Modal Logic Summer term 2016

Exercise Sheet 4

Solutions due: 13.07.2016 Rev. 12945

Exercise 1 Complexity of $\mathbf{S5_n}$

Recall that **S5** is the normal modal logic that is generated by $\{(\mathrm{T}), (4), (\mathrm{B})\}$ and recall that **S5** is sound and complete w.r.t. the class of frames $(\mathcal{W}, \mathcal{R})$ where $\mathcal{R}$ is an equivalence relation. We have seen that $SAT(\mathbf{S5})$ (the satisfiability problem of **S5**) is NP-complete.

This is not true for $\mathbf{S5_n}$, the logic of frames that have n equivalence relations $\mathcal{R}_1, \ldots, \mathcal{R}_n$, for all $n \geq 2$. These logics are generated by n copies of the axioms (T), (4) and (B), where the axioms

$$\begin{array}{ll} (\mathrm{T}_i) & p \to \Diamond_i p \\ (4_i) & \Diamond_i \Diamond_i p \to \Diamond_i p \\ (\mathrm{B}_i) & p \to \Box_i \Diamond_i p \end{array}$$

axiomatize the relation $\mathcal{R}_i$ to be an equivalence relation.

1. Show that $SAT(\mathbf{S5_2})$ is PSPACE-hard by reducing $SAT(\mathbf{T})$ to $SAT(\mathbf{S5_2})$.

2. Prove that this implies PSPACE-hardness of $\mathbf{S5_n}$ for all $n \geq 2$.

3. Is it possible to reduce $SAT(\mathbf{S5_2})$ to $SAT(\mathbf{T})$? If so, what does this tell us about PSPACE-completeness of $\mathbf{S5_2}$?

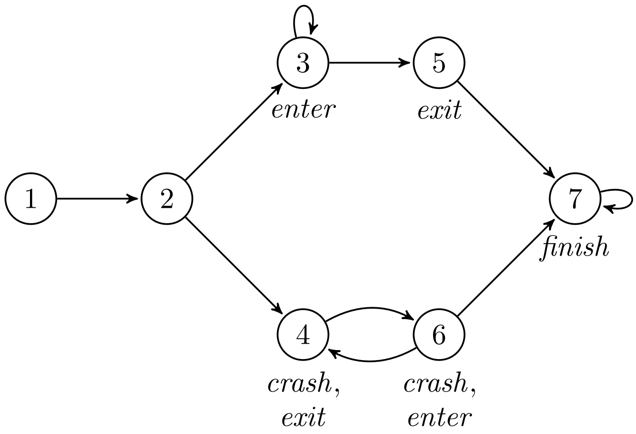
Exercise 2 CTL

Assume that we have the propositional variables $P = \{crash, enter, exit, finish\}$, coming with the intuition that we consider a system that may *crash*, that may *enter* and *exit* critical sections and that may *finish* the execution.

1. Define CTL-formulas that formalize the following statements:

   - "The system can evolve in such a way that it never crashes."
   - "No matter how the system evolves, no crash ever happens."
   - "There is a way for the system to crash in two consecutive worlds, but it is not possible, that the system crashes once it has finished the execution."

2. Now consider the following *serial* model $\mathcal{M} = (\mathcal{W}, \mathcal{R}, \mathcal{V})$:



For each of your formulas ψ_i from the previous item, check, whether ψ_i is satisfied at the worlds 1, 3, 4; that is, for each of your formulas ψ_i, verify whether $\mathcal{M}, 1 \models_{CTL} \psi_i$, $\mathcal{M}, 3 \models_{CTL} \psi_i$, $\mathcal{M}, 4 \models_{CTL} \psi_i$. Justify your answers.

Exercise 3 CTL as fragment of the μ-calculus

We have seen that CTL can be embedded into the modal μ-calculus by using the following translation rules:

$$e(A(\psi\ U\ \varphi)) = \mu X.\ e(\varphi) \lor (e(\psi) \land \Box X)$$
$$e(E(\psi\ U\ \varphi)) = \mu X.\ e(\varphi) \lor (e(\psi) \land \Diamond X)$$

1. Use these rules to derive μ-calculus formulas that are equivalent to the CTL formulas $AF\varphi$, $EF\varphi$, $AG\varphi$ and $EG\varphi$.

2. Use the results of the previous item to translate all your formulas from Exercise 2.1. to the μ-calculus.

3. Let us again consider the model $\mathcal{M}$ from Exercise 2.2. For each of the μ-calculus formulas ϑ_i that you obtained in the previous item, compute the truth-set of ϑ_i in $\mathcal{M}$, i.e., compute $[\![\vartheta_i]\!]_\epsilon$ in $\mathcal{M}$.

