Succinctness Results for some Extensions of Multimodal Logic over K and S5

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June 4, 2014
Previous Work

Comparing Logics

Logic

A logic \( L = (\Phi, \models, \mathcal{M}) \)

- \( \Phi \): non-empty set of formulae
- \( \models \): satisfaction relation
- \( \mathcal{M} \): non-empty class of models

\( \mathcal{M} \models \varphi \) for some \( \mathcal{M} \in \mathcal{M}, \varphi \in \Phi \)
Comparing Logics

Logic

A logic \( L = (\Phi, \models, M) \)
- \( \Phi \): non-empty set of formulae
- \( \models \): satisfaction relation
- \( M \): non-empty class of models

Expressivity

\( L_1 = (\Phi_1, \models_1, M), L_2 = (\Phi_2, \models_2, M). \)

\( L_2 \) is at least as expressive as \( L_1 \) \((L_1 \leq_M L_2)\) iff

\[
\forall \varphi_1 \in \Phi_1 \exists \varphi_2 \in \Phi_2 \forall M \in M. \ M \models_1 \varphi_1 \iff M \models_2 \varphi_2
\]

\( \varphi_1 \equiv_M \varphi_2 \) (equivalence)
Succinctness

Let $L_1 = (\Phi_1, \models_1, M)$, $L_2 = (\Phi_2, \models_2, M)$, $L_1 \leq_M L_2$ and $F$ be a class of functions.

$L_1$ is $F$-succinct in $L_2$ on $M$ ($L_1 \leq^F_M L_2$) iff

$$\exists f \in F \ \forall \varphi_1 \in \Phi_1 \ \exists \varphi_2 \equiv_M \varphi_2 \in \Phi_2. \ |\varphi_2| \leq f(|\varphi_1|)$$

$L_1$ is exponentially more succinct than $L_2$ iff $L_1 \leq^F_M L_2$ and $F \not\subseteq SUBEXP$ ($L_1 \not\leq^{SUBEXP}_M L_2$).
Succinctness

Let $L_1 = (\Phi_1, \models_1, M)$, $L_2 = (\Phi_2, \models_2, M)$, $L_1 \leq_M L_2$ and $F$ be a class of functions. $L_1$ is $F$-succinct in $L_2$ on $M$ ($L_1 \leq^F_M L_2$) iff

$$\exists f \in F \forall \varphi_1 \in \Phi_1 \exists \varphi_1 \equiv_M \varphi_2 \in \Phi_2. |\varphi_2| \leq f(|\varphi_1|)$$

$L_1$ is exponentially more succinct than $L_2$ iff $L_1 \leq^F_M L_2$ and $F \not\subseteq SUBEXP$ ($L_1 \not\leq^{SUBEXP}_M L_2$).

- $L_1 \leq^F_M L_2$ and $L_2 \leq^F_M L_1$ (or $L_1 \not\leq^F_M L_2$ and $L_2 \not\leq^F_M L_1$) can be true at the same time.
- Succinctness is not necessarily transitive.
Multimodal Logic $\mathcal{ML}$

Syntax of $\mathcal{ML}$

$$\varphi ::= \top \mid \bot \mid p \mid \neg p \mid \varphi \lor \varphi \mid \varphi \land \varphi \mid \langle r \rangle \varphi \mid [r] \varphi$$

with propositional symbols $p$ and relational symbols $r \in R$. 
Multimodal Logic $\mathcal{ML}$

Syntax of $\mathcal{ML}$

$\varphi ::= \top | \bot | p | \neg p | \varphi \lor \varphi | \varphi \land \varphi | \langle r \rangle \varphi | [r] \varphi$

with propositional symbols $p$ and relational symbols $r \in R$.

Negation normal form ($\overline{\varphi}$ negation for $\varphi \in \mathcal{ML}$):

- $\overline{\top} = \bot$
- $\overline{p} = \neg p$
- $\overline{\varphi_1 \land \varphi_2} = \overline{\varphi_1} \lor \overline{\varphi_2}$
- $\overline{\langle r \rangle \varphi} = [r] \overline{\varphi}$
- $\ldots$
Multimodal Logic $\mathcal{ML}$

Syntax of $\mathcal{ML}$

$$\varphi ::= T | \bot | p | \neg p | \varphi \lor \varphi | \varphi \land \varphi | \langle r \rangle \varphi | [r] \varphi$$

with propositional symbols $p$ and relational symbols $r \in R$.

