Modal Logic Summer term 2016

Exercise Sheet 3

Solutions due: 23.06.2016

Rev. 12709

Exercise 1 Frame Definability

1. Recall that we refer to the normal modal logic generated by the Löb-axiom

$$\text{(L)} \qquad \Box(\Box p \rightarrow p) \rightarrow \Box p$$

as *provability logic* and that we denote this logic by **L**. We know that **L** is sound and complete w.r.t. the class FTT of all frames that have a finite transitive tree as relation.

Prove Lemma 3.37 from the lecture, i.e. show that there is no FOL-formula that locally corresponds to (L).

2. Proposition 3.39 from the lecture implies that modal formulas locally correspond to their respective *second-order* translation. Prove Proposition 3.39 by showing that for all modal formulas φ that use n propositional variables $p_1, \ldots, p_n$, all *frames* $\mathcal{F} = (\mathcal{W}, \mathcal{R})$ and all worlds $w \in \mathcal{W}$,

   (a) $\mathcal{F}, w \Vdash \varphi$ iff $\mathcal{F}, \eta \models_{MSO} \forall P_1, \ldots, P_n . ST_x(\varphi)$ for $\eta = [w/x]$,

   (b) $\mathcal{F} \Vdash \varphi$ iff $\mathcal{F} \models_{MSO} \forall P_1, \ldots, P_n . ST_x(\varphi)$.

Exercise 2 Filtration

Prove Item 2. of Theorem 4.2 from the lecture, i.e. prove that for all finite sets of formulas Γ that are closed under taking subformulas, all models M, all formulas $\varphi \in \Gamma$ and all worlds w of $\mathcal{M}$, we have

$$\mathcal{M}, w \models \varphi \text{ iff } \mathcal{M}^f, [w] \models \varphi,$$

where $\mathcal{M}^f$ denotes the filtration of $\mathcal{M}$ through Γ.

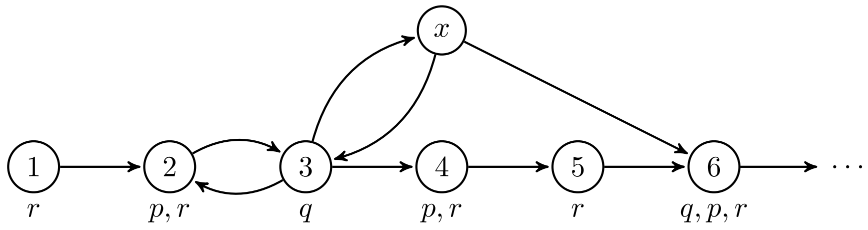
Exercise 3 Selection

1. Finish the proof of Theorem 4.6 from the lecture by proving that for all formulas ψ,

$$\mathcal{M}, w \models \psi \text{ iff } \mathcal{M}^{\psi,w}, w \models \psi,$$

where $\mathcal{M}^{\psi,w}$ is obtained from $\mathcal{M}$ by selection, i.e. by restricting the model to the worlds that are relevant for the satisfaction of ψ at world w.

2. Consider the model $\mathcal{M} = (\mathcal{W}, \mathcal{R}, \mathcal{V})$



given by

$$\begin{aligned}
\mathcal{W} &= \{x\} \cup \{i \mid i > 0\} \\
\mathcal{R} &= \{(i, i+1) \mid i > 0\} \cup \{(3,2), (3,x), (x,3), (x,6)\} \\
\mathcal{V}(p) &= \{j \mid j \text{ is even}\} \\
\mathcal{V}(q) &= \{j \mid \exists i \in \mathbb{N}. j = 3i\} \\
\mathcal{V}(r) &= \mathbb{N}_{>0} \setminus \{3\}
\end{aligned}$$

Let $\psi = \Diamond(\neg\Diamond r \vee \Diamond(\Diamond p \wedge \Diamond q))$ and compute the result of applying the selection function from the proof of Theorem 4.6 to ψ and 1, i.e. compute $s(\psi, 1)$. Use this set to define the model $\mathcal{M}^{\psi,1}$ that is obtained from $\mathcal{M}$ by selection.

