mathbf{C}|$
- a set of morphisms $\mathrm{Hom}(A, B)$ for each pair $A, B \in |\mathbf{C}|$ such that
    - each $\mathrm{Hom}(A, A)$ contains an identity morphism $\mathrm{id}_A : A \to A$
    - compatible morphsims can be composed: $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C) \implies g \circ f \in \mathrm{Hom}(A, C)$

Plus the laws:

$$f \circ \mathrm{id} = f \qquad \mathrm{id} \circ f = f \qquad f \circ (g \circ h) = (f \circ g) \circ h$$

Examples: The category of sets and functions (**Set**); the category of sets and relations (**Rel**); also: **Grp**, **Ring**, $\mathbf{Vec}_k$, **Top**, **Meas**, **CMS**, **Cat**, **Cpo**, **CLat**, **Hilb**, etc, etc, etc.

Categories with $|\mathrm{Hom}(A, B)| \leqslant 1$ $(A, B \in |\mathbf{C}|)$ are exactly (large) preorders: $\mathrm{Hom}(A, B) \neq \emptyset$ iff $A \leqslant B$

Three Misconceptions

Because of the categorical origins of ETCS, three misconceptions commonly arise

- *The underlying motive is to replace set theory with category theory.* It is not: it is set theory
- *ETCS demands more mathematical sophistication than others (such as ZFC).* This is false but understandable. Here we strive for the most elementary presentation
- *There is a circularity: in order to axiomatize sets categorically, we must already know what a set is (for each* $\mathrm{Hom}(A, B)$ *must be a set).* In fact, ETCS is categorical in style, but it does not depend on having a general definition of category

  Put it differently: both ZFC and ETCS are (classical!) first order theories

Summary of Axioms

1. Composition of functions is associative and has identities
2. There is a set with exactly one element
3. There is a set with no elements
4. A function is determined by its effect on elements
5. Given sets X and Y, one can form their Cartesian product $X \times Y$
6. Given sets X and Y, one can form the set of functions from X to Y
7. Given $f : X \to Y$ and $y \in Y$, one can form the inverse image $f^{-1}(y)$
8. The subsets of a set X correspond to the functions from X to $\{0, 1\}$
9. The natural numbers form a set
10. Every surjection has a right inverse

Elements as Functions

In standard set theory: functions $f \in Y^X$ are subsets of $X \times Y$, which are total and single-valued:

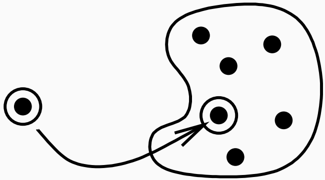
$$f \in Y^X \iff \forall x \in X.\, \exists y \in Y.\, (x, y) \in f \,\wedge$$
$$\forall x \in X.\, \forall y, y' \in Y.\, (x, y) \in f \wedge (x, y') \in f \to y = y'$$

In ECTS, an element of a set X is identified with the function $1 \to X$ selecting this element

Here, 1 is the one-element set $\{\bullet\}$

As a slogan:

Elements are a special case of functions



In a similar way we select: a curve on a plane $\mathbb{R} \to \mathbb{R}^2$, a loop on a plane $S^1 \to \mathbb{R}^2$, a sequence of elements $\mathbb{N} \to X$, etc

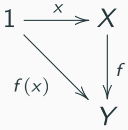
Evaluation as Composition

Now that elements are identified with functions, what is evaluation?

Evaluation as Composition

Now that elements are identified with functions, what is evaluation?

$$f(x) = f \circ x$$



That is:

Evaluation is a special case of composition

Setting the Stage

We assume

- some things called sets
- for each set X and set Y, some things called functions from X to Y, with functions f from X to Y written as $f : X \to Y$ or $X \xrightarrow{f} Y$
- for each set X, set Y and set Z, an operation assigning to each $f : X \to Y$ and $g : Y \to Z$ a function $g \circ f : X \to Z$
- for each set X, a function $\mathrm{id}_X : X \to X$

Axiom 1: Associativity and Identity Laws

First of all, we restate the laws of a category:

- for all sets W, X, Y, Z and functions

  $$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z,$$

  we have $h \circ (g \circ f) = (h \circ g) \circ f$
- For all sets X, Y and functions $f : X \to Y$, we have $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$

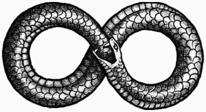
If we wish to omit the identity functions from the list of primitive concepts, we must replace the second item by the statement that for all sets X, there exists a function $\mathrm{id}_X : X \to X$ such that $g \circ \mathrm{id}_X = g$ for all $g : X \to Y$ and $\mathrm{id}_X \circ f = f$ for all $f : W \to X$. These conditions characterize id_X uniquely

Axiom 2: One-Element Set

We would like to say ‘there exists a one-element set’

Axiom 2: One-Element Set

We would like to say ‘there exists a one-element set’

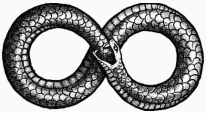


But, wait, we wanted to define what an ‘element’ is using a one-element set!

