Succinctness Results for some Extensions of Multimodal Logic over K and S5

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Previous Work

- T. French et al.: On the succinctness of some modal logics (2013)
- S. Figueira, D. Gorín: On the size of shortest modal descriptions (2010)

Comparing Logics

Logic

A logic $L = (\Phi, \models, \mathbb{M})$

Φ: non-empty set of formulae

$\models$: satisfaction relation

$\mathbb{M}$: non-empty class of models

$\mathcal{M} \models \varphi$ for some $\mathcal{M} \in \mathbb{M}, \varphi \in \Phi$

Comparing Logics

Logic

A logic $L = (\Phi, \models, \mathbb{M})$

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Expressivity

$L_1 = (\Phi_1, \models_1, \mathbb{M})$, $L_2 = (\Phi_2, \models_2, \mathbb{M})$.
L_2 is at least as expressive as L_1 ($L_1 \leq_{\mathbb{M}} L_2$) iff

$$\forall \varphi_1 \in \Phi_1 \, \exists \varphi_2 \in \Phi_2 \, \forall \mathcal{M} \in \mathbb{M} \, . \, \mathcal{M} \models_1 \varphi_1 \Leftrightarrow \mathcal{M} \models_2 \varphi_2$$

$\varphi_1 \equiv_{\mathbb{M}} \varphi_2$ (equivalence)

Succinctness

Succinctness

Let $L_1 = (\Phi_1, \models_1, \mathbb{M})$, $L_2 = (\Phi_2, \models_2, \mathbb{M})$, $L_1 \leq_{\mathbb{M}} L_2$ and F be a class of functions.

L_1 is F-succinct in L_2 on $\mathbb{M}$ ($L_1 \leq^F_{\mathbb{M}} L_2$) iff

$$\exists f \in F \, \forall \varphi_1 \in \Phi_1 \, \exists \varphi_1 \equiv_{\mathbb{M}} \varphi_2 \in \Phi_2 \, . \, |\varphi_2| \leq f(|\varphi_1|)$$

L_1 is exponentially more succinct than L_2 iff $L_1 \leq^F_{\mathbb{M}} L_2$ and $F \not\subseteq \mathit{SUBEXP}$ ($L_1 \not\leq^{\mathit{SUBEXP}}_{\mathbb{M}} L_2$).

Succinctness

Succinctness

Let $L_1 = (\Phi_1, \models_1, \mathbb{M})$, $L_2 = (\Phi_2, \models_2, \mathbb{M})$, $L_1 \leq_{\mathbb{M}} L_2$ and F be a class of functions.
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L_1 is exponentially more succinct than L_2 iff $L_1 \leq^F_{\mathbb{M}} L_2$ and $F \not\subseteq SUBEXP$ ($L_1 \not\leq^{SUBEXP}_{\mathbb{M}} L_2$).

- $L_1 \leq^F_{\mathbb{M}} L_2$ and $L_2 \leq^F_{\mathbb{M}} L_1$ (or $L_1 \not\leq^F_{\mathbb{M}} L_2$ and $L_2 \not\leq^F_{\mathbb{M}} L_1$) can be true at the same time.
- Succinctness is not necessarily transitive.

Multimodal Logic $\mathcal{ML}$

Syntax of $\mathcal{ML}$

$$\varphi ::= \top \mid \bot \mid p \mid \neg p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \langle r \rangle \varphi \mid [r]\varphi$$

with propositional symbols p and relational symbols $r \in R$.

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Negation normal form ($\overline{\varphi}$ negation for $\varphi \in \mathcal{ML}$):

- $\overline{\top} = \bot$
- $\overline{p} = \neg p$
- $\overline{\varphi_1 \wedge \varphi_2} = \overline{\varphi_1} \vee \overline{\varphi_2}$
- $\overline{\langle r \rangle \varphi} = [r]\overline{\varphi}$
- $\dots$

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Formula size:

- $|\top| = |\bot| = |p| = |\neg p| = 1$
- $|\varphi \vee \psi| = |\varphi \wedge \psi| = 1 + |\varphi| + |\psi|$
- $|\langle r \rangle \varphi| = |[r]\varphi| = 1 + |\varphi|$

$[\forall_\Gamma]\,\mathcal{ML}$

Syntax of $[\forall_\Gamma]\,\mathcal{ML}$

$$\varphi ::= \top \mid \bot \mid p \mid \neg p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \langle r\rangle\varphi \mid [r]\varphi \mid [\forall_\Gamma]\,\varphi \mid \langle\forall_\Gamma\rangle\varphi$$

with propositional symbols p and relational symbols $r \in R$ and sets of relational symbols $\Gamma \subseteq R$.

$\mathcal{ML}$-equivalence:

$$[\forall_\Gamma]\,\psi \equiv \bigwedge_{r\in\Gamma}[r]\psi$$

$[\exists_\Gamma]\,\mathcal{ML}$

Syntax of $[\exists_\Gamma]\,\mathcal{ML}$

$$\varphi ::= \top \mid \bot \mid p \mid \neg p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \langle r \rangle \varphi \mid [r]\varphi \mid [\exists_\Gamma]\,\varphi \mid \langle \exists_\Gamma \rangle \varphi$$

with propositional symbols p and relational symbols $r \in R$ and sets of relational symbols $\Gamma \subseteq R$.

