Reiterman's Theorem on Finite Algebras for a Monad

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Profinite equations are an indispensable tool for the algebraic classification of formal languages. Reiterman's theorem states that they precisely specify pseudovarieties, i.e. classes of finite algebras closed under finite products, subalgebras and quotients. In this paper, Reiterman's theorem is generalized to finite Eilenberg-Moore algebras for a monad $\mathbf{T}$ on a category $\mathscr{D}$: we prove that a class of finite $\mathbf{T}$-algebras is a pseudovariety iff it is presentable by profinite equations. As a key technical tool, we introduce the concept of a profinite monad $\widehat{\mathbf{T}}$ associated to the monad $\mathbf{T}$, which gives a categorical view of the construction of the space of profinite terms.

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1 INTRODUCTION

One of the main principles of both mathematics and computer science is the specification of structures in terms of equational properties. The first systematic study of equations as mathematical objects was pursued by Birkhoff [6] who proved that a class of algebraic structures over a finitary signature Σ can be specified by equations between Σ-terms if and only if it is closed under quotient algebras (a.k.a. homomorphic images), subalgebras, and products. This fundamental result, known as the *HSP theorem*, lays the ground for universal algebra and has been extended and generalized in many directions over the past 80 years, including categorical approaches via Lawvere theories [4, 12] and monads [15].

While Birkhoff's seminal work and its categorifications are concerned with general algebraic structures, in many computer science applications the focus is on *finite* algebras. For instance, in automata theory, regular languages (i.e. the behaviors of classical finite automata) can be characterized as precisely the languages recognizable by finite monoids. This algebraic point of view leads to important insights, including decidability results. As a prime example, Schützenberger's theorem [21] asserts that star-free regular languages correspond to *aperiodic* finite monoids, i.e. monoids where the unique idempotent power x^ω of any element x satisfies $x^\omega = x \cdot x^\omega$. As an immediate application, one obtains the decidability of star-freeness. However, the identity $x^\omega = x \cdot x^\omega$ is not an equation in Birkhoff's sense since the operation $(-)^\omega$ is not a part of the signature of monoids. Instead, it is an instance of a *profinite equation*, a topological generalization of Birkhoff's concept introduced by Reiterman [19]. (Originally, Reiterman worked with the equivalent concept of an

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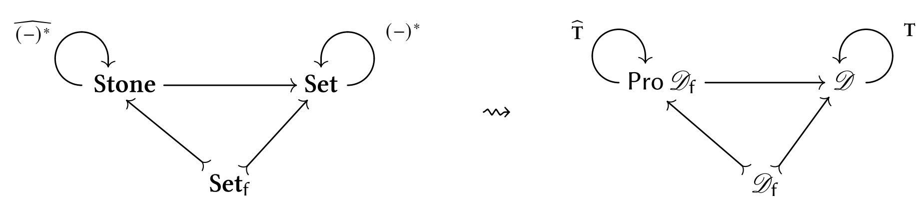
implicit equation, cf. Section 5.) Given a set X of variables and $x \in X$, the expression x^ω can be interpreted as an element of the Stone space $\widehat{X^*}$ of *profinite words*, constructed as the cofiltered limit of all finite quotient monoids of the free monoid X^*. Analogously, over general signatures Σ one can form the Stone space of *profinite Σ-terms.* Reiterman proved that a class of finite Σ-algebras can be specified by profinite equations (i.e. pairs of profinite terms) if and only if it is closed under quotient algebras, subalgebras, and finite products. This result establishes a finite analogue of Birkhoff's HSP theorem.

