Adjunctions and Monads

Thorsten Wißmann, Seminar "Categories in Programming", June 3, 2015

1 Adjunctions

> Adjoint functors arise everywhere
>
> [Lan98]

However, this talk is based on [Awo10, Chapter 9 and 10]. So far, we have seen many kinds of one-to-one-correspondences:

Products

$$\frac{C \longrightarrow A \quad C \longrightarrow B}{C \longrightarrow A \times B}$$

Currying

$$\frac{A \times B \longrightarrow C}{A \longrightarrow C^B}$$

∀-Quantification

$$\frac{\gamma \vdash \psi, \quad y \notin \mathsf{FV}(\gamma)}{\gamma \vdash \forall y.\psi}$$

with γ, ψ Formulas of FOL, $\mathsf{FV}(\gamma) \subseteq X, y \notin X$, $\mathsf{FV}(\gamma) \subseteq X \cup \{y\}$.

All examples are instances of the same principle: *adjoint functors.*

Definition 1. For two functors $F : \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{C}$, we have an *adjunction* $F \dashv U$ if there is a natural transformation $\eta : \mathrm{Id}_\mathcal{C} \to UF$ with the *universal mapping property (UMP)*:

$$\forall \;\; C \xrightarrow{\;g\;} UD \;\exists!\;\; FC \overset{g}{\dashrightarrow} D \;\; \text{such that} \quad \begin{array}{ccc} UFC & \xrightarrow{Ug} & UD \\ \eta_C \uparrow & \nearrow_{f} & \\ C & & \end{array}$$

Example 2. Denote by $F : \mathsf{Set} \to \mathsf{Mon}$ the *free monoid* construction, $X \mapsto X^*$, the set of finite words over X, together with concatenation, and the forgetful $U : \mathsf{Mon} \to \mathsf{Set}$. The universal property of the says: For a monoid M, any function $X \to UM$ in Set extends to a unique monoid homomorphism $FX \to M$.

Proposition 3. *For $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$, the following are equivalent:*

- $F \dashv U$
- *There is a bijection $\mathcal{C}(X, UY) \cong \mathcal{D}(FX, Y)$, that is natural in X and Y:*

$$\frac{FX \longrightarrow Y}{X \longrightarrow UY}$$

- *There is a $\epsilon : FU \to \mathrm{Id}_\mathcal{D}$ (the* counit*) with the UMP:*

$$\forall\; FX \xrightarrow{g} Y \quad \exists!\; X \xrightarrow{f} UY : \quad g = \epsilon_Y \cdot Ff$$

Products

$F : \mathcal{C} \times \mathcal{C} \to \mathcal{C}, (X, Y) \mapsto X \times Y$
$U : \mathcal{C} \to \mathcal{C} \times \mathcal{C}, C \mapsto (C, C)$

$$\frac{(f, g) : (C, C) \longrightarrow (A, B)}{\langle f, g \rangle : C \longrightarrow A \times B}$$

Currying

Fixed $B \in \mathcal{C}$, $F, U : \mathcal{C} \to \mathcal{C}$: $FX = X \times B, UX = X^B$.
$\eta_X : X \to (X \times B)^B$
$\epsilon_X : (X^B \times B) \to X$

∀-Quantification

$\mathrm{Form}(\bar{x})$ = FOL-Formulas with free variables from $\bar{x}$ ordered by $\vdash$.
$F : \mathrm{Form}(\bar{x}) \to \mathrm{Form}(\bar{x}, y)$
$U : \mathrm{Form}(\bar{x}, y) \to \mathrm{Form}(\bar{x})$
$U : \psi \mapsto \forall y.\psi$

Example 4. Any kind of free construction functor F and forgetful functor U.

Example 5. For posets $\mathcal{C}$ and $\mathcal{D}$, the pair $F \dashv U$ is called *Galois connection.*

Proposition 6. *Right adjoints preserve limits and monomorphisms.*

2 Monads

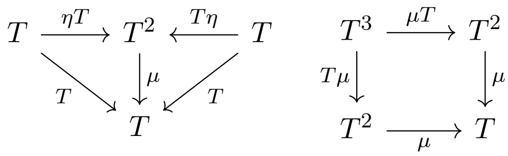
Monads are just monoids in the category of endofunctors

Monads generalize the notion of *algebraic theories* (Signature + Equations):

Definition 7. A *Monad* on a category $\mathcal{C}$ is a triple*

$$\big(T : \mathcal{C} \to \mathcal{C}, \eta : \mathrm{Id}_{\mathcal{C}} \to T, \mu : TT \to T\big)$$

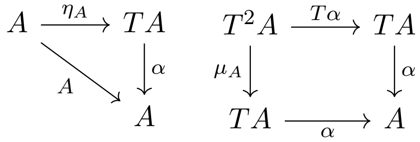
such that the following diagrams (called *unit law* and *associativity law*) hold:



Example 8. $TX = X^*$, $\eta_X : x \mapsto x$, and $\mu_X : X^{**} \to X^*$ concats.

Theorem 9. *For adjunction $F \dashv U$ with unit η, $T = UF$ is a monad with unit η and $\mu = U\epsilon F : UFUF \to UF$.*

Definition 10 (Eilenberg-Moore category)**.** Define $\mathcal{C}^T$ has of algebras $(A, \alpha : TA \to A)$ s.t.

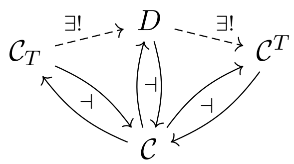


commutes as objects, and T-algebra homomorphisms as morphisms.

Theorem 11. *Every monad (T, η, μ) arises from an adjunction $F : \mathcal{C} \rightleftarrows \mathcal{C}^T : U$.*

Proofsketch. $FX = (TX, \mu_X : TTX \to TX)$ is an EM-algebra by the monad laws and $Ff = Tf$ a homomorphism by naturality of μ. $U(A, \alpha) = A$. The unit of the adjunction is η. The counit is $\epsilon_{(A,\alpha)} = \alpha$. An $f : X \to U(Y, y)$ induces $\epsilon_{(Y,y)} \cdot Tf : (TX, \mu_X) \to (Y, y)$. □

Remark 12. There are multiple ways to decompose a monad T into two adjoint functors. Another possibility is through the Kleisli-Category $\mathcal{C}_T$ a full subcategory of $\mathcal{C}^T$ consisting of free algebras only. $\mathbf{obj}\mathsf{Adj}(T) = (\mathcal{D}, F : \mathcal{D} \rightleftarrows \mathcal{C} : U)$; $\mathsf{Adj}(T)(\mathcal{D}, \mathcal{E})$ = Functors preserving U and F.



Example 13.

- For an (1-sorted) algebraic theory (Σ, E), i.e. Signature + Equations, TX = Σ-Terms over X modulo E. Set^T = Models for (Σ, E). E.g. for $TX = X^*$, $\mathsf{Set}^T = \mathsf{Mon}$.
- For $\mathcal{C}$ with coproducts and $C \in \mathcal{C}$, set $TX = X + C$: $\mathcal{C}^T = C/\mathcal{C}$, $\mathcal{C}_T$ = "computations with exceptions from C".
- For $\mathcal{C}$ a poset, monads on $\mathcal{C}$ are *closure operators*.

References

[Awo10] Steve Awodey. *Category Theory.* Oxford Logic Guides. OUP Oxford, 2010.

[Lan98] Saunders Mac Lane. *Categories for the Working Mathematician.* Graduate Texts in Mathematics. Springer New York, 1998.

*and was originally called *triple*