$\mathcal{ML}$-equivalence:

$$[\exists_\Gamma]\,\psi \equiv \bigvee_{r \in \Gamma} [r]\psi$$

$[\varphi]\mathcal{ML}$

Syntax of $[\varphi]\mathcal{ML}$

$$\varphi ::= \top \mid \bot \mid p \mid \neg p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \langle r \rangle \varphi \mid [r]\varphi \mid [\varphi]\,\psi \mid \langle \varphi \rangle \psi$$

with propositional symbols p and relational symbols $r \in R$.

$\mathcal{ML}$-equivalence:

$$\begin{aligned}
[\varphi]p &\equiv \varphi \rightarrow p \\
[\varphi](\psi_1 \vee \psi_2) &\equiv [\varphi]\psi_1 \vee [\varphi]\psi_2 \\
[\varphi]\overline{\psi} &\equiv \varphi \rightarrow \overline{[\varphi]\psi} \\
[\varphi][r]\psi &\equiv \varphi \rightarrow [r][\varphi]\psi \\
[\varphi_1][\varphi_2]\psi &\equiv [\varphi_1 \wedge [\varphi_1]\varphi_2]\psi
\end{aligned}$$

Multimodal Logic $\mathcal{ML}$

Interpretation of formulae: Kripke Models $\mathcal{M} = (W, R, V)$

W: non-empty carrier set

R: set of binary relations $(\{r, b, g, \ldots\})$

V: Valuation

Successors: $succs_r(w) = \{v \mid (w, v) \in r\}$

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Semantics of $\mathcal{ML}$

$$
\begin{array}{ll}
\mathcal{M}, w \models \top & \\
\mathcal{M}, w \models p & \Leftrightarrow w \in V(p) \\
\mathcal{M}, w \models \neg p & \Leftrightarrow w \notin V(p) \\
\mathcal{M}, w \models \varphi \wedge \psi & \Leftrightarrow \mathcal{M}, w \models \varphi \text{ and } \mathcal{M}, w \models \psi \\
\mathcal{M}, w \models \varphi \vee \psi & \Leftrightarrow \mathcal{M}, w \models \varphi \text{ or } \mathcal{M}, w \models \psi \\
\mathcal{M}, w \models \langle r \rangle \varphi & \Leftrightarrow \mathcal{M}, v \models \varphi \text{ for some } v \in succs_r(w) \\
\mathcal{M}, w \models [r] \varphi & \Leftrightarrow \mathcal{M}, v \models \varphi \text{ for every } v \in succs_r(w)
\end{array}
$$

Separation and Description Problem

Separation

$\mathcal{M} = (W, R, V)$ model, $S, D \subseteq W$ non-empty sets

φ separates S and D in $\mathcal{M}$ iff $\forall s \in S \,.\, \mathcal{M}, s \models \varphi$ and $\forall d \in D \,.\, \mathcal{M}, d \not\models \varphi$

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Proof strategy for L_1 being exponentially more succinct than L_2:

1. Find a family of formulae $\varphi_n \in L_2$ with size exponential in n
2. Find a family of models $\mathcal{M}_n$, $S_n, D_n \subseteq W_n$ with φ_n being the smallest formulae separating S_n and D_n for all n.
3. If $\psi_n \in L_1$, $\psi_n \equiv_{\mathbb{M}} \varphi_n$ for all n, is of size linear in n then $L_1 \not\leq^{SUBEXP}_{\mathbb{M}} L_2$

Bisimulation Games

Model $\mathcal{M} = (W, R, V)$, $w, v \in W$.

Bisimulation Game

The game $\mathcal{G}(w, v)$ is played between two players (Spoiler, Duplicator).

Rules are:

- (p): Spoiler picks p with $w \in V(p)$ and $v \notin V(p)$ and wins.
- $(\overline{p})$: Spoiler picks p with $w \notin V(p)$ and $v \in V(p)$ and wins.
- $\langle r, w' \rangle$: Spoiler picks $r \in R$ and one $w' \in succs_r(w)$. Duplicator has to pick one $v' \in succs_r(v)$ or loses.
  Continuation in game $\mathcal{G}(w', v')$.
- $[r, v']$: Spoiler picks $r \in R$ and one $v' \in succs_r(v)$. Duplicator has to pick one $w' \in succs_r(w)$ or loses.
  Continuation in game $\mathcal{G}(w', v')$.

Winning Strategies

If Duplicator has a winning strategy in $\mathcal{G}(w, v)$ on the model $\mathcal{M}$ then

$$\forall \psi \in \mathcal{ML} \,.\, \mathcal{M}, w \vDash \psi \Leftrightarrow \mathcal{M}, v \vDash \psi$$

If Spoiler has a winning strategy in in $\mathcal{G}(w, v)$ then

$$\exists \varphi \in \mathcal{ML} \,.\, \mathcal{M}, w \vDash \varphi \text{ and } \mathcal{M}, v \nvDash \varphi$$

and $\min_{\varphi}(|\varphi|)$ gives a lower bound for the size of formulae describing w.

In the game $\mathcal{G}(w, w)$ Spoiler cannot have a winning strategy.

Uniform Strategy Trees

Winning strategies for Spoiler in bisimulation games:

Uniform Strategy Trees

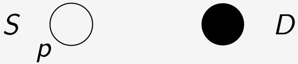
- nodes: $\langle r, S' \rangle$, $[r, D']$, (p), $(\overline{p})$, $(\vee)$, $(\wedge)$
  r relational symbol, $S, S', D, D' \subseteq W$
- essentially a syntax tree for a formula
- a formula separates S and D if its corresponding uniform strategy tree wins the game $\mathcal{G}(S, D)$

Uniform Strategy Trees

In a uniform strategy tree, winning $\mathcal{G}(S, D)$, with root x the following properties hold:

Winning Uniform Strategy Tree

If $x = (p)$ then $S \cap V(p) = S$ and $D \cap V(p) = \emptyset$



$\varphi = p$ separates S and D

Uniform Strategy Trees

In a uniform strategy tree, winning $\mathcal{G}(S, D)$, with root x the following properties hold:

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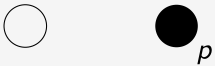
If $x = (\overline{p})$ then $S \cap V(p) = \emptyset$ and $D \cap V(p) = D$.