In this paper, we develop a categorical approach to Reiterman's theorem and the theory of profinite equations. The idea is to replace monoids (or general algebras over a signature) by Eilenberg-Moore algebras for a monad $\mathbf{T}$ on an arbitrary base category $\mathscr{D}$. As an important technical device, we introduce a categorical abstraction of the space of profinite words. To this end, we consider a full subcategory $\mathscr{D}_\mathsf{f}$ of $\mathscr{D}$ of "finite" objects and form the category $\operatorname{Pro}\mathscr{D}_\mathsf{f}$, the free completion of $\mathscr{D}_\mathsf{f}$ under cofiltered limits. We then show that the monad $\mathbf{T}$ naturally induces a monad $\widehat{\mathbf{T}}$ on $\operatorname{Pro}\mathscr{D}_\mathsf{f}$, called the *profinite monad* of $\mathbf{T}$, whose free algebras $\widehat{\mathbf{T}}X$ serve as domains for profinite equations. For example, for $\mathscr{D} = \mathbf{Set}$ and the full subcategory $\mathbf{Set}_\mathsf{f}$ of finite sets, we get $\operatorname{Pro}\mathbf{Set}_\mathsf{f} = \mathbf{Stone}$, the category of Stone spaces. Moreover, if $\mathbf{T}X = X^*$ is the finite-word monad (whose algebras are precisely monoids), then $\widehat{\mathbf{T}}$ is the monad of profinite words on $\mathbf{Stone}$; that is, $\widehat{\mathbf{T}}$ associates to each finite Stone space (i.e. a finite set with the discrete topology) X the space $\widehat{X^*}$ of profinite words on X. Our overall approach can thus be summarized by the following diagram, where the skewed functors are inclusions and the horizontal ones are forgetful functors.



It turns out that many familiar properties of the space of profinite words can be developed at the abstract level of profinite monads and their algebras. Our main result is the

Generalized Reiterman Theorem. A class of finite $\mathbf{T}$-algebras is presentable by profinite equations if and only if it is closed under quotient algebras, subalgebras, and finite products.

Here, *profinite equations* are modelled categorically as finite quotients $e\colon \widehat{T}X \twoheadrightarrow E$ of the object $\widehat{T}X$ of generalized profinite terms. If the category $\mathscr{D}$ is $\mathbf{Set}$ or, more generally, a category of first-order structures, we will see that this abstract concept of an equation is equivalent to the familiar one: $\widehat{T}X$ is a topological space and quotients e as above can be identified with sets of pairs (s, t) of profinite terms $s, t \in \widehat{T}X$. Thus, our categorical results instantiate to the original Reiterman theorem [19] ($\mathscr{D} = \mathbf{Set}$), but also to its versions for ordered algebras ($\mathscr{D} = \mathbf{Pos}$) and for first-order structures due to Pin and Weil [17].

Our proof of the Generalized Reiterman Theorem is purely categorical and relies on general properties of (codensity) monads, free completions and locally finitely copresentable categories. It does not employ any topological methods, as opposed to all known proofs of Reiterman's theorem and its variants. The insight that topological reasoning can be completely avoided in the profinite world is quite surprising, and we consider it as one of the main contributions of our paper.

Related work. This paper is the full version of an extended abstract [8] presented at FoSSaCS 2016. Besides providing complete proofs of all results, the presentation is significantly more general than in *op. cit.*: there we restricted ourselves to base categories $\mathscr{D}$ which are varieties of (possibly ordered) algebras, and the development of the profinite monad and its properties used results from topology. In contrast, the present paper works with general categories $\mathscr{D}$ and develops all required profinite concepts in full categorical abstraction.

An important application of the Generalized Reiterman Theorem and the profinite monad can be found in algebraic language theory [24]: we showed that given a category $\mathscr{C}$ dually equivalent to Pro $\mathscr{D}_\mathrm{f}$, the concept of a profinite equational class of finite $\mathbf{T}$-algebras dualizes to the concept of a *variety of* $\mathbf{T}$*-recognizable languages in* $\mathscr{C}$. For instance, for $\mathscr{D} = \mathbf{Set}$ and Pro $\mathscr{D}_\mathrm{f} = \mathbf{Stone}$, the classical Stone duality yields the category $\mathscr{C} = \mathbf{BA}$ of boolean algebras, and for the monad $\mathbf{T}X = X^*$ on $\mathbf{Set}$ the dual correspondence gives Eilenberg's fundamental *variety theorem* for regular languages [9]. Using our duality-theoretic approach we established a categorical generalization of Eilenberg's theorem and showed that it instantiates to more than a dozen Eilenberg-type results known in the literature, along with a number of new correspondence results.

Recently, an abstract approach to HSP-type theorems [16] has been developed that not only provides a common roof over Birkhoff's and Reiterman's theorem, but also applies to classes of algebras with additional underlying structure, such as ordered, quantitative, or nominal algebras. The characterization of pseudovarieties in terms of pseudoeuqations given in Proposition 3.8 is a special case of the HSP theorem in *op. cit.*

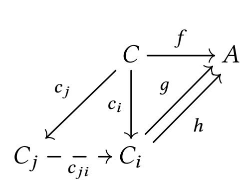
2 PROFINITE COMPLETION

In this preliminary section, we review the profinite completion (commonly known as pro-completion) of a category and describe it for the category Σ-Str of structures over a first-order signature Σ.