Exercise 4 Model size

Theorem 4.12 from the lecture states that there is for each $\Lambda \in \{\mathbf{K}, \mathbf{T}, \mathbf{B}, \mathbf{K4}, \mathbf{S4}\}$ and each $n \in \mathbb{N}$ a formula φ_n s.t.

1. $|\varphi_n| \in \mathcal{O}(n^2)$,
2. φ_n is Λ-satisfiable, and
3. for all models $\mathcal{M} = (\mathcal{W}, \mathcal{R}, \mathcal{V})$ of φ_n, we have $|\mathcal{W}| \geq 2^n$.

Prove this by devising a series of formulas φ_n such that for each $n \in \mathbb{N}$, φ_n defines a full binary tree of depth n and by verifying properties 1., 2. and 3. for your series of formulas.

Exercise 5 *Sahlqvist Formulas* 5 points

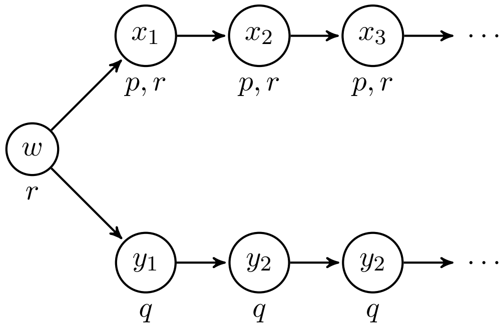
1. Are the following formulas Sahlqvist formulas?

$$\begin{aligned}
\psi_1 &:= \Diamond\Box p \to \Box\Diamond p \\
\psi_2 &:= \Box\Diamond p \to \Diamond\Box p \\
\psi_3 &:= (p \wedge \Diamond\Diamond p) \to \Diamond p \\
\psi_4 &:= (p \wedge \Box p \wedge \Box\Box p) \to \Diamond p
\end{aligned}$$

2. For any formula from the previous item that is a Sahlqvist formula, define a first-order formula that expresses the same property on frames.

Exercise 6 *Filtration* 9 points

1. Consider the model $\mathcal{M} = (\mathcal{W}, \mathcal{R}, \mathcal{V})$



given by

$$\begin{aligned}
\mathcal{W} &= \{w\} \cup \{x_i \mid i > 0\} \cup \{y_i \mid i > 0\} \\
\mathcal{R} &= \{(w, x_1), (w, y_1)\} \cup \{(x_i, x_{i+1}) \mid i > 0\} \cup \{(y_i, y_{i+1}) \mid i > 0\} \\
\mathcal{V}(p) &= \{x_i \mid i > 0\} \\
\mathcal{V}(q) &= \{y_i \mid i > 0\} \\
\mathcal{V}(r) &= \{w\} \cup \{x_i \mid i > 0\}
\end{aligned}$$

Let $\Gamma = \{\Diamond p, \Diamond q, \Diamond r\}$. Compute $\sim_\Gamma$ and use this relation to find

- the smallest filtration of $\mathcal{M}$ through Γ, and
- the largest filtration of $\mathcal{M}$ through Γ.

Is there an upper bound on the number of worlds that filtrations of an arbitrary model $\mathcal{M}'$ through Γ may have?

2. Prove that for all models $\mathcal{M}$ that are based on a symmetric frame and all sets of formulas Γ,

   (a) the largest filtration of $\mathcal{M}$ through Γ *not* necessarily has a symmetric relation.

   (b) the smallest filtration of $\mathcal{M}$ through Γ has a symmetric relation.

   To prove Item (a), it suffices to find a counterexample. Use Item (b) to adapt the proof of Theorem 4.5 from the lecture to show that $SAT(\mathbf{B})$ and $VAL(\mathbf{B})$ are decidable.

Exercise 7 *Tableaux* — 6 points

Use the Tableau-algorithm from the lecture to decide whether the following formulas are *valid*:

1. $\Box(p \to (\Diamond q \land \Diamond r)) \land \Diamond p \land \Diamond q \land \Box\Box((q \land r) \to p)$
2. $\Box(p \to (\Diamond q \lor \Diamond r)) \land (\Diamond p \lor \Diamond\neg p) \land \Box\Box(p \land \neg r)$