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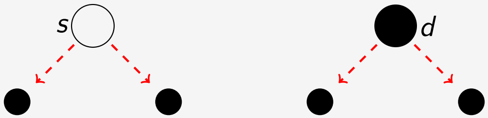
If $x = \langle r, S' \rangle$ then $S' \cap \mathrm{succs}_r(s) \neq \emptyset$ for every $s \in S$ and if $\mathrm{succs}_r(D) \neq \emptyset$ then there is an edge $x \rightarrow y$ and y is the root of a uniform strategy tree, winning the game $\mathcal{G}(S', \mathrm{succs}_r(D))$.



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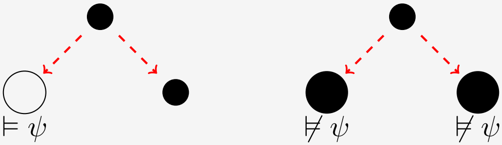

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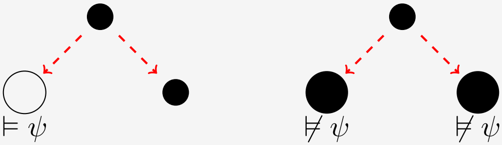

$\varphi = \langle r \rangle \psi$ separates S and D.

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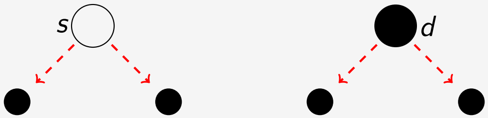
If $x = [r, D']$ then $D' \cap \text{succs}_r(d) \neq \emptyset$ for every $d \in D$ and if $\text{succs}_r(S) \neq \emptyset$ then there is an edge $x \to y$ and y is the root of a uniform strategy tree, winning the game $\mathcal{G}(\text{succs}_r(S), D')$.



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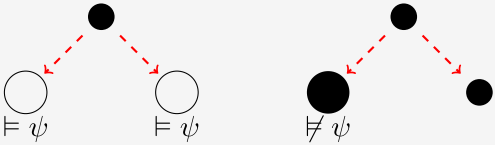

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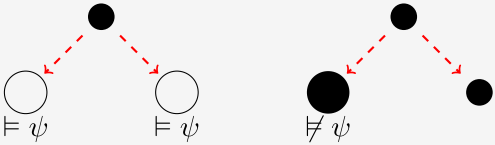


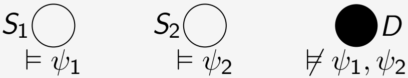
$\varphi = [r]\psi$ separates S and D.

Uniform Strategy Trees

In a uniform strategy tree, winning $\mathcal{G}(S, D)$, with root x the following properties hold:

Winning Uniform Strategy Tree

If $x = (\vee)$ then $S = S_1 \cup S_2$ and there are nodes y_1 and y_2 with edges $x \xrightarrow{S_i} y_i$ and y_i is the root of a uniform strategy tree, winning the game $\mathcal{G}(S_i, D)$, $i = 1, 2$.



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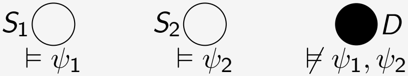


$\varphi = \psi_1 \vee \psi_2$ separates S and D.

Uniform Strategy Trees

In a uniform strategy tree, winning $\mathcal{G}(S, D)$, with root x the following properties hold:

Winning Uniform Strategy Tree

If $x = (\wedge)$ then $D = D_1 \cup D_2$ and there are nodes y_1 and y_2 with edges $x \xrightarrow{D_i} y_i$ and y_i is the root of a uniform strategy tree, winning the game $\mathcal{G}(S, D_i)$

$\varphi = \psi_1 \wedge \psi_2$ separates S and D.

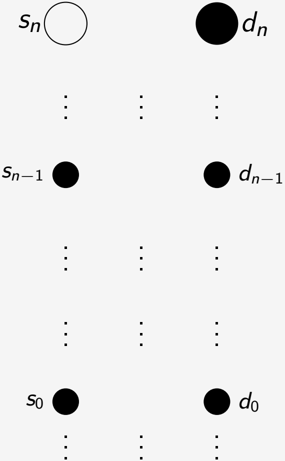
Proof Strategy

Proof strategy for L_1 being exponentially more succinct than $L_2 = \mathcal{ML}$:

1. Find a family of formulae $\varphi_n \in L_2$ with size exponential in n.
2. Find a family of models $(\mathcal{M}_n, S_n, D_n)$ with the uniform strategy tree of minimum size for the game $\mathcal{G}(S_n, D_n)$ being the syntax tree for φ_n.
3. If $\psi_n \in L_1$, $\psi_n \equiv_{\mathbb{M}} \varphi_n$ for all n, is of size linear in n then $L_1 \not\leq_{\mathbb{M}}^{SUBEXP} L_2$.