Remark 2.1. Recall that a category is *cofiltered* if every finite subcategory has a cone in it. For example, every cochain (i.e. a poset dual to an ordinal number) is cofiltered. A *cofiltered limit* is a limit of a diagram with a small cofiltered diagram scheme. A functor is *cofinitary* if it preserves cofiltered limits. An object A of a category $\mathscr{C}$ is called *finitely copresentable* if the functor $\mathscr{C}(-, A): \mathscr{C} \to \mathbf{Set}^{\mathrm{op}}$ is cofinitary. The latter means that for every limit cone $c_i: C \to C_i$ ($i \in \mathcal{I}$) of a cofiltered diagram,

(1) each morphism $f: C \to A$ factorizes through some $c_i: C \to C_i$ as $f = g \cdot c_i$, and

(2) the morphism $g: C_i \to A$ is *essentially unique*, i.e. given another factorization $f = h \cdot c_i$, there is a connecting morphism $c_{ji}: C_j \to C_i$ with $g \cdot c_{ji} = h \cdot c_{ji}$:



The dual concept is that of a *filtered colimit*.

Notation 2.2. (1) The free completion of a category $\mathscr{C}$ under cofiltered limits, i.e. the *pro-completion*, is denoted by

$$\mathrm{Pro}\,\mathscr{C}.$$

This is a category with cofiltered limits together with a full embedding $E: \mathscr{C} \rightarrowtail \mathrm{Pro}\,\mathscr{C}$ satisfying the following universal property:

(1a) Every functor $F\colon \mathscr{C} \to \mathscr{K}$ into a category $\mathscr{K}$ with cofiltered limits admits a cofinitary extension $\overline{F}\colon \operatorname{Pro}\mathscr{C} \to \mathscr{K}$, i.e. the triangle below commutes:

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{E} & \operatorname{Pro}\mathscr{C} \\ & \searrow^{F} & \downarrow{\scriptstyle \overline{F}} \\ & & \mathscr{K} \end{array}$$

(1b) The functor $\overline{F}$ is *essentially unique*, i.e. for every cofinitary extension G of F there exists a unique natural isomorphism $i\colon \overline{F} \xrightarrow{\cong} G$ with $iE = \mathsf{id}_F$.

More precisely, the full embedding E *is* the pro-completion, but we will often simply refer to $\operatorname{Pro}\mathscr{C}$ as the pro-completion instead.

(2) Dually, the free completion of $\mathscr{C}$ under filtered colimits, i.e. the *ind-completion*, is denoted by

$$\operatorname{Ind}\mathscr{C}.$$

Some standard results on ind- and pro-completions can be found in the Appendix.

Example 2.3. (1) Let $\mathbf{Set}_\mathrm{f}$ be the category of finite sets and functions. Its pro-completion is the category

$$\operatorname{Pro}\mathbf{Set}_\mathrm{f} = \mathbf{Stone}$$

of *Stone spaces*, i.e. compact topological spaces in which distinct elements can be separated by clopen subsets. Morphisms are the continuous functions. The embedding $\mathbf{Set}_\mathrm{f} \rightarrowtail \mathbf{Stone}$ identifies finite sets with finite discrete spaces. This is a consequence of the Stone duality [10] between **Stone** and the category **BA** of boolean algebras, and its restriction to finite sets and finite Boolean algebras. In fact, since **BA** is a finitary variety, it is the ind-completion of its full subcategory $\mathbf{BA}_\mathrm{f}$ of finitely presentable objects, which are precisely the finite Boolean algebras. Therefore

$$\operatorname{Pro}\mathbf{Set}_\mathrm{f} = (\operatorname{Ind}\mathbf{Set}_\mathrm{f}^{\mathrm{op}})^{\mathrm{op}} \cong (\operatorname{Ind}\mathbf{BA}_\mathrm{f})^{\mathrm{op}} \cong \mathbf{BA}^{\mathrm{op}} \cong \mathbf{Stone}.$$

(2) For the category of finite posets and monotone functions, denoted by $\mathbf{Pos}_\mathrm{f}$, we obtain the category

$$\operatorname{Pro}\mathbf{Pos}_\mathrm{f} = \mathbf{Priest}$$

of *Priestley spaces*, i.e. ordered Stone spaces such that any two distinct elements can be separated by clopen upper sets. Morphisms in **Priest** are continuous monotone functions. This follows from the Priestley duality [18] between **Priest** and bounded distributive lattices. The argument is analogous to item (1): finite, equivalently finitely presentable, distributive lattices dualize to finite posets with discrete topology.