Formula size:

- $|T| = |\bot| = |p| = |\neg p| = 1$
- $|\varphi \lor \psi| = |\varphi \land \psi| = 1 + |\varphi| + |\psi|$
- $|\langle r \rangle \varphi| = |[r] \varphi| = 1 + |\varphi|$
Syntax of $[\forall \Gamma] \mathcal{ML}$

\[
\varphi ::= \top | \bot | p | \neg p | \varphi \lor \varphi | \varphi \land \varphi | \langle r \rangle \varphi | [r] \varphi | [\forall \Gamma] \varphi | \langle \forall \Gamma \rangle \varphi
\]

with propositional symbols $p$ and relational symbols $r \in R$ and sets of relational symbols $\Gamma \subseteq R$.

$\mathcal{ML}$-equivalence:

\[
[\forall \Gamma] \psi \equiv \bigwedge_{r \in \Gamma} [r] \psi
\]
Syntax of $[\exists \Gamma] \mathcal{ML}$

\[ \varphi ::= T \mid \bot \mid p \mid \neg p \mid \varphi \lor \varphi \mid \varphi \land \varphi \mid \langle r \rangle \varphi \mid [r] \varphi \mid [\exists \Gamma] \varphi \mid \langle \exists \Gamma \rangle \varphi \]

with propositional symbols $p$ and relational symbols $r \in R$ and sets of relational symbols $\Gamma \subseteq R$.

$\mathcal{ML}$-equivalence:

\[ [\exists \Gamma] \psi \equiv \bigvee_{r \in \Gamma} [r] \psi \]
Syntex of $\mathcal{ML}$

$\varphi ::= \top \mid \bot \mid p \mid \neg p \mid \varphi \lor \varphi \mid \varphi \land \varphi \mid \langle r \rangle \varphi \mid [r] \varphi \mid [\varphi] \psi \mid \langle \varphi \rangle \psi$

with propositional symbols $p$ and relational symbols $r \in R$.

$\mathcal{ML}$-equivalence:

$[\varphi]p \equiv \varphi \to p$

$[\varphi](\psi_1 \lor \psi_2) \equiv [\varphi] \psi_1 \lor [\varphi] \psi_2$

$[\varphi] \psi \equiv \varphi \to [\varphi] \psi$

$[\varphi][r] \psi \equiv \varphi \to [r][\varphi] \psi$

$[\varphi_1][\varphi_2] \psi \equiv [\varphi_1 \land [\varphi_1][\varphi_2]] \psi$
Multimodal Logic $\mathcal{ML}$

Interpretation of formulae: Kripke Models $\mathcal{M} = (W, R, V)$

- $W$: non-empty carrier set
- $R$: set of binary relations ($\{r, b, g, \ldots\}$)
- $V$: Valuation

Successors: $\text{succs}_r(w) = \{v \mid (w, v) \in r\}$
Interpretation of formulae: Kripke Models $\mathcal{M} = (W, R, V)$

- $W$: non-empty carrier set
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Successors: $\text{succs}_r(w) = \{v \mid (w, v) \in r\}$

### Semantics of $\mathcal{ML}$

- $\mathcal{M}, w \models T$
- $\mathcal{M}, w \models p$ $\iff$ $w \in V(p)$
- $\mathcal{M}, w \models \neg p$ $\iff$ $w \notin V(p)$
- $\mathcal{M}, w \models \varphi \land \psi$ $\iff$ $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \varphi \lor \psi$ $\iff$ $\mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \langle r \rangle \varphi$ $\iff$ $\mathcal{M}, v \models \varphi$ for some $v \in \text{succs}_r(w)$
- $\mathcal{M}, w \models [r] \varphi$ $\iff$ $\mathcal{M}, v \models \varphi$ for every $v \in \text{succs}_r(w)$
Separation and Description Problem