Axiom 2: One-Element Set

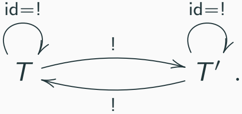
We would like to say 'there exists a one-element set'

But, wait, we wanted to define what an 'element' is using a one-element set!



To avoid circularity, we need a notion of a terminal object: a set T is said to be terminal if there is exactly one function $! : X \to T$ from any set X

We do not expect a terminal object to be unique, e.g. we do not want $\{$⚽$\} = \{$🎂$\}$ But, all terminal objects are isomorphic:



By 1 we mean arbitrary, but fixed terminal set

Axiom 3: Empty Set

There is a set with no elements

That is, there is a set X such that $1 \xrightarrow{f} X$ for no f

Axiom 4: Functional Extensionality

A function is determined by its effect on elements:

$$\forall x \in X.\, f(x) = g(x) \implies f = g$$

This is called functional extensionality. Why would it fail?

Axiom 4: Functional Extensionality

A function is determined by its effect on elements:

$$\forall x \in X.\, f(x) = g(x) \implies f = g$$

This is called functional extensionality. Why would it fail?

Let us take as functions $X \xrightarrow{f} Y$, suitably typed combinatory logic terms i.e. well-typed terms over

$$\begin{aligned} I &: \alpha \to \alpha \\ K &: \alpha \to \beta \to \alpha \\ S &: (\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma \end{aligned}$$

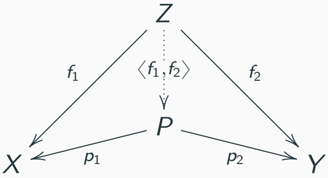
modulo equations

$$Ix = x \qquad Kxy = x \qquad Sxyz = (xz)(yz)$$

Now, $S(KI)I\,x = I\,x$ for all x, but $S(KI)I \neq I$.

Axiom 5: Cartesian Products

Let X and Y be sets. A product of X and Y is a set P together with functions $X \xleftarrow{p_1} P \xrightarrow{p_2} Y$, such that: for all Z and functions $X \xleftarrow{f_1} Z \xrightarrow{f_2} Y$, there is a unique $\langle f_1, f_2 \rangle : Z \to P$ making diagram



commute

We demand that a product $X \xleftarrow{\mathrm{pr}_1} X \times Y \xrightarrow{\mathrm{pr}_2} Y$ of any two sets X and Y exists. Again, it is unique only up to isomorphism, e.g. $X \times Y$ and $Y \times X$ are not equal, but isomorphic under $\mathrm{swap} : X \times Y \cong Y \times X$.

Alternatively, we can axiomatize products:

$$\mathrm{pr}_1 \circ \langle f, g \rangle = f \qquad \mathrm{pr}_2 \circ \langle f, g \rangle = g \qquad \langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle = \mathrm{id}$$

$$h \circ \langle f, g \rangle = \langle h \circ f, h \circ g \rangle$$

Halfway There

1. Composition of functions is associative and has identities
2. There is a set with exactly one element
3. There is a set with no elements
4. A function is determined by its effect on elements
5. Given sets X and Y, one can form their Cartesian product $X \times Y$
6. Given sets X and Y, one can form the set of functions from X to Y
7. Given $f : X \to Y$ and $y \in Y$, one can form the inverse image $f^{-1}(y)$
8. The subsets of a set X correspond to the functions from X to $\{0, 1\}$
9. The natural numbers form a set
10. Every surjection has a right inverse

Axiom 6: Sets of Functions

In ZFC functions $X \xrightarrow{f} Y$ form a set $Y^X \subseteq \mathcal{P}(X \times Y)$ by definition

Here, we have postulated function, sets and defined what an element of a set is. We are left to introduce Y^X and establish a correspondence between its elements $1 \to Y^X$ and functions $X \xrightarrow{f} Y$