Models

Recursively defined models:



$\mathcal{G}(S_n, D_n)$

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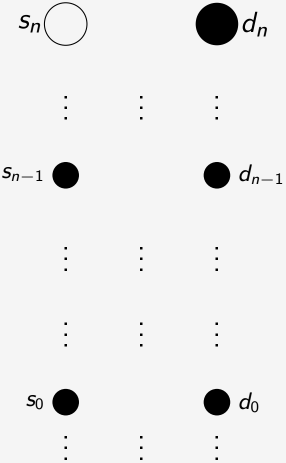

$\mathcal{G}(S_n, D_n)$

$\downarrow$ $\mathcal{UST}(\mathcal{M})$

Models

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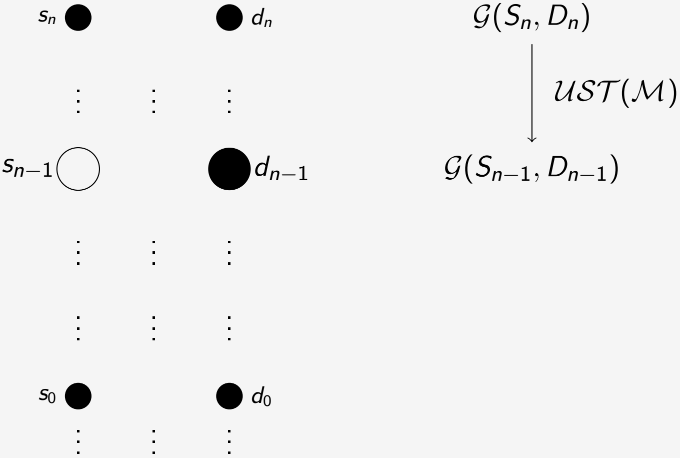

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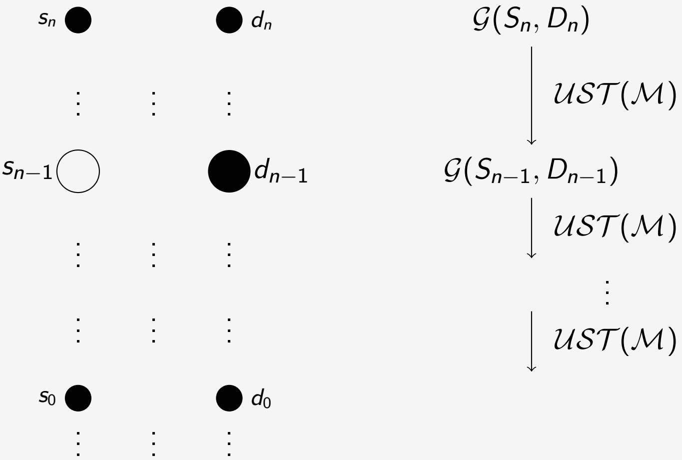

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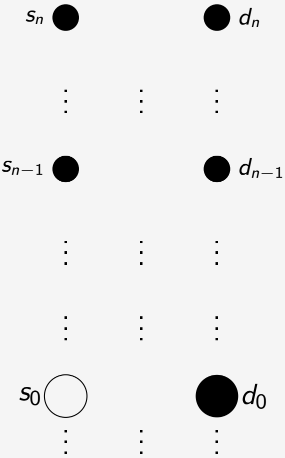

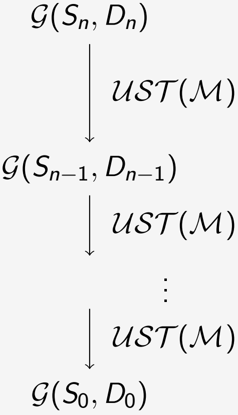

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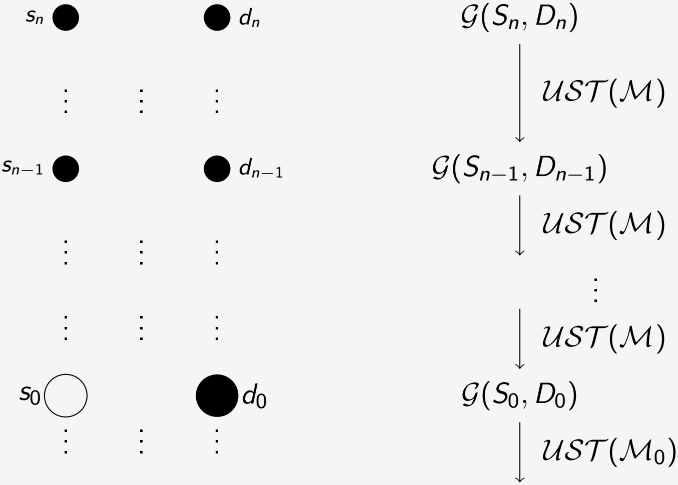

Minimality of Uniform Strategy Trees

Minimality as local property by construction

- Only a single (non-trivial) move is possible for Spoiler in $\mathcal{G}(S, D)$.
- It is easy to show that all alternative moves in $\mathcal{G}(S, D)$ are less than optimal in terms of size

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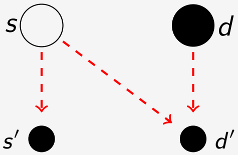
Successors in **K**

In a game $\mathcal{G}(s, d)$ with $d' \in succs_r(s)$ and $d' \in succs_r(d)$ Spoiler loses with the move $\langle r, d' \rangle$ (Analogously with a $[r]$-move).



