Unifying Notions of Feedback

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Grand unification:

Guarded Traced Categories
Why Semantics?
A Well-known Scenario

How do we know that automata

\[ q_1 /\text{one.osf} \]
\[ q_2 /\text{two.osf} \]
\[ q_1 /\text{three.osf} \]

are equivalent?

Because

\[ J_{q_1 /\text{one.osf}} K_{q_2 /\text{two.osf}} = J_{q_1 /\text{three.osf}} K_{q_2 /\text{two.osf}} \]

semantic brackets
dinaturality identity

\[ /\text{two.osf} /\text{one.osf} /\text{nine.osf} \]
A Well-known Scenario

How do we know that automata

\[
\begin{align*}
q_1 & \xrightarrow{a} q_2 \\
q_2 & \xrightarrow{b} q_1
\end{align*}
\]

\[
\begin{align*}
q' & \xrightarrow{a} q' \\
q' & \xrightarrow{b} q_2 \\
q_2 & \xrightarrow{a} q_3
\end{align*}
\]

are equivalent?

Because \([q_1] = (ab)^*\), \([q'_1] = a(ba)^*b + 1\) and: \((ab)^* = a(ba)^*b + 1\)
Semantics in Computer Science

Semantics
Effectiveness (of presentation, decidability)
Semantics in Computer Science

- Effectiveness (of presentation, decidability)
- Efficiency (complexity of computations, efficient data structures)
- Semantics
Semantics in Computer Science

- Effectiveness (of presentation, decidability)
- Efficiency (complexity of computations, efficient data structures)
- Algorithms, implementations, concrete programming languages, compilers, interpreters, simulators, proof assistants, ...
- Semantics
Bouncing ball is a simple Newtonian system specified by differential equation $\ddot{h} = -g$ ($g \approx 9.8$) whose solution is

$$h(t) = h_0 + v_0 t - \frac{gt^2}{2}$$

with initial values:
- $v_0 = 0$, $h_0 \neq 0$ (peak height)
- $h_0 = 0$, $v_0 \neq 0$ (zero height)

Features:
- deterministic
- hybrid: the velocity changes discretely at the bottom $v \leftrightarrow -cv$, but it changes continuously in the meanwhile
- Zeno behaviour: the state of rest is only reachable in the limit
Hybrid Automata

The following hybrid automata "A" and "B" capture the bouncing ball behaviour:

These automata are not equivalent under standard semantics, because

\[ [A] = (\text{[Diagram A]}), \text{ but } [B] = (\text{[Diagram B]}) \]
Impact of Semantics

- Knowing the semantics of automata, we can minimize them, transform, prove equivalence.

We can transfer knowledge between different models, as the theories of nondeterministic, probabilistic, push-down, etc., etc, automata have a lot in common.

We can optimize programs, e.g.

```plaintext
while b do ...
if b then ...
else /* dead code */
done
while b do ...
```

and verify them (since, we know what they mean!)
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and verify them (since, we know what they mean!)
- Principled semantic foundations improve design of languages, software and hardware systems (types, compositionality, Curry-Howard correspondence, etc)
Why Category Theory?
[…] Kategorientheorie – ein sehr komplexes Gebiet mit tiefen mathematischen Wurzeln, und mit relativ wenigen Experten auf diesem Gebiet

— Anonymous referee
A category $\mathbf{C}$ consists of wires (=objects) $|\mathbf{C}|$ and boxes (=morphisms) $\mathbf{C}(A, B)$ with $A, B \in |\mathbf{C}|$, which can be combined:
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- A category is **monoidal** if morphisms can be tensored:

- A category is **symmetric** if wires can be crossed:

Boxes are intuitively: programs, processes, components, automata; wires are types, communication channels.
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Traced categories additionally allow feedback loops, called traces:

Traced categories provide a unifying framework for

- Iteration (roughly: while-loops)
- Recursion (roughly: fixpoint combinators of $\lambda$-calculus)
- Knot theory
- Operator theory (e.g. traces model quantum measurements)
Why Guarded Traces?
Consider again the regular expressions \((ab)^*\) and \(a(ba)^*b + 1\). Here, Kleene star \(e^*\) is the unique fixpoint of

\[ x \mapsto ex + 1 \]

Equation \((ab)^* = a(ba)^*b + 1\) is true, because \(a(ba)^*b + 1\) is a fixpoint of the same map.
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\begin{align*}
a(ba)^*b + 1 &= a((ba)(ba)^* + 1)b + 1 \\
&= a(ba)(ba)^*b + a1b + 1 \\
&= (ab)a(ba)^*b + ab + 1
\end{align*}
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$$x \mapsto ex + 1$$

Equation $(ab)^* = a(ba)^*b + 1$ is true, because $a(ba)^*b + 1$ is a fixpoint of the same map:

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This only works because the map $x \mapsto abx + 1$ is guarded.

Contrastingly, the map $x \mapsto (a + 1)x + 1$ is unguarded and has infinitely many fixpoints.
Automata can be organized into a traced symmetric category, e.g. \((ab)^*\) is expressed as

- Guarded traces are simpler and better behaved
- They often have special useful properties, e.g. uniqueness
- Sometimes, there is no candidate for a total trace operator
Guarded (monoidal) categories are defined in terms of decorated diagrams, subject to the axioms:

Only the paths from left bars to right bars are considered guarded.
A guarded category is \textit{guarded traced} if it supports \textit{guarded traces}:

Intuitively, we are allowed to close the loop because it only runs through the interface of $f$ indicated by the black bars.
With guarded categories we mediate between symmetric monoidal categories and traced symmetric ones with

- **vacuous guardedness**: there are no guarded paths at all (symmetric monoidal categories)
- **total guardedness**: every path is guarded (traced symmetric categories)
- motivating examples (such as automata) occur properly between these two extremes
Guardedness in hybrid semantics means progressiveness, i.e. consuming non-zero time.

The behaviour of bouncing ball is described in terms of velocity and height \( \langle v, h \rangle \in R^2 \):

In contrast to general traces, guarded ones are computed as least fixpoints.
Further Examples

- Complete metric spaces: guardedness = contractiveness, fixpoints are computed via Banach’s fixpoint theorem.
- Infinite-dimensional Hilbert spaces under vacuous guardedness, traces = traces of bounded linear operators.
- Infinite trace semantics: guardedness by actions, traces are greatest fixpoints, e.g. $a^* + a^\omega$ is the canonical fixpoint of $x \mapsto ax + 1$.

and not just $a^*$
Q: Can we make sense of the following diagram?
Q: Can we make sense of the following diagram?
**Completeness Problem**

**Q:** Can we make sense of the following diagram?

**A:** Yes, because we can redraw it as

![Diagram](image-url)
Q: What about this one:
**Completeness Problem**

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![Diagram with nodes f, g, and h connected by lines, indicating a harmless unguarded path.]

**Open Problem:**
resolve the discrepancy between geometric guardedness and structural guardedness.
**Completeness Problem**

**Q:** What about this one:

![Diagram](image)

Morally, this diagram should be OK, but we cannot rearrange it so as to be able to derive it from the axioms of guardedness.
Completeness Problem

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Morally, this diagram should be OK, but we cannot rearrange it so as to be able to derive it from the axioms of guardedness.

Open Problem: resolve the discrepancy between geometric guardedness and structural guardedness.
Conclusions

- Guarded traced categories are abound in semantics and beyond
- Abstract notion of guardedness unifies various principles behind partiality of feedback: delay, progress, contractivity, etc
- Guarded categories come together with a semantically justified unifying diagrammatic metalanguage, suitable for visual programming and modelling (e.g. hybrid systems)
- Abstract guardedness helps to identify well-behaved notions of feedback (unique, least, greatest), which come together with the corresponding reasoning principles (co-induction, Scott induction)