More generally (take $Z = 1$), we require an "isomorphism"

$$Z \to Y^X \underset{f \mapsto \mathsf{curry}(f)}{\overset{f \mapsto \mathsf{uncury}(f)}{\rightleftarrows}} Z \times X \to Y$$

that is natural in Z, i.e. $\mathsf{curry}(f \circ \langle h, \mathsf{id} \rangle) = (\mathsf{curry}\, f) \circ h$

This induces the evaluation function $\mathsf{ev} = \mathsf{uncurry}(\mathsf{id}) : Y^X \times X \to Y$

It follows that $\mathsf{ev} \circ \langle \hat{f}, x \rangle = f \circ x$ where $\hat{f} : 1 \to Y^X$ is the element corresponding to $X \xrightarrow{f} Y$

Axiom 7: Natural Numbers

Like 1, $X \times Y$, and Y^X, we postulate natural numbers using the same recipe:

1. we fix a set $\mathbb{N}$
2. we fix zero $1 \xrightarrow{o} \mathbb{N}$ and successor $\mathbb{N} \xrightarrow{s} \mathbb{N}$ functions
3. we impose a characteristic property on $\mathbb{N}$

The characteristic property of natural numbers is primitive recursion: given $X \xrightarrow{f} Y$ and $X \times \mathbb{N} \times Y \xrightarrow{g} Y$, there is unique $w : X \times \mathbb{N} \to Y$ such that

$$w(x, o) = f(x) \qquad\qquad w(x, s(n)) = g(x, n, w(x, n))$$

Axiom 7: Natural Numbers

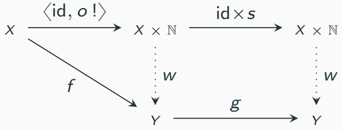
Like 1, $X \times Y$, and Y^X, we postulate natural numbers using the same recipe:

1. we fix a set $\mathbb{N}$
2. we fix zero $1 \xrightarrow{o} \mathbb{N}$ and successor $\mathbb{N} \xrightarrow{s} \mathbb{N}$ functions
3. we impose a characteristic property on $\mathbb{N}$

The characteristic property of natural numbers is primitive recursion: given $X \xrightarrow{f} Y$ and $X \times \mathbb{N} \times Y \xrightarrow{g} Y$, there is unique $w : X \times \mathbb{N} \to Y$ such that

$$w(x, o) = f(x) \qquad\qquad w(x, s(n)) = g(x, n, w(x, n))$$

In fact, we can weaken it slightly by restricting to $Y \xrightarrow{g} Y$:



Intermediate Total

So far we obtained a model for simply-typed λ-calculus with primitive recursion, e.g. we can zip two streams by zip : $X^{\mathbb{N}} \times X^{\mathbb{N}} \to (X \times Y)^{\mathbb{N}}$:

$$\begin{aligned} \text{zip} &= \text{curry}(\text{zip'} : X^{\mathbb{N}} \times Y^{\mathbb{N}} \times \mathbb{N} \to X \times Y) \\ \text{zip'}(\sigma, \sigma', o) &= \sigma(o) \\ \text{zip'}(\sigma, \sigma', s(x)) &= \text{zip'}(\sigma', \text{curry}(\text{ev} \circ (\text{id} \times s))(\sigma), x) \end{aligned}$$

For example, $\text{zip}([0, 2, 4, \ldots], [1, 3, 5, \ldots]) = [0, 1, 2, 3, 4, 5, \ldots]$, for e.g.

$$\begin{aligned} &\text{zip'}([0, 2, 4, \ldots], [1, 3, 5, \ldots], 2) \\ &\quad = \text{zip'}([1, 3, 5, \ldots], [2, 4, \ldots], 1) \\ &\quad = \text{zip'}([2, 4, \ldots], [3, 5, \ldots], 0) \\ &\quad = 2. \end{aligned}$$

Big Ideas (from Category Theory)

- *Definitions by universal properties*: we introduce things not only by constructing them, but also by declaring them to be extremal solutions to conditions they must satisfy with respect to the rest
- *Definitions up to isomorphisms*: we can not avoid the situation that what we define is unique only up to isomorphism. However, this is also desirable: in mathematical speech we speak e.g. of the trivial group—it does not make much sense to ask 'which trivial group?' Properties not stable under isomorphisms are sometimes dubbed evil

Big Ideas (from Category Theory)