**Separation**

\( \mathcal{M} = (W, R, V) \) model, \( S, D \subseteq W \) non-empty sets

\( \varphi \) separates \( S \) and \( D \) in \( \mathcal{M} \) iff \( \forall s \in S . \mathcal{M}, s \models \varphi \) and \( \forall d \in D . \mathcal{M}, d \not\models \varphi \)
Separation and Description Problem

Separation

\[ \mathcal{M} = (W, R, V) \] model, \( S, D \subseteq W \) non-empty sets
\[ \varphi \] separates \( S \) and \( D \) in \( \mathcal{M} \) iff \( \forall s \in S. \mathcal{M}, s \models \varphi \) and \( \forall d \in D. \mathcal{M}, d \not\models \varphi \)

Proof strategy for \( L_1 \) being exponentially more succinct than \( L_2 \):

1. Find a family of formulae \( \varphi_n \in L_2 \) with size exponential in \( n \)
2. Find a family of models \( \mathcal{M}_n, S_n, D_n \subseteq W_n \) with \( \varphi_n \) being the smallest formulae separating \( S_n \) and \( D_n \) for all \( n \).
3. If \( \psi_n \in L_1, \psi_n \equiv_M \varphi_n \) for all \( n \), is of size linear in \( n \) then \( L_1 \not\leq_{SUBEXP} L_2 \)
Model $\mathcal{M} = (W, R, V)$, $w, v \in W$.

**Bisimulation Game**

The game $G(w, v)$ is played between two players (Spoiler, Duplicator). Rules are:

- $(p)$: Spoiler picks $p$ with $w \in V(p)$ and $v \not\in V(p)$ and wins.
- $(\overline{p})$: Spoiler picks $p$ with $w \not\in V(p)$ and $v \in V(p)$ and wins.
- $\langle r, w' \rangle$: Spoiler picks $r \in R$ and one $w' \in \text{succs}_r(w)$. Duplicator has to pick one $v' \in \text{succs}_r(v)$ or loses. Continuation in game $G(w', v')$.
- $[r, v']$: Spoiler picks $r \in R$ and one $v' \in \text{succs}_r(v)$. Duplicator has to pick one $w' \in \text{succs}_r(w)$ or loses. Continuation in game $G(w', v')$. 
Winning Strategies

If Duplicator has a winning strategy in $G(w, v)$ on the model $M$ then

$$\forall \psi \in ML. M, w \models \psi \iff M, v \models \psi$$

If Spoiler has a winning strategy in $G(w, v)$ then

$$\exists \varphi \in ML. M, w \models \varphi \text{ and } M, v \not\models \varphi$$

and $\min(|\varphi|)$ gives a lower bound for the size of formulae describing $w$.

In the game $G(w, w)$ Spoiler cannot have a winning strategy.
Uniform Strategy Trees

Winning strategies for Spoiler in bisimulation games:

Uniform Strategy Trees

- nodes: $\langle r, S' \rangle$, $[r, D']$, $(p)$, $(\overline{p})$, $(\lor)$, $(\land)$
  - $r$ relational symbol, $S, S', D, D' \subseteq W$
- essentially a syntax tree for a formula
- a formula separates $S$ and $D$ if its corresponding uniform strategy tree wins the game $G(S, D)$
Uniform Strategy Trees

In a uniform strategy tree, winning $\mathcal{G}(S, D)$, with root $x$ the following properties hold:

**Winning Uniform Strategy Tree**

If $x = (p)$ then $S \cap V(p) = S$ and $D \cap V(p) = \emptyset$

$\varphi = p$ separates $S$ and $D$
Uniform Strategy Trees

In a uniform strategy tree, winning $G(S, D)$, with root $x$ the following properties hold:

**Winning Uniform Strategy Tree**

If $x = (\overline{p})$ then $S \cap V(p) = \emptyset$ and $D \cap V(p) = D$.