Notation 2.4 (First-order structures)**.** We will often work with the category

$$\Sigma\text{-}\mathbf{Str}$$

of Σ-structures and Σ-homomorphisms for a first-order many-sorted signature Σ. Given a set $\mathcal{S}$ of sorts, an $\mathcal{S}$*-sorted signature* Σ consists of (1) operation symbols $\sigma\colon s_1, \dots, s_n \to s$ where $n \in \mathbb{N}$, the sorts s_i form the domain of σ and s is its codomain, and (2) relation symbols $r\colon s_1, \dots, s_m$ where $m \in \mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. A Σ*-structure* is an $\mathcal{S}$-sorted set

$$A = (A^s)_{s \in \mathcal{S}} \quad \text{in} \quad \mathbf{Set}^{\mathcal{S}}$$

with (1) an operation $\sigma_A\colon A^{s_1} \times \cdots \times A^{s_n} \to A^s$ for every operation symbol $\sigma\colon s_1, \dots, s_n \to s$, and (2) a relation $r_A \subseteq A^{s_1} \times \dots A^{s_n}$ for every relation symbol $r\colon s_1, \dots, s_n$. A Σ*-homomorphism* is an

$\mathcal{S}$-sorted function $f: A \to B$ which preserves operations and relations in the usual sense. We denote by Σ-$\mathbf{Str}_\mathrm{f}$ the full subcategory of Σ-$\mathbf{Str}$ given by all Σ-structures A where each A^s is finite.

When $\mathcal{S}$ is a singleton, the notion of Σ-structures boils down to a more common situation. Namely, the arity of an operation symbol is given solely by $n \in \mathbb{N}$ and that of a relation symbol by $m \in \mathbb{N}^+$. A Σ-structure is a set A equipped with an operation $\sigma_A: A^n \to A$ for every n-ary operation symbol σ and with a relation $r_A \subseteq A^m$ for every m-ary relation symbol r.

Assumption 2.5. Throughout the paper, we assume that every signature has a finite set of sorts and finitely many relation symbols. There is no restriction on the number of operation symbols.

Remark 2.6. (1) The category Σ-$\mathbf{Str}$ is complete with limits created at the level of $\mathbf{Set}^{\mathcal{S}}$. More precisely, consider a diagram D in Σ-$\mathbf{Str}$ indexed by $\mathcal{I}$. Let $U^s: \mathbf{Set}^{\mathcal{S}} \to \mathbf{Set}$ be the projection sending B to B^s, and let

$$b_i^s: B^s \to D_i^s \quad (i \in \mathcal{I})$$

form limit cones of the diagrams $U^s D$ in $\mathbf{Set}$ for every $s \in \mathcal{S}$. Then the limit of D is the $\mathcal{S}$-sorted set $B := (B^s)$, with operations $\sigma_B: B^{s_1} \times \cdots \times B^{s_n} \to B^s$ uniquely determined by the requirement that each $b_i: B \to D_i$ preserves σ, and with relations $r_B \subseteq B^{s_1} \times \cdots \times B^{s_n}$ consisting of all n-tuples $(x_1, \ldots, x_n)$ that each function $b_i^{s_1} \times \cdots \times b_i^{s_n}$ maps into r_{D_i} for all $i \in \mathcal{I}$. The limit cone is given by $(b_i^s)_{s \in \mathcal{S}}: B \to D_i$ for $i \in \mathcal{I}$.