$[\forall_\Gamma]\,\mathcal{ML}$ over **K**

$$\left[\forall_{\{r,b\}}\right] \psi \equiv_{\mathbf{K}} [r]\psi \wedge [b]\psi$$

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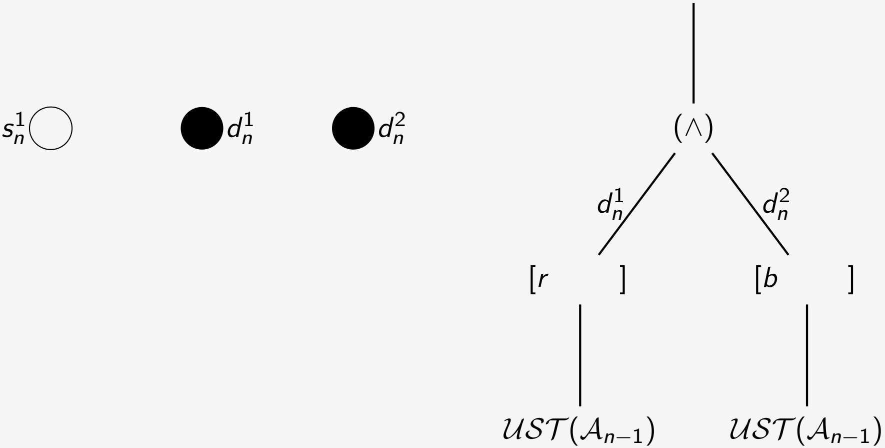

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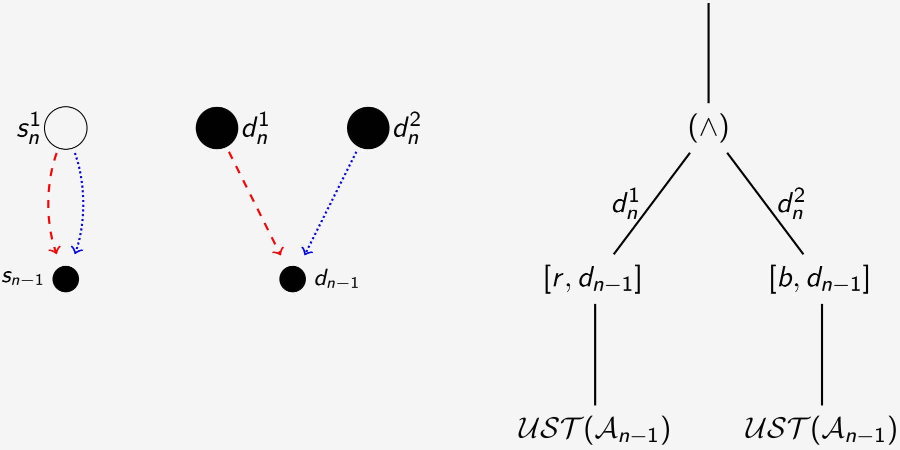

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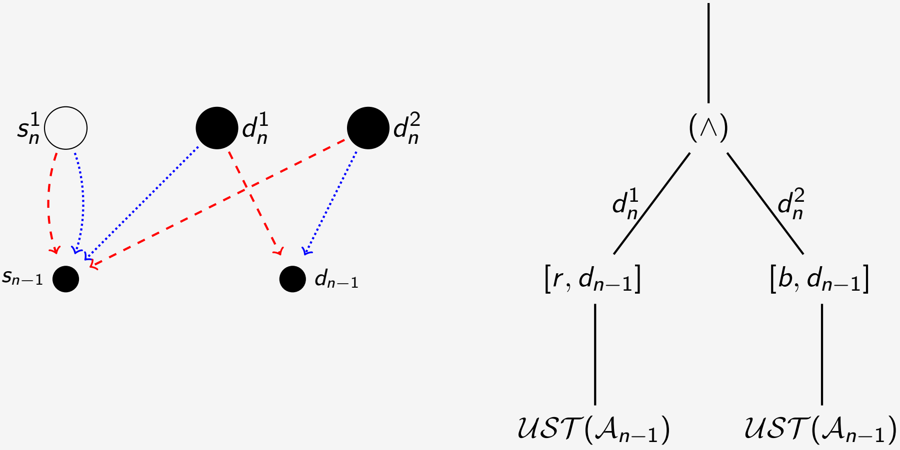

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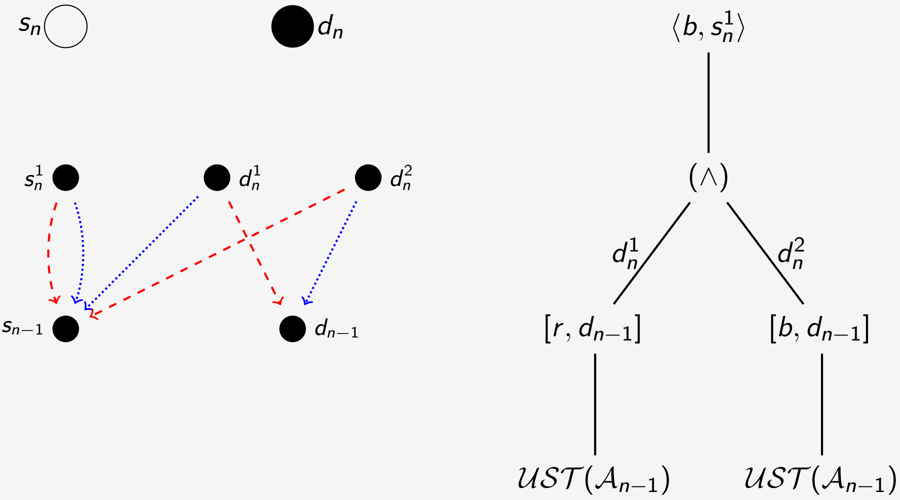

