Modal Logic   Summer term 2016

Exercise Sheet 2

Solutions due: 01.06.2016

Rev. 12341

Exercise 1 (Normal) Logics

1. Prove that if $(\Lambda_j)_{j\in K}$ is a family of normal modal logics, then $\bigcap_{j\in K}\Lambda_j$ is a normal modal logic too.
2. Recall that for any class of frames S, $\Lambda_S = \{\varphi \mid \Vdash_S \varphi\}$. Prove that
   (a) for each class of frames S, Λ_S is a logic;
   (b) for each class of frames S, Λ_S is a normal modal logic.

Exercise 2 Soundness

We know from Exercise 1.2. that for each class of frames S, Λ_S contains (K), (Dual) and all propositional tautologies and furthermore is closed under modus ponens, uniform substitution and generalization.

1. Use this knowledge to prove that
   (a) **K** is sound w.r.t. the class of all frames;
   (b) **L** is sound w.r.t. the class of all finite transitive trees.
2. Recall the following axioms from the lecture:

$$\begin{aligned} &(\mathrm{T}) \quad p \to \Diamond p && \text{(reflexivity)} \\ &(4) \quad \Diamond\Diamond p \to \Diamond p && \text{(transitivity)} \\ &(\mathrm{B}) \quad p \to \Box\Diamond p && \text{(symmetry)} \end{aligned}$$

   and recall that **S5** is the normal modal logic generated by $\{(\mathrm{T}), (4), (\mathrm{B})\}$. Now consider the axiom

$$(5) \quad \Diamond p \to \Box\Diamond p \qquad \text{(euclidity)}$$

   and let **KT5** be the normal modal logic generated by $\{(\mathrm{T}), (5)\}$. Show that for all classes of frames S, **S5** is sound w.r.t. S iff **KT5** is sound w.r.t. S. That is, show that for all classes of frames S,

$$\Vdash_S (\mathrm{T}), \Vdash_S (4) \text{ and } \Vdash_S (\mathrm{B}) \quad \text{iff} \quad \Vdash_S (\mathrm{T}) \text{ and } \Vdash_S (5).$$

Exercise 3 Completeness and consistency

Prove the second item of Proposition 3.13 from the lecture, that is, prove that for all logics Λ and all classes of frames S,

Λ is complete with respect to S iff every Λ-consistent formula is satisfiable in S.

Exercise 4 Hintikka properties

Prove items 3. and 4. of Proposition 3.16 from the lecture by showing that the following Hintikka properties hold for all maximal Λ-consistent sets Γ:

- Γ is closed under modus ponens, that is, if $\{\varphi, \varphi \to \psi\} \subseteq \Gamma$, then $\psi \in \Gamma$;
- $\Lambda \subseteq \Gamma$.

Exercise 5 *Soundness of* S5 — 4 points

Let S5 be the class of all reflexive, transitive, symmetric frames, that is, the class of all frames $\mathcal{F} = (\mathcal{W}, \mathcal{R})$ for which $\mathcal{R}$ is an equivalence relation. Also recall that **S5** is the normal modal logic generated by the set of axioms $\{(4), (\mathrm{T}), (\mathrm{B})\}$. Prove that **S5** is sound w.r.t. S5, i.e. prove that $\Vdash_{\mathrm{S5}} (4)$, $\Vdash_{\mathrm{S5}} (\mathrm{T})$ and $\Vdash_{\mathrm{S5}} (\mathrm{B})$.

Note: The solutions of Exercises 2. and 5. together prove Lemma 3.10 from the lecture.

Exercise 6 *Consistency and* Λ*-MCSs* — 9 points

1. Complete the proof of Proposition 3.13 from the lecture by showing that for all logics Λ and all classes of frames S,

   Λ is sound with respect to S iff every formula that is satisfiable in S is Λ-consistent.

2. Complete the proof of Proposition 3.16 from the lecture by showing that the following Hintikka properties hold for all maximal Λ-consistent sets Γ:
   - for all formulas ψ, we have $\psi \in \Gamma$ or $\neg\psi \in \Gamma$, but not both;
   - for all formulas ψ and φ, we have $\psi \land \varphi \in \Gamma$ iff $\psi \in \Gamma$ and $\varphi \in \Gamma$.

Exercise 7 *Canonical models* — 7 points

We consider the normal modal logic **KF** that is generated by the axiom

$$(\mathrm{F}) \quad \Box p \leftrightarrow \Diamond p.$$

Find a suitable class of frames $S_{\mathbf{KF}}$ for which **KF** is sound and complete.

1. Show that **KF** is sound with respect to your candidate $S_{\mathbf{KF}}$, that is, *prove* that indeed $\Vdash_{S_{\mathbf{KF}}} (\mathrm{F})$.
2. Use the canonical model construction to show that **KF** is complete with respect to your candidate $S_{\mathbf{KF}}$. In detail:
   (a) Adapt Lemma 3.25 from the lecture by showing that for all normal modal logics Λ, if $(\mathrm{F}) \in \Lambda$, then the canonical model $\mathcal{M}^{\Lambda}$ for Λ is based on a frame from $S_{\mathbf{KF}}$.
   (b) Adapt Theorem 3.23 from the lecture to show that **KF** is complete with respect to $S_{\mathbf{KF}}$.

Optional task: Consider the density axiom (C4) (i.e. the converse of (4)):

$$\text{(C4)} \quad \Box\Box p \rightarrow \Box p$$

Show that the logic **KC4** that is generated by $\{(\text{C4})\}$ is sound and complete w.r.t. the class of all dense frames, i.e. frames $(\mathcal{W}, \mathcal{R})$ with the property that

$$\forall w, v \in \mathcal{W}.(w, v) \in \mathcal{R} \Rightarrow \exists u.((w, u) \in \mathcal{R} \wedge (u, v) \in \mathcal{R}).$$

Use the canonical model construction to show completeness.