$\varphi = \overline{p}$ separates $S$ and $D$
Uniform Strategy Trees

In a uniform strategy tree, winning $G(S, D)$, with root $x$ the following properties hold:

**Winning Uniform Strategy Tree**

If $x = \langle r, S' \rangle$ then $S' \cap \text{succs}_r(s) \neq \emptyset$ for every $s \in S$ and if $\text{succs}_r(D) \neq \emptyset$ then there is an edge $x \rightarrow y$ and $y$ is the root of a uniform strategy tree, winning the game $G(S', \text{succs}_r(D))$. 
Uniform Strategy Trees

In a uniform strategy tree, winning $G(S, D)$, with root $x$ the following properties hold:

**Winning Uniform Strategy Tree**

If $x = \langle r, S' \rangle$ then $S' \cap \text{succs}_r(s) \neq \emptyset$ for every $s \in S$ and if $\text{succs}_r(D) \neq \emptyset$ then there is an edge $x \rightarrow y$ and $y$ is the root of a uniform strategy tree, winning the game $G(S', \text{succs}_r(D))$. 

$\models \phi \neq \models \phi \neq \models \phi$
Uniform Strategy Trees

In a uniform strategy tree, winning $G(S, D)$, with root $x$ the following properties hold:

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If $x = \langle r, S' \rangle$ then $S' \cap \text{succs}_r(s) \neq \emptyset$ for every $s \in S$ and if $\text{succs}_r(D) \neq \emptyset$ then there is an edge $x \rightarrow y$ and $y$ is the root of a uniform strategy tree, winning the game $G(S', \text{succs}_r(D))$.

$$\varphi = \langle r \rangle \psi \text{ separates } S \text{ and } D.$$
Uniform Strategy Trees

In a uniform strategy tree, winning $G(S, D)$, with root $x$ the following properties hold:

**Winning Uniform Strategy Tree**

If $x = [r, D']$ then $D' \cap \text{succs}_r(d) \neq \emptyset$ for every $d \in D$ and if $\text{succs}_r(S) \neq \emptyset$ then there is an edge $x \rightarrow y$ and $y$ is the root of a uniform strategy tree, winning the game $G(\text{succs}_r(S), D')$. 

![Diagram of Uniform Strategy Trees](image)
Uniform Strategy Trees

In a uniform strategy tree, winning $G(S, D)$, with root $x$ the following properties hold:

**Winning Uniform Strategy Tree**

If $x = [r, D']$ then $D' \cap \text{succs}_r(d) \neq \emptyset$ for every $d \in D$ and if $\text{succs}_r(S) \neq \emptyset$ then there is an edge $x \rightarrow y$ and $y$ is the root of a uniform strategy tree, winning the game $G(\text{succs}_r(S), D')$. 

\[\models \psi\] 

\[\not\models \psi\]
Uniform Strategy Trees

In a uniform strategy tree, winning $G(S, D)$, with root $x$ the following properties hold:

**Winning Uniform Strategy Tree**

If $x = [r, D']$ then $D' \cap \text{succs}_r(d) \neq \emptyset$ for every $d \in D$ and if $\text{succs}_r(S) \neq \emptyset$ then there is an edge $x \rightarrow y$ and $y$ is the root of a uniform strategy tree, winning the game $G(\text{succs}_r(S), D')$.

$\varphi = [r]\psi$ separates $S$ and $D$. 
Uniform Strategy Trees

In a uniform strategy tree, winning $G(S, D)$, with root $x$ the following properties hold:

**Winning Uniform Strategy Tree**

If $x = (\lor)$ then $S = S_1 \cup S_2$ and there are nodes $y_1$ and $y_2$ with edges $x \xrightarrow{S_i} y_i$ and $y_i$ is the root of a uniform strategy tree, winning the game $G(S_i, D)$, $i = 1, 2$.

\[
S_1 \models \psi_1 \quad S_2 \models \psi_2 \quad \not\models \psi_1, \psi_2
\]
Uniform Strategy Trees

In a uniform strategy tree, winning $G(S, D)$, with root $x$ the following properties hold:

**Winning Uniform Strategy Tree**

If $x = (\lor)$ then $S = S_1 \cup S_2$ and there are nodes $y_1$ and $y_2$ with edges $x \xrightarrow{S_i} y_i$ and $y_i$ is the root of a uniform strategy tree, winning the game $G(S_i, D)$, $i = 1, 2$.