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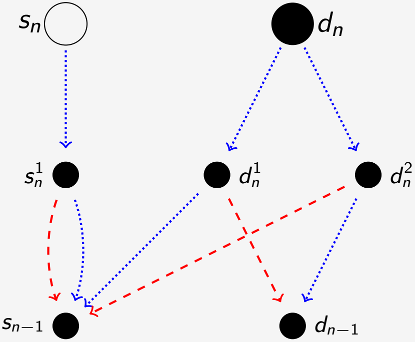
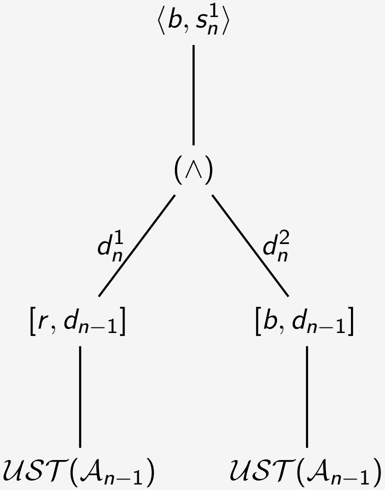
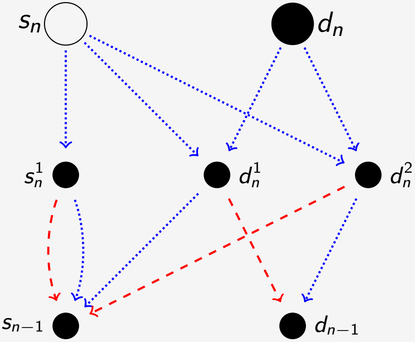
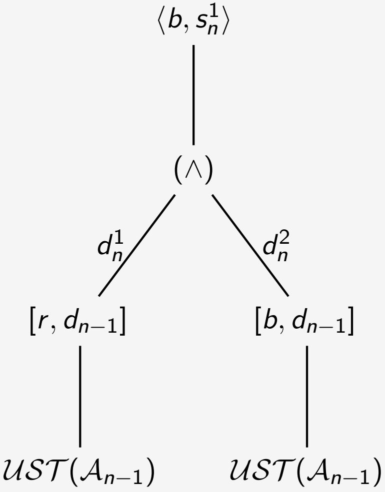




$[\forall_\Gamma]\mathcal{ML}$ over $\mathbf{K}$





Results for **K**

Previous Results

- French, T. et al.: Proof that $[\forall_\Gamma]\mathcal{ML}$ and $[\exists_\Gamma]\mathcal{ML}$ are exponentially more succinct than $\mathcal{ML}$ over **K** based on models with 2 relational symbols and 1 propositional symbol.
- Lutz, C.: Proof that $[\varphi]\mathcal{ML}$ is exponentially more succinct than $\mathcal{ML}$ over **K** based on models with 2 relational symbols

My Results

$[\forall_\Gamma]\mathcal{ML}$, $[\exists_\Gamma]\mathcal{ML}$ and $[\varphi]\mathcal{ML}$ are exponentially more succinct than $\mathcal{ML}$ over **K** based on models with only 2 relational symbols

S5

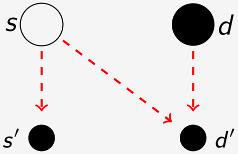
Handling Reflexivity, Symmetry, Transitivity

S5

Handling Reflexivity, Symmetry, Transitivity

Cliques

Let $s \in S$, $d \in D$ in some game $\mathcal{G}(S, D)$ over some **S5**-model. If s and d are members of the same r-clique for some relational symbol r then there is no winning strategy for Spoiler beginning with a $\langle r \rangle$- or $[r]$-move.



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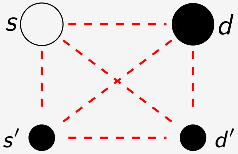

S5

Handling Reflexivity, Symmetry, Transitivity

Subgames

Let t be the uniform strategy tree of minimum size for $\mathcal{G}(S, D)$ over some model and t' the uniform strategy tree of minimum size for $\mathcal{G}(S \cup S', D \cup D')$. Assuming the games can be won by Spoiler then $|t| \leq |t'|$.

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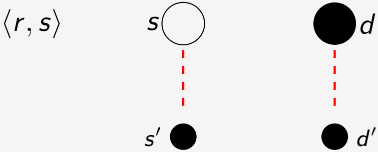

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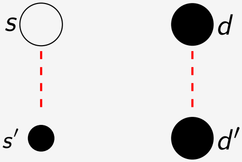
$\langle r, s \rangle$



S5

Handling Reflexivity, Symmetry, Transitivity

r-equivalent Games

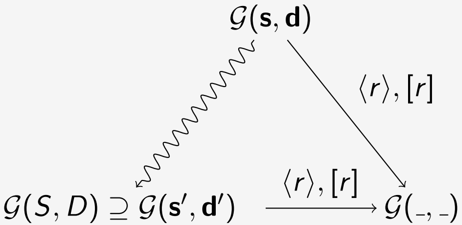
Let $\mathcal{G}(s, d)$ and $\mathcal{G}(s', d')$ be games over some **S5**-model and $s' \in succs_r(s)$, $d' \in succs_r(d)$. Then every $\langle r \rangle$- or $[r]$-move applied to both games will result in the same game. We call two such games r-equivalent and any strategy starting with a $\langle r \rangle$- or $[r]$-move is applicable to both games.