(2) The category Σ-$\mathbf{Str}$ is also cocomplete. Indeed, let Σ_{op} be the subsignature of all operation symbols in Σ. Then Σ_{op}-$\mathbf{Str}$ is a monadic category over $\mathbf{Set}^{\mathcal{S}}$. Since epimorphisms split in $\mathbf{Set}^{\mathcal{S}}$, all monadic categories are cocomplete, see e.g. [1]. The category Σ-$\mathbf{Str}$ has colimits obtained from the corresponding colimits in Σ_{op}-$\mathbf{Str}$ by taking the smallest relations making each of the colimit injections a Σ-homomorphism.

Notation 2.7. The category of Stone topological Σ-structures and continuous Σ-homomorphisms is denoted by

$$\mathbf{Stone}(\Sigma\text{-}\mathbf{Str}).$$

A *topological* Σ*-structure* is an $\mathcal{S}$-sorted topological space $A = (A^s)$ endowed with a Σ-structure such that every operation $\sigma_s: A^{s_1} \times \cdots \times A^{s_n} \to A$ is continuous and for every relation symbol r the relation $r_A \subseteq A^{s_1} \times \cdots \times A^{s_n}$ is a closed subset.

Remark 2.8. The category $\mathbf{Stone}(\Sigma$-$\mathbf{Str})$ is complete with limits formed on the level of $\mathbf{Set}^{\mathcal{S}}$. This follows from the construction of limits in $\mathbf{Stone}^{\mathcal{S}}$ and in Σ-$\mathbf{Str}$. Thus, the forgetful functor from $\mathbf{Stone}(\Sigma$-$\mathbf{Str})$ to Σ-$\mathbf{Str}$ preserves limits.

The following proposition describes the pro-completion of Σ-$\mathbf{Str}_\mathrm{f}$. It is a categorical reformulation of results by Pin and Weil [17] on topological Σ-structures, and also appears in Johnstone's book [10, Prop. & Rem. VI.2.4] for the special case of single-sorted algebras. We provide a full proof for the convenience of the reader.

Definition 2.9. A Stone topological Σ-structure is called *profinite* if it is a cofiltered limit in $\mathbf{Stone}(\Sigma$-$\mathbf{Str})$ of finite Σ-structures.

Proposition 2.10. *The category* $\mathrm{Pro}(\Sigma$-$\mathbf{Str}_\mathrm{f})$ *is the full subcategory of* $\mathbf{Stone}(\Sigma$-$\mathbf{Str})$ *on all profinite* Σ*-structures.*

Proof. (1) We first observe that cofiltered limits of finite sets in $\mathbf{Stone}$ have the following property: If $b_i: B \to B_i$ $(i \in \mathcal{I})$ is a cofiltered limit cone such that all B_i are finite, then for every $i \in \mathcal{I}$ there exists a connecting morphism of our diagram $h: B_j \to B_i$ with the same image as b_i:

$$b_i[B] = h[B_j]. \tag{2.1}$$

Since under Stone duality finite Stone spaces dualizes to finite boolean algebras, it suffices to verify the dual statement about filtered colimits of finite Boolean algebras: if $c_i : C_i \to C$ ($i \in \mathcal{I}$) is a filtered colimit cocone of finite Boolean algebras, then for every i there exists a connecting morphism $f : C_i \to C_j$ with the same kernel as c_i. But this is clear: given any pair $x, y \in C_i$ merged by c_i, there exists a connecting morphism f merging x and y, since filtered colimits are formed on the level of **Set**. Due to $C_i \times C_i$ being finite, we can choose one f for all such pairs.

(2) The argument is similar for cofiltered limits of finite Σ-structures in **Stone**(Σ-**Str**): Consider a limit cone

$$b_i : B \to B_i \qquad (i \in \mathcal{I})$$

of a cofiltered diagram D in **Stone**(Σ-**Str**). For every $i \in \mathcal{I}$, we verify that there is a connecting morphism $h : B_j \to B_i$ with sorts h^s for $s \in \mathcal{S}$ such that

$$b_i^s[B^s] = h^s[B_j^s] \qquad \text{for all } s \in \mathcal{S}, \tag{2.2}$$

and

$$b_i^{s_1} \times \cdots \times b_i^{s_n}[r_B] = h^{s_1} \times \cdots \times h^{s_n}[r_B] \qquad \text{for all } r : s_1, \ldots, s_n \text{ in } \Sigma. \tag{2.3}$$

Indeed, if we only consider (2.2) then the existence of such an h follows from (1) by the assumption that $\mathcal{S}$ is finite and that $\mathcal{I}$ is cofiltered. For every sort s, we have a cofiltered limit $b_j^s : B^s \to B_j^s$ in **Stone**, thus we can apply (1) and obtain a connecting morphism $h : B_j \to B_i$. Again, $\mathcal{S}$ is finite, so the choice of h can be made independent of $s \in \mathcal{S}$.