$\models \psi_1$

$\models \psi_2$

$\not\models \psi_1, \psi_2$

$\varphi = \psi_1 \lor \psi_2$ separates $S$ and $D$. 
Uniform Strategy Trees

In a uniform strategy tree, winning $\mathcal{G}(S, D)$, with root $x$ the following properties hold:

Winning Uniform Strategy Tree

If $x = (\wedge)$ then $D = D_1 \cup D_2$ and there are nodes $y_1$ and $y_2$ with edges $x \xrightarrow{D_i} y_i$ and $y_i$ is the root of a uniform strategy tree, winning the game $\mathcal{G}(S, D_i)$

\[
\begin{align*}
S & \models \psi_1, \psi_2 \\
& \not\models \psi_1 \\
& \not\models \psi_2 \\
\varphi &= \psi_1 \land \psi_2 \text{ separates } S \text{ and } D.
\end{align*}
\]
Proof Strategy

Proof strategy for $L_1$ being exponentially more succinct than $L_2 = \mathcal{ML}$:

1. Find a family of formulae $\varphi_n \in L_2$ with size exponential in $n$.
2. Find a family of models $(\mathcal{M}_n, S_n, D_n)$ with the uniform strategy tree of minimum size for the game $G(S_n, D_n)$ being the syntax tree for $\varphi_n$.
3. If $\psi_n \in L_1$, $\psi_n \equiv_{\mathcal{M}} \varphi_n$ for all $n$, is of size linear in $n$ then $L_1 \not\leq_{\mathcal{M}}^{\text{SUBEXP}} L_2$. 
Models

Recursively defined models:

\[
\begin{align*}
G(S_n, D_n) & \quad s_n & d_n \\
& \vdots & \vdots & \vdots \\
& s_{n-1} & d_{n-1} \\
& \vdots & \vdots & \vdots \\
& s_0 & d_0 \\
& \vdots & \vdots & \vdots 
\end{align*}
\]
Models

Recursively defined models:

$s_n$  

$\vdots$  

$s_{n-1}$  

$\vdots$  

$s_0$  

$G(S_n, D_n)$  

$UST(M)$
Recursively defined models:

\[ G(S_n, D_n) \]

\[ \text{UST}(M) \]

\[ G(S_{n-1}, D_{n-1}) \]
Models

Recursively defined models:

\[ s_n \quad d_n \]

\[ \vdots \quad \vdots \quad \vdots \]

\[ s_{n-1} \quad d_{n-1} \]

\[ \vdots \quad \vdots \quad \vdots \]

\[ s_0 \quad d_0 \]

\[ \vdots \quad \vdots \quad \vdots \]

\[ G(S_n, D_n) \]

\[ UST(\mathcal{M}) \]

\[ G(S_{n-1}, D_{n-1}) \]

\[ UST(\mathcal{M}) \]

\[ \vdots \]

\[ UST(\mathcal{M}) \]
Models

Recursively defined models:

\[
\begin{align*}
&G(S_n, D_n) \\
&\downarrow \text{UST}(M) \\
&G(S_{n-1}, D_{n-1}) \\
&\downarrow \text{UST}(M) \\
&\vdots \\
&\downarrow \text{UST}(M) \\
&G(S_0, D_0)
\end{align*}
\]
Models

Recursively defined models:

\[ G(S_n, D_n) \]
\[ \xrightarrow{\text{UST}(\mathcal{M})} \]
\[ G(S_{n-1}, D_{n-1}) \]
\[ \xrightarrow{\text{UST}(\mathcal{M})} \]
\[ \vdots \]
\[ \xrightarrow{\text{UST}(\mathcal{M})} \]
\[ G(S_0, D_0) \]
\[ \xrightarrow{\text{UST}(\mathcal{M}_0)} \]
Minimality of Uniform Strategy Trees

Minimality as local property by construction

- Only a single (non-trivial) move is possible for Spoiler in $G(S, D)$.
- It is easy to show that all alternative moves in $G(S, D)$ are less than optimal in terms of size.
Minimality of Uniform Strategy Trees

Minimality as local property by construction
- Only a single (non-trivial) move is possible for Spoiler in $G(S, D)$.
- It is easy to show that all alternative moves in $G(S, D)$ are less than optimal in terms of size.