S5

Handling Reflexivity, Symmetry, Transitivity

Strategies on r-equivalent Games

Let $\mathcal{G}(s, d)$ be a game over some **S5**-model, t a uniform strategy tree for this game, $\mathcal{G}(S, D)$ the game after one or more moves in t and $\mathcal{G}(s', d')$ a subgame of $\mathcal{G}(S, D)$. If the uniform strategy tree of minimum size for $\mathcal{G}(s', d')$ begins with a $\langle r \rangle$- or $[r]$-move and $\mathcal{G}(s', d')$ and $\mathcal{G}(s, d)$ are r-equivalent then t is not a uniform strategy tree of minimum size for $\mathcal{G}(s, d)$.

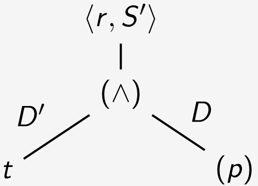


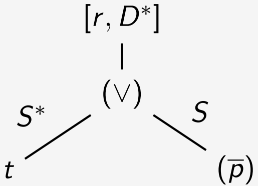
S5

Handling Reflexivity, Symmetry, Transitivity

Reflexivity and propositional Symbols

Let $\langle r, S' \rangle$ (or $[r, D^*]$) be the first move in the minimum uniform strategy tree for the game $\mathcal{G}(S, D)$ in the **S5**-model $(\mathcal{M}, S, D)$, and let $\mathcal{G}(S', D \cup D')$ (or $\mathcal{G}(S \cup S^*, D^*)$) be the game we would have to play after this move ($S \subseteq succs_r(S)$, $D \subseteq succs_r(D)$, reflexivity of **S5**). If $S' \models p$ and $D \models \overline{p}$ (or $D^* \models p$ and $S \models \overline{p}$) then we can use the strategies shown below where t is the minimum uniform strategy tree for the game $\mathcal{G}(S', D')$ (or $\mathcal{G}(S^*, D^*)$).





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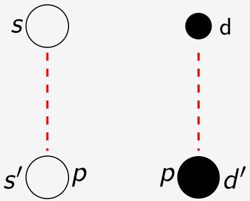

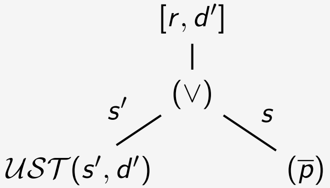

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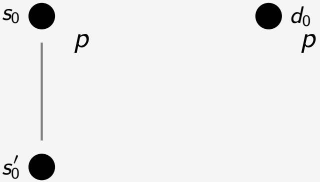
Size

Any uniform strategy tree of this kind is at most three time larger than the uniform strategy tree of minimum size.

Recursive **S5**-Model



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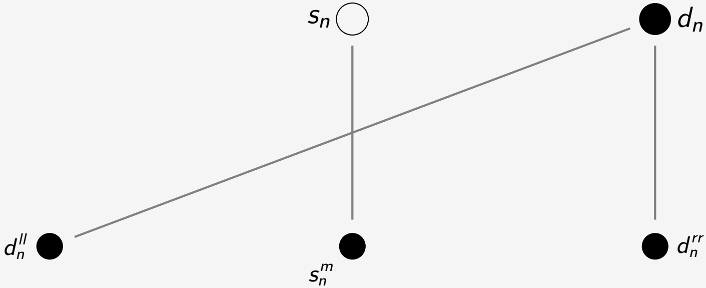

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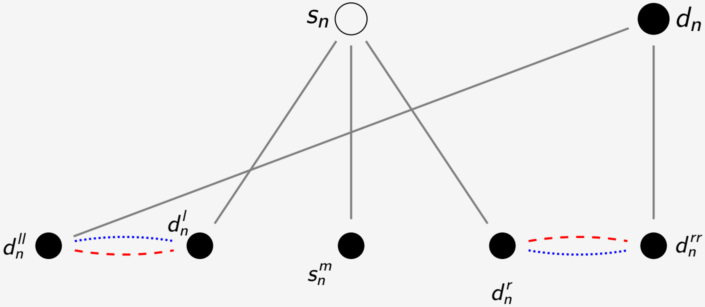

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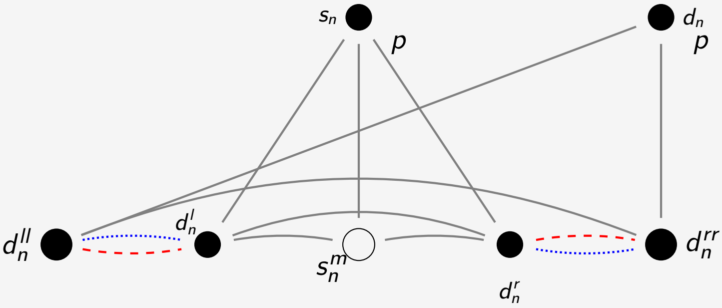

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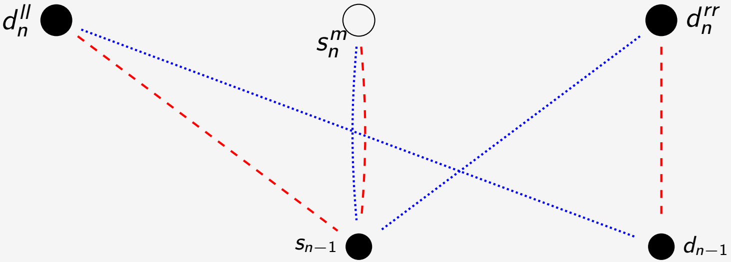