- *Definitions by universal properties*: we introduce things not only by constructing them, but also by declaring them to be extremal solutions to conditions they must satisfy with respect to the rest
- *Definitions up to isomorphisms*: we can not avoid the situation that what we define is unique only up to isomorphism. However, this is also desirable: in mathematical speech we speak e.g. of the trivial group—it does not make much sense to ask 'which trivial group?' Properties not stable under isomorphisms are sometimes dubbed evil

The upshot of this: in contrast to TT, we do not build a hierarchy of types, e.g. everything that behaves as a product is a product

Impredicativity

From **nLab**: *'A definition is impredicative if it refers to a totality which includes the thing being defined'*

A standard example is the definition of a least upper bound $\bigsqcup S$ of a set S of real numbers, because $\bigsqcup S$ is characterized as the least element of the set of all upper bounds of S and that already includes $\bigsqcup S$

Impredicativity can be harmless: the tallest guy in the room

The definitions, we gave so far are arguably harmlessly impredicative: instead of introducing a product of A and B, we could introduce the product $A \times B$, which is by definition a construction over A and B. This is the way of type theory

As a slogan

In TT we construct things; In CT we (loosely) specify things

Axiom 8: Inverse Images

Given a function $f : X \to I$ and an element $i \in I$, we require existence of an inverse image or fibre $f^{-1}(i)$, defined as follows, to exist:

- The inclusion function $j : f^{-1}(i) \subseteq X$ has the property that $f \circ j$ is constantly i
- Moreover, whenever $r : Y \to X$ is a function such that $f \circ r$ is constantly i, the image of r must lie within $f^{-1}(i)$; that is, $r = j \circ \bar{r}$ for some $\bar{r} : Y \to f^{-1}(i)$ (necessarily unique)

Axiom 8: Inverse Images

Given a function $f : X \to I$ and an element $i \in I$, we require existence of an inverse image or fibre $f^{-1}(i)$, defined as follows, to exist:

- The inclusion function $j : f^{-1}(i) \subseteq X$ has the property that $f \circ j$ is constantly i
- Moreover, whenever $r : Y \to X$ is a function such that $f \circ r$ is constantly i, the image of r must lie within $f^{-1}(i)$; that is, $r = j \circ \bar{r}$ for some $\bar{r} : Y \to f^{-1}(i)$ (necessarily unique)

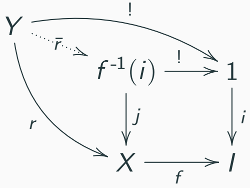
Intuition from TT: $X = \sum_{i:I} f[r[i]] = \sum_{i:I} \sum_{x:f[i]} \bar{r}[x] \quad (h[x] \overset{def}{=} h^{-1}(x))$

Axiom 8: Inverse Images

Given a function $f : X \to I$ and an element $i \in I$, we require existence of an inverse image or fibre $f^{-1}(i)$, defined as follows, to exist:

- The inclusion function $j : f^{-1}(i) \subseteq X$ has the property that $f \circ j$ is constantly i
- Moreover, whenever $r : Y \to X$ is a function such that $f \circ r$ is constantly i, the image of r must lie within $f^{-1}(i)$; that is, $r = j \circ \bar{r}$ for some $\bar{r} : Y \to f^{-1}(i)$ (necessarily unique)

Intuition from TT: $X = \sum_{i:I} f[r[i]] = \sum_{i:I} \sum_{x:f[i]} \bar{r}[x] \quad (h[x] \stackrel{def}{=} h^{-1}(x))$
Intuition from CT: pullback



Axiom 9: Subset Classifier

Fasten your seat belts, we are going to call on something really sophisticated and powerful: subset classifier

Axiom 9: Subset Classifier

Fasten your seat belts, we are going to call on something really sophisticated and powerful: subset classifier

What we are missing so far is a conversion between subsets, i.e. injective functions $Y \to X$ and predicates i.e. functions $X \to 2$ where 2 is a truth-value object consisting of true and false

Axiom 9: Subset Classifier

Fasten your seat belts, we are going to call on something really sophisticated and powerful: subset classifier

What we are missing so far is a conversion between subsets, i.e. injective functions $Y \to X$ and predicates i.e. functions $X \to 2$ where 2 is a truth-value object consisting of true and false

This we will enable the comprehension notation

$$X' \stackrel{def}{=} \{x \in X \mid p(x)\} \qquad \text{e.g.} \qquad \text{Even} \stackrel{def}{=} \{n \in \mathbb{N} \mid n \bmod 2 = 0\}$$