   Hint: Notice that we have $|\mathcal{W}| = 7$ so that by Kleene's fixpoint theorem, we know that for any least fixpoint formula $\mu X.\ \psi$,

   $$[\![\mu X.\ \psi]\!]_\epsilon = ([\![\psi]\!]_\epsilon^X)^7(\emptyset)$$

   and for any greatest fixpoint formula $\nu X.\ \psi$,

   $$[\![\nu X.\ \psi]\!]_\epsilon = ([\![\psi]\!]_\epsilon^X)^7(\mathcal{W}),$$

   that is, in $\mathcal{M}$, the semantics of fixpoint formulas stabilizes after at most 7 unfolding steps.

Exercise 4 Fairness and unfairness in the μ-calculus

A *fairness property* states that some event happens infinitely often while an *unfairness property* states that an event happens only finitely often (i.e. that from some time on, it does not happen any more).

Consider the following μ-calculus formulas, to be interpreted over *serial* frames:

$$\begin{aligned}
\psi_1 &= \mu X.\, p \vee \Box X \\
\psi_2 &= \mu X.\nu Y.\, \Box X \vee (\neg p \wedge \Box Y) \\
\psi_3 &= \nu X.\mu Y.\, (p \wedge \Box X) \vee (\neg p \wedge \Box Y) \\
\psi_4 &= \nu X.\mu Y.\, (\neg p \wedge \Box X) \vee (p \wedge \Box Y)
\end{aligned}$$

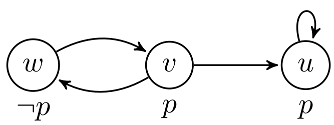
Identify

1. a formula that expresses a fairness property for p, and
2. a formula that expresses an unfairness property for p.

Justify your answer.

Exercise 5 Model checking games

Consider the model $\mathcal{N} = (\mathcal{W}, \mathcal{R}, \mathcal{V})$ given by the following diagram:



Use the model checking game algorithm from the lecture to decide whether the following statements are true:

1. $\mathcal{N}, w \models \nu X.\mu Y.\, (p \wedge \Box X) \vee (\neg p \wedge \Box Y)$
2. $\mathcal{N}, v \models \nu X.\mu Y.\, (\neg p \wedge \Box X) \vee (p \wedge \Box Y)$

For each of the statements $i \in \{1, 2\}$, define the corresponding parity game G_i, solve the game and give a winning strategy s_i for either ∀belard or ∃loise.

Exercise 6 *CTL II* 5 points

We assume the conventions from Exercise 2.

1. Define CTL-formulas that formalize the following statements:
   - "There is a way for the system to crash and there is way for the system to finish without crashing."
   - "There is a way for the system to eventually enter a critical section, but once the system entered a critical section, it will not crash or enter another critical section until it left the first critical section; also, the system is guaranteed to eventually finish execution."
2. Consider the model $\mathcal{M}$ from Exercise 2.2. and check for each of your formulas ψ_i from the previous item, whether ψ_i is satisfied at the worlds 1, 3, 4 in $\mathcal{M}$; that is, for each of your formulas ψ_i, verify whether $\mathcal{M}, 1 \models_{CTL} \psi_i$, $\mathcal{M}, 3 \models_{CTL} \psi_i$, $\mathcal{M}, 4 \models_{CTL} \psi_i$. Justify your answers.

Exercise 7 *CTL as fragment of the μ-calculus II* 6 points

1. Use the results of Exercise 3.1. to translate all your formulas from Exercise 6.1. to the μ-calculus.
2. Let us again consider the model $\mathcal{M}$ from Exercise 2.2. For *one* of the μ-calculus formulas ϑ_i that you obtained in the previous item, compute the truth-set of ϑ_i in $\mathcal{M}$, i.e., compute $[\![\vartheta_i]\!]_\epsilon$ in $\mathcal{M}$.

Exercise 8 *Model checking games II* 9 points

Consider the model $\mathcal{N}$ from Exercise 5 and use the model checking game algorithm from the lecture to decide whether the following statements are true:

1. $\mathcal{N}, w \models \psi_1$
2. $\mathcal{N}, v \models \psi_2$
3. $\mathcal{N}, u \models \psi_2$,

where $\psi_1 = \mu X.\ p \lor \Box X$ and $\psi_2 = \mu X.\nu Y.\ \Box X \lor (p \land \Box Y)$. For each statement, define the corresponding parity game, solve the game and give a strategy for either ∀belard or ∃loise that wins the relevant node.

Hint: The last two statements correspond to the same game, but to different nodes in this game.

Also notice that $ad(\psi_1) = ad(\psi_2) = 1$ while $ad(\nu Y.\ \Box\psi_2 \lor (p \land \Box Y)) = 0$.