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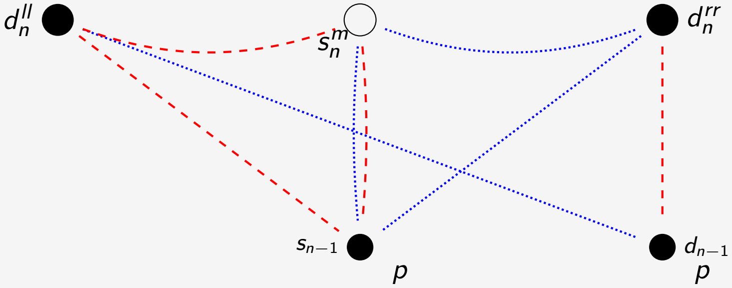

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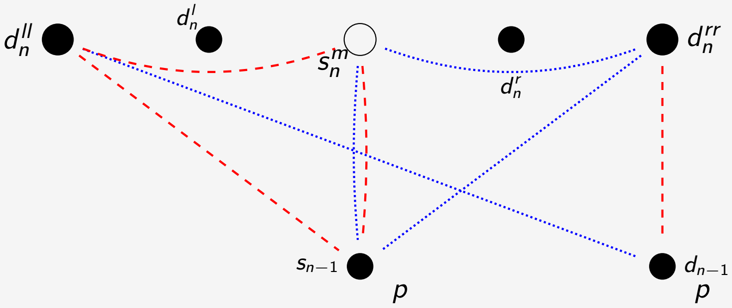

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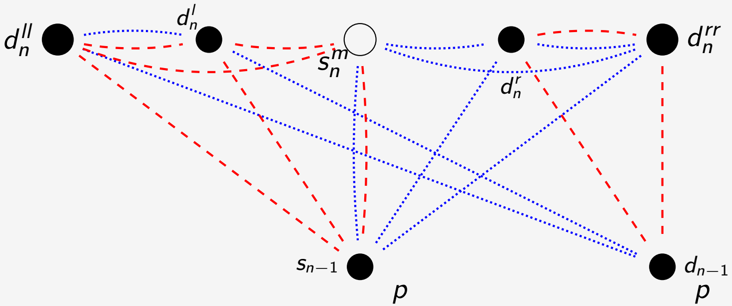

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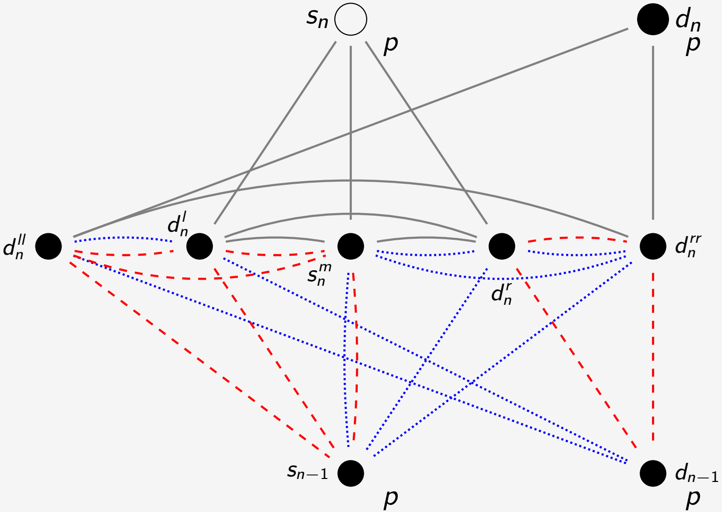

Results for **S5**

Succinctness Results for $[\forall_\Gamma]\,\mathcal{ML}$, $[\exists_\Gamma]\,\mathcal{ML}$, $[\varphi]\,\mathcal{ML}$

In **S5**, the Logics $[\forall_\Gamma]\,\mathcal{ML}$, $[\exists_\Gamma]\,\mathcal{ML}$ and $[\varphi]\,\mathcal{ML}$, with at least three relational symbols and one propositional symbol, are exponentially more succinct than $\mathcal{ML}$.

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Family of formulae for $\mathcal{ML}$:

$$\varphi_0 = \langle g\rangle\overline{p}$$
$$\varphi_n = \langle g\rangle\left(\overline{p} \wedge \left([b]\left(\overline{p} \vee \varphi_{n-1}\right) \wedge [r]\left(\overline{p} \vee \varphi_{n-1}\right)\right)\right)$$

and $[\forall_\Gamma]\,\mathcal{ML}$:

$$\psi_0 = \varphi_0$$
$$\psi_n = \langle g\rangle\left(\overline{p} \wedge \left(\left[\forall_{\{b,r\}}\right]\left(\overline{p} \vee \psi_{n-1}\right)\right)\right) \equiv \varphi_n$$

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Family of formulae for $\mathcal{ML}$:

$$\varphi'_0 = [g]p$$
$$\varphi'_n = [g]\left(p \vee \left([b]\left(\overline{p} \vee \varphi'_{n-1}\right) \vee [r]\left(\overline{p} \vee \varphi'_{n-1}\right)\right)\right)$$

and $[\exists_\Gamma]\,\mathcal{ML}$:

$$\psi'_0 = \varphi'_0$$
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with $\phi_{n-1} = \langle b\rangle\left(p \wedge \varphi_{n-1}^*\right)$

and $[\varphi]\,\mathcal{ML}$:

$$\psi_0^* = \varphi_0^*$$
$$\psi_n^* = \langle g\rangle\left(\overline{p} \wedge \langle\langle\langle b\rangle\left(p \wedge \psi_{n-1}^*\right)\rangle\langle b\rangle p\rangle\langle r\rangle p\right)$$