Axiom 9: Subset Classifier

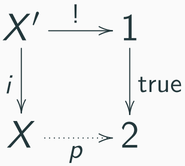
Fasten your seat belts, we are going to call on something really sophisticated and powerful: subset classifier

What we are missing so far is a conversion between subsets, i.e. injective functions $Y \to X$ and predicates i.e. functions $X \to 2$ where 2 is a truth-value object consisting of true and false

This we will enable the comprehension notation

$$X' \stackrel{def}{=} \{x \in X \mid p(x)\} \qquad \text{e.g.} \qquad \mathrm{Even} \stackrel{def}{=} \{n \in \mathbb{N} \mid n \bmod 2 = 0\}$$

For every injection $i : X' \to X$, we require existence of p such that i is an inverse image of $\mathrm{true} : 1 \to 2$ under p:



Powerset and Relations

Function sets together with the subobject classifiers imply powersets:

$$\mathcal{P}(X) \stackrel{def}{=} 2^X$$

A relation can thus be equivalently defined

- as an element of $\mathcal{P}(X \times Y)$
- as a function $X \to \mathcal{P}(Y)$
- as a function $X \times Y \to 2$

Subset Classifier: Impredicativity

Subset classifier is genuinely impredicative

In particular, we can introduce universal quantification

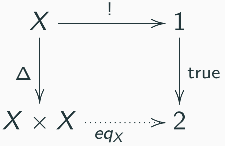
$$\frac{p : 1 \to 2^{A \times B}}{\forall_A \; p : 1 \to 2^{B}}$$

for every A

We thus refer to the totality of all predicates 2^X to define individual elements $1 \to 2^X$ for any concrete X

Subset Classifier: Internal Equality

Equality of functions can be internalized:



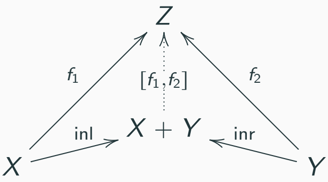
The parallel to TT is as follows:

CT	TT
external equality ($=$)	judgmental equality
internal equality (eq)	propositional equality

Since in our case $=$ and eq are essentially equivalent, the alluded TT is extensional

Coproducts and Excluded Middle

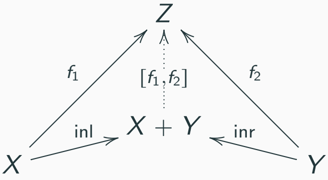
A coproduct of X and Y is a set $X + Y$ together with functions $X \xrightarrow{\mathsf{inl}} X + Y \xleftarrow{\mathsf{inr}} Y$, such that: for all Z and functions $X \xrightarrow{f_1} Z \xleftarrow{f_2} Y$, there is a unique $[f_1, f_2] : X + Y \to Z$ making diagram



commute

Coproducts and Excluded Middle

A coproduct of X and Y is a set $X + Y$ together with functions $X \xrightarrow{\text{inl}} X + Y \xleftarrow{\text{inr}} Y$, such that: for all Z and functions $X \xrightarrow{f_1} Z \xleftarrow{f_2} Y$, there is a unique $[f_1, f_2] : X + Y \to Z$ making diagram

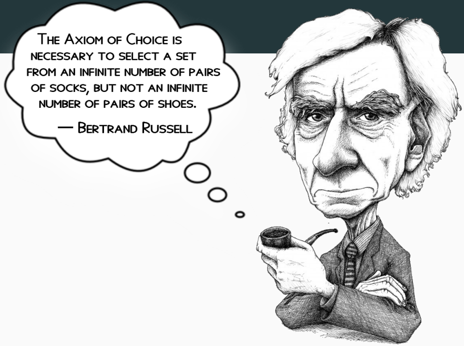


commute

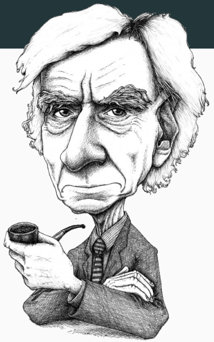
Hard Consequence: $X + Y$ always exists

Easy Consequence: $2 = 1 + 1$, specifically true, false : $1 \to 2$ are the only elements of 2. This makes ETCS a classical set theory