Next consider (2.3) for a fixed relation symbol $r : s_1, \ldots, s_n$. Form the diagram D_r in **Stone** with the above diagram scheme $\mathcal{I}$ and with objects

$$D_r i = r_{B_i} \text{ (a finite discrete space).}$$

Connecting morphisms are the domain-codomain restrictions of all connecting morphisms $B_j \xrightarrow{h} B_k$: since h preserves the relation r, we have

$$h^{s_1} \times \cdots \times h^{s_n}[r_{B_j}] \subseteq r_{B_k},$$

and we form the corresponding connecting morphism $\overline{h} : r_{B_j} \to r_{B_k}$ of D_r. From the description of limits in Σ-**Str** in Remark 2.6 and the fact that limits in **Stone**(Σ-**Str**) are preserved by the forgetful functor into Σ-**Str** by Remark 2.8 we deduce that the limit of D_r in **Stone** is the space $r_B \subseteq B^{s_1} \times \cdots \times B^{s_n}$ and the limit cone $r_B \to r_{B_j}$, $j \in \mathcal{I}$, is formed by domain-codomain restrictions of $b_j^{s_1} \times \cdots \times b_j^{s_n}$ for $j \in \mathcal{I}$. Apply (1) to this cofiltered limit to find a connecting morphism $h : B_j \to B_i$ of D satisfying (2.3) for any chosen relation symbol r of Σ. Since we only have finitely many relation symbols by Assumption 2.5, we conclude that h can be chosen to satisfy (2.3).

(3) Denote the full subcategory formed by profinite Σ-structures by

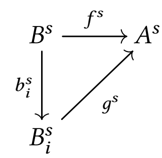
$$\mathscr{L} \subseteq \mathbf{Stone}(\Sigma\text{-}\mathbf{Str}).$$

In order to prove that $\mathscr{L}$ forms the pro-completion of Σ-$\mathbf{Str}_f$, we verify the conditions given in Corollary A.5. By construction, conditions (1) and (2) hold. It remains to prove condition (3): every finite Σ-structure A is finitely copresentable in $\mathscr{L}$. Hence, consider a limit cone

$$b_i : B \to B_i \quad (i \in \mathcal{I})$$

of a cofiltered diagram D in $\mathscr{L}$. Due to the definition of $\mathscr{L}$, we may assume that all B_i are finite. We need to show that for every homomorphism $f = (f^s)_{s \in \mathcal{S}} : B \to A$ into a finite Σ-structure $A = (A^s)_{s \in \mathcal{S}}$, there is an essentially unique factorization through some b_i. For every sort s, we have a projection $V^s : \mathscr{L} \to$ **Stone**, and the cofiltered diagram $V^s D$ has the limit cone $f_i^s : B^s \to B_i^s$ ($i \in \mathcal{I}$).

Since each A^s is finite, the fact that **Stone** is the pro-completion of $\mathbf{Set}_f$ implies that for every sort s there is $i \in \mathcal{I}$ and an essentially unique factorization of f^s as follows



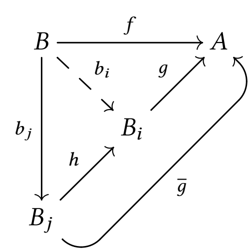
By Assumption 2.5 the set $\mathcal{S}$ is finite, so we can choose i independent of s and thus obtain a continuous $\mathcal{S}$-sorted function

$$g = (g^s) \colon B_i \to A \quad \text{in } \mathbf{Stone}^{\mathcal{S}}$$

which factorizes f, i.e. $f = g \cdot b_i$.

All we still need to prove is that we can choose our i and g so that, moreover, g is a Σ-homomorphism. The essential uniqueness of g then follows from the corresponding property of g in **Stone**.

