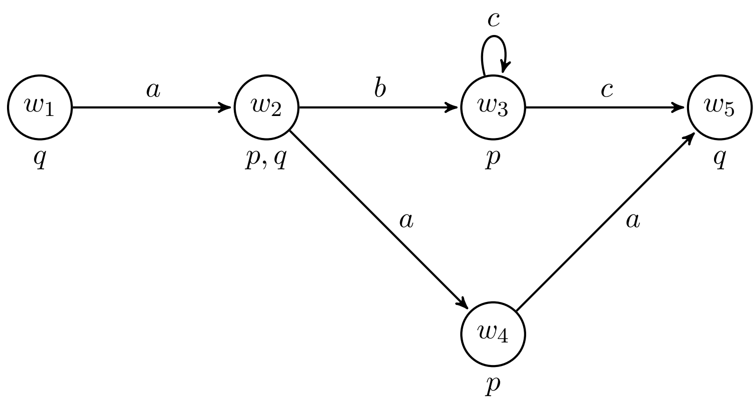
Modal Logic     Summer term 2016

# Exercise Sheet 1

Solutions due: 11.05.2016     Rev. 12109

## Exercise 1 Satisfaction, satisfiability, validity

1. Let $P = \{p, q\}$ and $I = \{a, b, c\}$. Consider the following model




given by $\mathcal{M} = (\mathcal{W}, (\mathcal{R}_a, \mathcal{R}_b, \mathcal{R}_c), \mathcal{V})$, where

$$\mathcal{W} = \{w_1, w_2, w_3, w_4, w_5\} \qquad \mathcal{R}_a = \{(w_1, w_2), (w_2, w_4), (w_3, w_5)\}$$
$$\mathcal{R}_b = \{(w_2, w_3)\} \qquad \mathcal{R}_c = \{(w_3, w_3), (w_3, w_5)\}$$
$$\mathcal{V}(p) = \{w_2, w_3, w_4\} \qquad \mathcal{V}(q) = \{w_1, w_2, w_5\}$$

- Find, for any two nodes $w_i, w_j \in \mathcal{W}$, a formula $\psi$ that distinguishes them, i.e. find a formula $\psi$ with $\mathcal{M}, w_i \models \psi$ but $\mathcal{M}, w_j \not\models \psi$.
- Are the following statements correct?
  (a) $\mathcal{M}, w_1 \models q \vee p$
  (b) $\mathcal{M}, w_1 \models \Diamond_a \Box_a p$
  (c) $\mathcal{M}, w_3 \models \Diamond_c \Diamond_c \neg p \wedge \Box_c (p \vee q)$
  (d) $\mathcal{M}, w_2 \models (\Diamond_a \Box_a q \wedge \Box_b \Diamond_c q) \rightarrow (\neg q \vee \Box_a \bot)$

2. Given a frame $\mathcal{F}$ and a class of frames $S$, a formula $\varphi$ is *valid* in

- $\mathcal{F}$ (written $\Vdash_{\mathcal{F}} \varphi$) iff for each model $\mathcal{M}$ based on $\mathcal{F}$ and each world $w$ of $\mathcal{F}$, we have $\mathcal{M}, w \models \psi$.
- $S$ (written $\Vdash_S \varphi$) iff $\Vdash_{\mathcal{F}} \varphi$ for all frames $\mathcal{F} \in S$.

(a) Prove that for all modal formulas $\psi$,

$\psi$ is valid in the class of all frames iff $\neg\psi$ is unsatisfiable.

(b) Show that the following formulas are not valid in the class of all frames:
- $\Box\bot$
- $p \rightarrow \Box\Diamond p$
- $\Diamond\Box p \rightarrow \Box\Diamond p$

For each formula, describe a class $S$ of frames such that the formula is valid in $S$.