Axiom 10: Choice

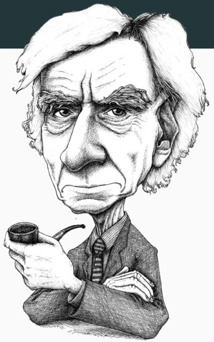


Axiom 10: Choice



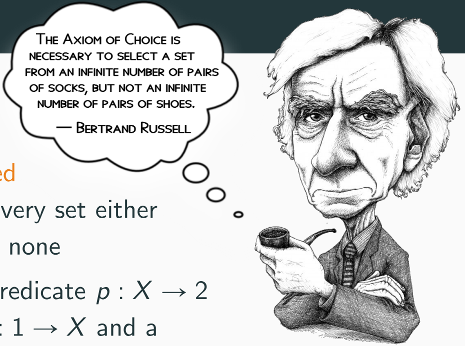
- As long as we believe in excluded middle, we should accept that every set either contains an element or contains none

Axiom 10: Choice



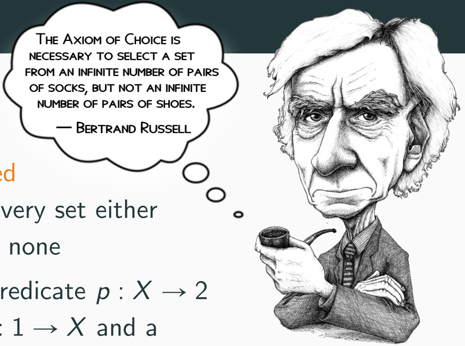
- As long as we believe in excluded middle, we should accept that every set either contains an element or contains none
- Therefore, for every surjective predicate $p : X \to 2$ we can construct an example $t : 1 \to X$ and a counterexample $f : 1 \to X$, that is $p \circ q = \mathrm{id} : 2 \to 2$ for $q(\mathrm{true}) = t$ and $q(\mathrm{false}) = f$

Axiom 10: Choice



- As long as we believe in excluded middle, we should accept that every set either contains an element or contains none
- Therefore, for every surjective predicate $p : X \to 2$ we can construct an example $t : 1 \to X$ and a counterexample $f : 1 \to X$, that is $p \circ q = \mathrm{id} : 2 \to 2$ for $q(\mathrm{true}) = t$ and $q(\mathrm{false}) = f$
- This can be extended to a classification of the elements of X over I classes by a surjective function $f : X \to I$ (e.g. with socks, for each fiber $|f^{-1}(i)| = 2$)

Axiom 10: Choice



- As long as we believe in excluded middle, we should accept that every set either contains an element or contains none
- Therefore, for every surjective predicate $p : X \to 2$ we can construct an example $t : 1 \to X$ and a counterexample $f : 1 \to X$, that is $p \circ q = \mathrm{id} : 2 \to 2$ for $q(\text{true}) = t$ and $q(\text{false}) = f$
- This can be extended to a classification of the elements of X over I classes by a surjective function $f : X \to I$ (e.g. with socks, for each fiber $|f^{-1}(i)| = 2$)

We thus require that every such f has a right inverse

Aftermath

So, what ETCS buys us?

- Axioms are genuine principles of math, not specific to a particular theory

Aftermath

So, what ETCS buys us?

- Axioms are genuine principles of math, not specific to a particular theory
- ETCS gently introduces into category theory. Interesting category classes are obtained by weakening the axioms: finitely complete, Cartesian closed, toposes, well-pointed toposes, Boolean toposes. In this sense, ETCS is just a well-pointed topos with a natural number object and choice (what's the problem?)

Aftermath

So, what ETCS buys us?

- Axioms are genuine principles of math, not specific to a particular theory
- ETCS gently introduces into category theory. Interesting category classes are obtained by weakening the axioms: finitely complete, Cartesian closed, toposes, well-pointed toposes, Boolean toposes. In this sense, ETCS is just a well-pointed topos with a natural number object and choice (what's the problem?)
- Incidentally, ETCS is properly weaker than ZFC. But ETCS axioms are well justified. Hence ETCS gives an insight into foundations

ETCS v.s. ZFC

- In ETCS (unlike ZFC) we do not have countable sums, e.g.

$$\mathbb{N} + \mathcal{P}(\mathbb{N}) + \mathcal{P}(\mathcal{P}(\mathbb{N})) + \dots$$

  does not exist (unless ETCS is inconsistent)
- We can obtain a system equivalent to ZFC by adding the replacement axiom. This is difficult (impossible?) to state categorically, but it says roughly that the image $f[X]$ of a set X under a definable function f is again a set

Questions?