Modal Logic Summer term 2016

Exercise Sheet 5

Solutions due: 21.07.2016 Rev. 12979

Exercise 1 Polynomial functors

Kripke-polynomial functors are constructed by the following grammar:

$$T : = Id \mid A \mid T_1 \times T_2 \mid T_1 + T_2 \mid \mathcal{P}_\omega(T) \mid T^A, \qquad (A \in \mathbf{Set})$$

We define the actions on objects and functions for the corresponding functors:

- the identity functor Id;

$$Id(X) = X \qquad\qquad Id(f) = f$$

- the constant functor A for any set A;

$$A(X) = A \qquad\qquad A(f) = id_A$$

- the coproduct of functors $T_1 + T_2$;

$$(T_1 + T_2)(X) = T_1X \uplus T_2X \qquad\qquad (T_1 + T_2)(f) = T_1f + T_2f,$$

  where for all functions $g : A \to B$ and $h : C \to D$, $(g + h) : (A + B) \to (C + D)$ is defined as follows (for all $x \in A + B$):

$$(g+h)(x) = \begin{cases} inl(g(y)) & \text{if } x = inl(y) \\ inr(h(y)) & \text{if } x = inr(y) \end{cases}$$

- the product of functors $T_1 \times T_2$;

$$(T_1 \times T_2)(X) = T_1X \times T_2X \qquad\qquad (T_1 \times T_2)(f) = T_1f \times T_2f,$$

  where for all functions $g : A \to B$ and $h : C \to D$, $(g \times h) : (A \times B) \to (C \times D)$ is defined as follows (for all $(x, y) \in A \times B$):

$$(g \times h)(x, y) = (g(x), h(y))$$

- the *finite* powerset construction $\mathcal{P}_\omega(T)$;

$$(\mathcal{P}_\omega(T))(X) = \{Y \mid Y \subseteq TX, Y \text{ is finite}\} = \mathcal{P}_\omega(TX)$$
$$((\mathcal{P}_\omega(T))(f))(Y) = \{Tf(x) \mid x \in TY\} = Tf[TY],$$

- the exponentiation functor T^A for any set A;

$$(T^A)(X) = \{f \mid f : A \to TX\}$$
$$((T^A)(f))(g) = Tf \circ g : A \to TZ \text{ for } f : X \to Z$$

1. Show that all of the above are indeed functors.

Exercise 2 Coalgebras

1. Describe T-coalgebras for the following polynomial functors, where $1 = \{*\}$ and $2 = \{\top, \bot\}$:

$$T = A \times Id$$
$$T = Id \times A \times Id$$
$$T = 1 + (A \times Id)$$
$$T = 2 \times (\mathcal{P}_\omega)^A$$

2. Give a non-empty example of a T-coalgebra for all the functors from the previous item.

Exercise 3 *Predicate Liftings and truth sets* 7 points

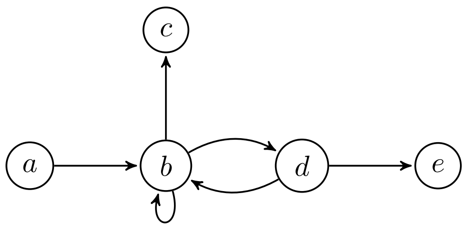
Let $T = \mathcal{P}$ so that T-coalgebras are Kripke frames and recall that we can define the semantics of $\Box$ and $\Diamond$ using the following predicate liftings:

$$[\![\Box]\!]_X(B) = \{A \in \mathcal{P}(X) \mid A \subseteq B\}$$
$$[\![\Diamond]\!]_X(B) = \{A \in \mathcal{P}(X) \mid A \cap B \neq \emptyset\},$$

where X is any set and $B \subseteq X$. Now we define two additional predicate liftings as follows:

$$[\![\blacksquare]\!]_X(B) = \{A \in \mathcal{P}(X) \mid A \cap B = \emptyset\}$$
$$[\![\blacklozenge]\!]_X(B) = \{A \in \mathcal{P}(X) \mid A \cap (X \setminus B) \neq \emptyset\},$$

1. What is the intuition of the obtained semantics of the two modal operators $\blacksquare$ and $\blacklozenge$?
2. Let $C = (\mathcal{W}, \gamma)$ be the T-coalgebra given by the following diagram,



i.e. $\mathcal{W} = \{a, b, c, d, e\}$ and the successor function $\gamma : \mathcal{W} \to \mathcal{P}(\mathcal{W})$ is defined as

$$\gamma(a) = \{b\} \qquad \gamma(b) = \{b, c, d\}$$
$$\gamma(c) = \gamma(e) = \emptyset \qquad \gamma(d) = \{b, e\}$$

Compute the sets $[\![\Box\Diamond\top]\!]^C$ and $[\![\blacklozenge\Box\blacksquare\top]\!]^C$.

Exercise 4 *The non-empty powerset functor* 3 points

Let $T = \mathcal{P}_+$ be the *non-empty* powerset functor defined by

$$\mathcal{P}_+(X) = \{Y \mid \emptyset \neq Y \subseteq X\}$$
$$(\mathcal{P}_+(f))(X) = \{f(x) \mid x \in X\}.$$

We define the semantics of $\Box$ and $\Diamond$ using the essentially the same predicate liftings as for $T = \mathcal{P}$, i.e.

$$[\![\Box]\!]_X(B) = \{A \in \mathcal{P}_+(X) \mid A \subseteq B\}$$
$$[\![\Diamond]\!]_X(B) = \{A \in \mathcal{P}_+(X) \mid A \cap B \neq \emptyset\},$$

for any set X and $B \subseteq X$.

1. What is the difference between $\mathcal{P}$-coalgebras and $\mathcal{P}_+$-coalgebras? Recall that $\mathcal{P}$-coalgebras are Kripke frames; how can we characterize $\mathcal{P}_+$-coalgebras?
2. Which modal logic do we obtain from the above predicate liftings?