Successors in $K$
In a game $G(s, d)$ with $d' \in \text{succs}_r(s)$ and $d' \in \text{succs}_r(d)$ Spoiler loses with the move $\langle r, d' \rangle$ (Analogously with a $[r]$-move).
∀Γ \mathcal{ML} \text{ over } \mathcal{K}

[∀\{r,b\}] \psi ≡_K [r]\psi \land [b]\psi
\[ \forall \{r, b\} \psi \equiv \mathbf{K} [r] \psi \land [b] \psi \]
∀Γ \mathcal{ML} \text{ over } K

\begin{align*}
&\text{UST}(A_{n-1}) \\
&\Upsilon
\end{align*}
∀Γ \mathcal{ML} over K
$\forall \Gamma \mathcal{ML}$ over $\textbf{K}$
∀Γ \mathcal{ML} \text{ over } \mathbf{K}
Results for $K$

Previous Results

- French, T. et al.: Proof that $\forall \Gamma \mathcal{ML}$ and $\exists \Gamma \mathcal{ML}$ are exponentially more succinct than $\mathcal{ML}$ over $K$ based on models with 2 relational symbols and 1 propositional symbol.
- Lutz, C.: Proof that $[\varphi] \mathcal{ML}$ is exponentially more succinct than $\mathcal{ML}$ over $K$ based on models with 2 relational symbols.

My Results

$[\forall \Gamma] \mathcal{ML}$, $[\exists \Gamma] \mathcal{ML}$ and $[\varphi] \mathcal{ML}$ are exponentially more succinct than $\mathcal{ML}$ over $K$ based on models with only 2 relational symbols.
Handling Reflexivity, Symmetry, Transitivity
Handling Reflexivity, Symmetry, Transitivity

Clique

Let $s \in S$, $d \in D$ in some game $G(S, D)$ over some $\textbf{S5}$-model. If $s$ and $d$ are members of the same $r$-clique for some relational symbol $r$ then there is no winning strategy for Spoiler beginning with a $\langle r \rangle$- or $[r]$-move.
S5

Handling Reflexivity, Symmetry, Transitivity

Cliques

Let \( s \in S, d \in D \) in some game \( G(S, D) \) over some S5-model. If \( s \) and \( d \) are members of the same \( r \)-clique for some relational symbol \( r \) then there is no winning strategy for Spoiler beginning with a \( \langle r \rangle \)- or \([r]\)-move.
S5

Handling Reflexivity, Symmetry, Transitivity

**Subgames**

Let $t$ be the uniform strategy tree of minimum size for $G(S, D)$ over some model and $t'$ the uniform strategy tree of minimum size for $G(S \cup S', D \cup D')$. Assuming the games can be won by Spoiler then $|t| \leq |t'|$. 
Handling Reflexivity, Symmetry, Transitivity

Subgames

Let $t$ be the uniform strategy tree of minimum size for $G(S, D)$ over some model and $t'$ the uniform strategy tree of minimum size for $G(S \cup S', D \cup D')$. Assuming the games can be won by Spoiler then $|t| \leq |t'|$. 

\[
\langle r, s \rangle \quad \begin{array}{c}
\text{s} \\
\text{s'}
\end{array} \quad \begin{array}{c}
\text{d} \\
\text{d'}
\end{array}
\]
Handling Reflexivity, Symmetry, Transitivity

**Subgames**

Let $t$ be the uniform strategy tree of minimum size for $G(S, D)$ over some model and $t'$ the uniform strategy tree of minimum size for $G(S \cup S', D \cup D')$. Assuming the games can be won by Spoiler then $|t| \leq |t'|$. 

\[
\langle r, s \rangle \quad s \quad d \\
|s' \quad \bullet | \quad |d' \quad \bullet |
\]
Handling Reflexivity, Symmetry, Transitivity

**r-equivalent Games**

Let $\mathcal{G}(s, d)$ and $\mathcal{G}(s', d')$ be games over some S5-model and $s' \in \text{succs}_r(s)$, $d' \in \text{succs}_r(d)$. Then every $\langle r \rangle$- or $[r]$-move applied to both games will result in the same game. We call two such games $r$-equivalent and any strategy starting with a $\langle r \rangle$- or $[r]$-move is applicable to both games.
Handling Reflexivity, Symmetry, Transitivity