Let $h \colon B_j \to B_i$ be a connecting map satisfying (2.2) and (2.3). Choose j in lieu of i and $\overline{g} = g \cdot h$ in lieu of g. We conclude that $\overline{g}$ is a morphism of $\mathbf{Stone}^{\mathcal{S}}$ factorizing f through the limit map b_j:



Moreover, we prove that $\overline{g}$ is a Σ-homomorphism:

(3a) For every operation symbol $\sigma \colon s_1 \dots s_n \to s$ in Σ and every n-tuple $(x_1, \dots, x_n) \in B_j^{s_1} \times \cdots \times B_j^{s_n}$ we have

$$\overline{g}^s \cdot \sigma_{B_j}(x_1, \dots, x_n) = \sigma_A\left(\overline{g}^{s_1}(x_1), \dots, \overline{g}^{s_n}(x_n)\right).$$

Indeed, choose $y_k \in B^{s_k}$ with $b_i^{s_k}(y_k) = h^{s_k}(x_k)$, $k = 1, \dots, n$, using (2.2). Then

$$\begin{aligned}
\overline{g}^s \cdot \sigma_{B_j}(x_1, \dots, x_n) &= g^s \cdot h^s \cdot \sigma_{B_j}(x_1, \dots, x_n) && \overline{g} = g \cdot h \\
&= g^s \cdot \sigma_{B_i}(h^{s_1}(x_1), \dots, h^{s_n}(x_n)) && h \text{ a } \Sigma\text{-homomorphism} \\
&= g^s \cdot \sigma_{B_i}(b_i^{s_1}(y_1), \dots, b_i^{s_n}(y_n)) && b_i^{s_k}(y_k) = h^{s_k}(x_k) \\
&= g^s \cdot b_i^s \cdot \sigma_B(y_1, \dots, y_n) && b_i \text{ a } \Sigma\text{-homomorphism} \\
&= \sigma_A(g^{s_1} \cdot b_i^{s_1}(y_1), \dots, g^{s_n} \cdot b_i^{s_n}(y_n)) && g \cdot b_i = f \text{ a } \Sigma\text{-homomorphism} \\
&= \sigma_A(g^{s_1} \cdot h^{s_1}(y_1), \dots, g^{s_n} \cdot h^{s_n}(y_n)) && b_i^{s_k}(y_k) = h^{s_k}(x_k) \\
&= \sigma_A(\overline{g}^{s_1}(x_1), \dots, \overline{g}^{s_n}(x_n)) && \overline{g} = g \cdot h.
\end{aligned}$$

(3b) For every relation symbol $r \colon s_1, \dots, s_n$ in Σ, we have that

$$(x_1, \dots, x_n) \in r_{B_j} \text{ implies } (\overline{g}^{s_1}(x_1), \dots, \overline{g}^{s_n}(x_n)) \in r_A.$$

Indeed, using (2.3), we can choose $(y_1, \dots, y_n) \in r_B$ with

$$(b_i^{s_1}(y_1), \dots, b_i^{s_n}(y_n)) = (h^{s_1}(x_1), \dots, h^{s_n}(x_n)).$$

Then the n-tuple

$$(\overline{g}^{s_1}(x_1), \dots, \overline{g}^{s_n}(x_n)) = (g^{s_1} \cdot b_i^{s_1}(y_1), \dots, g^{s_n} \cdot b_i^{s_n}(y_n))$$

lies in r_A because $g \cdot b_i = f$ is a Σ-homomorphism. □

Notation 2.11. Let $\mathscr{D}$ be a full subcategory of Σ-**Str**. We denote by

$$\mathbf{Stone}\mathscr{D}$$

the full subcategory of **Stone**(Σ-**Str**) on all Stone topological Σ-structures whose Σ-structure lies in $\mathscr{D}$. Moreover, let $\mathscr{D}_\mathrm{f}$ denote the full subcategory of $\mathscr{D}$ on all finite objects, i.e. $D \in \mathscr{D}_\mathrm{f}$ if each D^s is finite.

Corollary 2.12. *Let $\mathscr{D}$ be a full subcategory of* Σ-**Str** *closed under cofiltered limits. Then* Pro $\mathscr{D}_\mathrm{f}$ *is the full subcategory of* **Stone**$\mathscr{D}$ *given by all profinite $\mathscr{D}$-structures, i.e. cofiltered limits of finite Σ-structures in $\mathscr{D}$.*

The proof is completely analogous to that of Proposition 2.10: the only fact we used in that proof was the description of cofiltered limits in Σ-**Str**.