## Exercise 2 Standard translation

1. Translate the following modal formulas to FOL, i.e. compute $ST_x(\psi_1)$ and $ST_x(\psi_2)$:
   - $\psi_1 := \Box p \to p$,
   - $\psi_2 := \Diamond\Diamond p \to \Diamond p$,
2. Prove item 1. of Proposition 1.12 from the lecture, i.e. prove that for all models $\mathcal{M}$, all worlds $w$ of $\mathcal{M}$ and all modal formulas $\psi$, we have

   $$\mathcal{M}, w \models \psi \text{ iff } \mathcal{M}, \eta \models_{FOL} ST_x(\psi) \text{ for } \eta = [w/x].$$

   Prove this by structural induction over $\psi$.

## Exercise 3 Generated submodels

1. Prove Proposition 2.6 from the lecture, i.e. prove that for all models $\mathcal{M}$, all generated submodels $\mathcal{M}'$ of $\mathcal{M}$ and all worlds $w$ of $\mathcal{M}'$, we have

   $$\mathcal{M}, w \equiv_{ML} \mathcal{M}', w.$$

   Use structural induction over formulas in your proof.
2. Use Proposition 2.6 to prove that in the basic modal language, we cannot define the "backwards diamond" $\Diamond^{-}\varphi$ with

   $\mathcal{M}, w \models \Diamond^{-}\varphi$ iff there is a $v \in \mathcal{W}$ s.t. $(v, w) \in \mathcal{R}$ and $\mathcal{M}, w' \models \phi$.

## Exercise 4 $p$-morphisms

1. Prove Proposition 2.8 from the lecture, i.e. prove that for all models $\mathcal{M}$ and $\mathcal{M}'$ s.t. there is a $p$-morphism $f$ from $\mathcal{M}$ to $\mathcal{M}'$, for each world $w$ of $\mathcal{M}$,

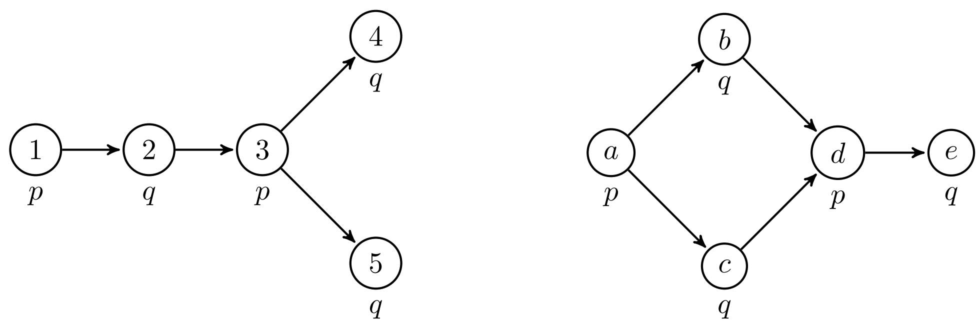
   $$\mathcal{M}, w \equiv_{ML} \mathcal{M}', f(w).$$

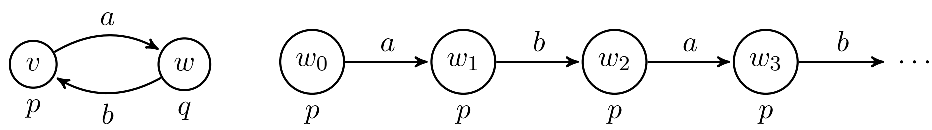
2. Finish the proof of Proposition 2.10 from the lecture by showing the following:

   The mapping $f : W' \to W$ defined by putting $f(w, u_1, \dots, u_n) = u_n$ is a $p$-morphism from $\mathcal{M}'$ to $\mathcal{M}$.

## Exercise 5 Bisimulations

1. Are the nodes 1 and $a$ bisimilar in the models given by the following diagrams? If so, find a suitable bisimulation; otherwise explain why there is no bisimulation relating the two nodes. How about $v$ and $w_0$?






2. Prove Theorem 2.12 from the lecture, i.e. prove that for all models $\mathcal{M}$ and $\mathcal{M}'$, all worlds $w$ of $\mathcal{M}$ and all worlds $w'$ of $\mathcal{M}'$, if $\mathcal{M}, w \simeq \mathcal{M}', w'$, then

$$\mathcal{M}, w \equiv_{ML} \mathcal{M}', w'.$$

## Exercise 6 *Basics, standard translation* 6 points

1. Let $S$, $S_R$, $S_S$ and $S_T$ be the classes of all frames, all reflexive frames, all symmetric frames and all transitive frames, respectively.