**Strategies on r-equivalent Games**

Let $G(s, d)$ be a game over some S5-model, $t$ a uniform strategy tree for this game, $G(S, D)$ the game after one or more moves in $t$ and $G(s', d')$ a subgame of $G(S, D)$. If the uniform strategy tree of minimum size for $G(s', d')$ begins with a $\langle r \rangle$- or $[r]$-move and $G(s', d')$ and $G(s, d)$ are $r$-equivalent then $t$ is not a uniform strategy tree of minimum size for $G(s, d)$.
S5

Handling Reflexivity, Symmetry, Transitivity

Reflexivity and propositional Symbols

Let \( \langle r, S' \rangle \) (or \([r, D^*]\)) be the first move in the minimum uniform strategy tree for the game \( G(S, D) \) in the S5-model \((M, S, D)\), and let \( G(S', D \cup D') \) (or \( G(S \cup S^*, D^*) \)) be the game we would have to play after this move \((S \subseteq \text{succs}_r(S), D \subseteq \text{succs}_r(D), \text{reflexivity of S5})\). If \( S' \models p \) and \( D \models \overline{p} \) (or \( D^* \models p \) and \( S \models \overline{p} \)) then we can use the strategies shown below where \( t \) is the minimum uniform strategy tree for the game \( G(S', D') \) (or \( G(S^*, D^*) \)).
Handling Reflexivity, Symmetry, Transitivity

Reflexivity and propositional Symbols

Let $\langle r, S' \rangle$ (or $[r, D^*]$) be the first move in the minimum uniform strategy tree for the game $G(S, D)$ in the S5-model $(\mathcal{M}, S, D)$, and let $G(S', D \cup D')$ (or $G(S \cup S^*, D^*)$) be the game we would have to play after this move ($S \subseteq \text{succs}_r(S)$, $D \subseteq \text{succs}_r(D)$, reflexivity of S5). If $S' \models p$ and $D \models \neg p$ (or $D^* \models p$ and $S \models \neg p$) then we can use the strategies shown below where $t$ is the minimum uniform strategy tree for the game $G(S', D')$ (or $G(S^*, D^*)$).
S5

Handling Reflexivity, Symmetry, Transitivity

Reflexivity and propositional Symbols

Let \( \langle r, S' \rangle \) (or \([r, D^*]\)) be the first move in the minimum uniform strategy tree for the game \( G(S, D) \) in the S5-model \((M, S, D)\), and let \( G(S', D \cup D') \) (or \( G(S \cup S^*, D^*) \)) be the game we would have to play after this move \((S \subseteq \text{succs}_r(S), D \subseteq \text{succs}_r(D), \text{reflexivity of S5})\). If \( S' \models p \) and \( D \models \bar{p} \) (or \( D^* \models p \) and \( S \models \bar{p} \)) then we can use the strategies shown below where \( t \) is the minimum uniform strategy tree for the game \( G(S', D') \) (or \( G(S^*, D^*) \)).
S5

Handling Reflexivity, Symmetry, Transitivity

**Reflexivity and propositional Symbols**

Let \( \langle r, S' \rangle \) (or \( [r, D^*] \)) be the first move in the minimum uniform strategy tree for the game \( G(S, D) \) in the S5-model \( (M, S, D) \), and let \( G(S', D \cup D') \) (or \( G(S \cup S^*, D^*) \)) be the game we would have to play after this move \( (S \subseteq \text{succs}_r(S), D \subseteq \text{succs}_r(D), \text{reflexivity of } S5) \). If \( S' \models p \) and \( D \models \overline{p} \) (or \( D^* \models p \) and \( S \models \overline{p} \)) then we can use the strategies shown below where \( t \) is the minimum uniform strategy tree for the game \( G(S', D') \) (or \( G(S^*, D^*) \)).

**Size**

Any uniform strategy tree of this kind is at most three time larger than the uniform strategy tree of minimum size.
Recursive **S5**-Model
Recursive S5-Model
Recursive S5-Model
Recursive S5-Model
Recursive $\textbf{S5}$-Model

\[ d_n^{ll} \quad \bullet \quad s_{n-1} \quad \bullet \quad s_n^m \quad \bullet \quad d_n \quad \bullet \quad d_n^{rr} \quad \bullet \quad d_{n-1} \]
Recursive S5-Model
Recursive S5-Model
Recursive S5-Model
Recursive S5-Model
Results for **S5**

**Succinctness Results for \( [\forall \Gamma] \mathcal{ML}, [\exists \Gamma] \mathcal{ML}, [\varphi] \mathcal{ML} \)**

In **S5**, the Logics \( [\forall \Gamma] \mathcal{ML}, [\exists \Gamma] \mathcal{ML} \) and \( [\varphi] \mathcal{ML} \), with at least three relational symbols and one propositional symbol, are exponentially more succinct than \( \mathcal{ML} \).
Results for S5

Succinctness Results for $[∀Γ]ML$, $[∃Γ]ML$, $[ϕ]ML$

In S5, the Logics $[∀Γ]ML$, $[∃Γ]ML$ and $[ϕ]ML$, with at least three relational symbols and one propositional symbol, are exponentially more succinct than $ML$.

Family of formulae for $ML$:

$$\varphi_0 = \langle g \rangle \overline{p}$$
$$\varphi_n = \langle g \rangle (\overline{p} \land ([b] (\overline{p} \lor \varphi_{n-1}) \land [r] (\overline{p} \lor \varphi_{n-1})))$$

and $[∀Γ]ML$:

$$\psi_0 = \varphi_0$$
$$\psi_n = \langle g \rangle (\overline{p} \land ([\forall\{b,r\}] (\overline{p} \lor \psi_{n-1}))) \equiv \varphi_n$$
Results for S5

Succinctness Results for $[\forall \Gamma] \mathcal{ML}$, $[\exists \Gamma] \mathcal{ML}$, $[\varphi] \mathcal{ML}$

In S5, the Logics $[\forall \Gamma] \mathcal{ML}$, $[\exists \Gamma] \mathcal{ML}$ and $[\varphi] \mathcal{ML}$, with at least three relational symbols and one propositional symbol, are exponentially more succinct than $\mathcal{ML}$.

Family of formulae for $\mathcal{ML}$:

$$\varphi'_0 = [g]p$$
$$\varphi'_n = [g] (p \lor ([b] (\neg p \lor \varphi'_{n-1}) \lor [r] (\neg p \lor \varphi'_{n-1}))))$$

and $[\exists \Gamma] \mathcal{ML}$:

$$\psi'_0 = \varphi'_0$$
$$\psi'_n = [g] (p \lor ([\exists \{b, r\}] (\neg p \lor \psi'_{n-1})))) \equiv \varphi'_n$$
Results for S5

Succinctness Results for $[\forall \Gamma] \mathcal{M}\mathcal{L}$, $[\exists \Gamma] \mathcal{M}\mathcal{L}$, $[\varphi] \mathcal{M}\mathcal{L}$

In S5, the Logics $[\forall \Gamma] \mathcal{M}\mathcal{L}$, $[\exists \Gamma] \mathcal{M}\mathcal{L}$ and $[\varphi] \mathcal{M}\mathcal{L}$, with at least three relational symbols and one propositional symbol, are exponentially more succinct than $\mathcal{M}\mathcal{L}$.

Family of formulae for $\mathcal{M}\mathcal{L}$:

$$\varphi_0^* = \langle g \rangle \overline{p}$$
$$\varphi_n^* = \langle g \rangle (\overline{p} \land \varphi_{n-1} \land \langle b \rangle (p \land \varphi_{n-1}) \land \langle r \rangle (p \land \varphi_{n-1} \land \langle b \rangle (p \land \varphi_{n-1}))$$
with $\varphi_{n-1} = \langle b \rangle (p \land \varphi_{*_{n-1}})$

and $[\varphi] \mathcal{M}\mathcal{L}$:

$$\psi_0^* = \varphi_0^*$$
$$\psi_n^* = \langle g \rangle (\overline{p} \land \langle \langle b \rangle (p \land \psi_{*_{n-1}}) \rangle \langle b \rangle p \langle r \rangle p)$$