   (a) Find a formula $\psi_1$ s.t. $\Vdash_{S_T} \psi_1$ but $\nVdash_{S_R} \psi_1$.

   (b) Find a satisfiable formula $\psi_2$ that is unsatisfiable in $S_S$ (so that there is no model built over a frame from $S_S$ that satisfies $\psi_2$ at some point).

   (c) Find a formula $\psi_3$ s.t. $\Vdash_{S_R} \psi_3$, but $\nVdash_S \psi_3$.

   Justify your solutions.

2. Translate the following modal formulas to FOL, i.e. compute $ST_x(\psi_3)$ and $ST_x(\psi_4)$:

   - $\psi_3 := \Diamond\Diamond p \wedge \Box(q \to \Diamond p)$,
   - $\psi_4 := \Diamond_4\Diamond_1 p \wedge \Box_3(q \to \Diamond_5 p)$, where $I = \mathbb{N}$.

## Exercise 7 *Invariance Results* 7 points

1. **Disjoint unions:** Prove Proposition 2.3 from the lecture, i.e. prove that for all families of pairwise disjoint uni-modal models $(\mathcal{M}_j = (\mathcal{W}_j, \mathcal{R}_j, \mathcal{V}_j))_{j \in K}$, for each $j \in K$ and all worlds $w$ of $\mathcal{M}_j$, we have

$$\mathcal{M}_j, w \equiv_{ML} (\biguplus_{j \in K} \mathcal{M}_j), w.$$

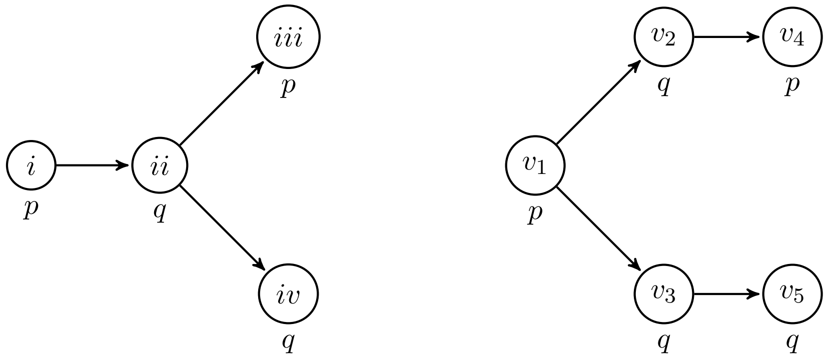
   Use structural induction over formulas in your proof.

2. **p-Morphisms:** Show that p-morphisms generalize disjoint unions and generated submodels, i.e. show that Proposition 2.8. from the lecture implies Propositions 2.3 and 2.6.

   *Hint:* It suffices to define suitable functions $f_d$ and $f_g$, and show them to be $p$-morphisms, where the function $f_d$ should, for each $i \in K$, map worlds from $\mathcal{M}_i$ to worlds from $(\biguplus_{j \in K} \mathcal{M}_j)$ while the function $f_g$ should map worlds from a generated submodel $\mathcal{M}'$ of $\mathcal{M}$ to worlds of the supermodel $\mathcal{M}$.

## Exercise 8 *Bisimulations* 7 points

1. Are the nodes $i$ and $v_1$ bisimilar in the models given by the following diagrams? If so, find a suitable bisimulation; otherwise explain why there is no bisimulation relating the two nodes.




2. Let $\mathcal{M} = (\mathcal{W}, (\mathcal{R}_i)_{i \in I}, V)$ be a model. Prove that

   (a) the identity relation $Id_{\mathcal{W}} = \{(w, w) \mid w \in W\}$ is a bisimulation on $\mathcal{M}$;

   (b) if $Z_1$ and $Z_2$ are bisimulations on $\mathcal{M}$, so is $Z_1 \cup Z_2$.