Example 2.13. For $\mathscr{D}$ = **Pos**, we get an alternative description of the category **Priest** of Example 2.3(2). For the signature Σ with a single binary relation, **Pos** is a full subcategory of Σ-**Str**. The category **Stone**(Σ-**Str**) is that of graphs on Stone spaces. By Corollary 2.12, Pro(**Pos**$_\mathrm{f}$) is the category of all profinite posets, i.e. Stone graphs that are cofiltered limits of finite posets. Note that every such limit $B = (V, E)$ is a poset: given $x \in V$ we have $(x, x) \in E$ because every object of the given cofiltered diagram has its relation reflexive. Analogously, E is transitive and (since limit cones are collectively monic) antisymmetric.

Moreover, B is a Priestley space: given $x, y \in V$ with $x \nleq y$, then there exists a member $b_i : B \to B_i$ of the limit cone with $b_i(x) \nleq b_i(y)$. Since B_i is finite, and thus carries the discrete topology, the upper set $b_i^{-1}(\uparrow x)$ is clopen, and it contains x but not y. Conversely, every Priestley space is a profinite poset, as shown by Speed [22].

Example 2.14. Johnstone [10, Thm. VI.2.9] proves that for a number of "everyday" varieties of algebras $\mathscr{D}$, we simply have

$$\text{Pro } \mathscr{D}_\mathrm{f} = \mathbf{Stone}\mathscr{D}.$$

This holds for semigroups, monoids, groups, vector spaces, semilattices, distributive lattices, etc. In contrast, for some important varieties Pro $\mathscr{D}_\mathrm{f}$ is a proper subcategory of **Stone**$\mathscr{D}$, e.g. for the variety of lattices or the variety of Σ-algebras where Σ consists of a single unary operation.

Remark 2.15. (1) The category Σ-**Str** has a factorization system $(\mathcal{E}, \mathcal{M})$ where $\mathcal{E}$ consists of all surjective Σ-homomorphisms (more precisely, every sort is a surjective function) and $\mathcal{M}$ consists of all injective Σ-homomorphisms reflecting all relations. That is, a Σ-homomorphism $f : X \to Y$ lies in $\mathcal{M}$ iff for every sort s the function $f^s : X^s \to Y^s$ is injective, and for every relation symbol $r : s_1, \dots, s_n$ in Σ and every n-tuple $(x_1, \dots, x_n) \in X^{s_1} \times \dots \times X^{s_n}$ one has

$$(x_1, \dots, x_n) \in r_X \quad \text{iff} \quad (f^{s_1}(x_1), \dots, f^{s_n}(x_n)) \in r_Y.$$

The $(\mathcal{E}, \mathcal{M})$-factorization of a Σ-homomorphism $g : X \to Z$ is constructed as follows. Define a Σ-structure Y by $Y^s = g^s[X^s]$ for all sorts $s \in \mathcal{S}$, let the operations of Y be the domain-codomain restriction of those of Z, and for every relation symbol $r : s_1, \dots, s_n$ define r_Y to be the restriction of r_Z to Y, i.e. $r_Y = r_Z \cap Y^{s_1} \times \dots \times Y^{s_n}$. Then the codomain restriction of g is a surjective Σ-homomorphism $e : X \twoheadrightarrow Y$, and the embedding $m : Y \rightarrowtail Z$ is a injective Σ-homomorphism reflecting all relations.

(2) Similarly, the category **Stone**(Σ-**Str**) has the factorization system $(\mathcal{E}, \mathcal{M})$ where $\mathcal{E}$ consists of all surjective morphisms and $\mathcal{M}$ of all relation-reflecting monomorphisms. Indeed, if $f: X \to Z$ is a continuous Σ-homomorphism, and if its factorization in Σ-**Str** is given by a Σ-structure Y and Σ-homomorphisms $e: X \twoheadrightarrow Y$ (surjective) and $m: Y \rightarrowtail Z$ (injective and relation-reflecting), then the Stone topology on Y inherited from Z yields, due to $Y = e[X]$ being closed in Z, the desired factorization in **Stone**(Σ-**Str**).

Remark 2.16. Recall that the *arrow category* $\mathscr{A}^{\to}$ of a category $\mathscr{A}$ has as objects all morphisms $f: X \to Y$ in $\mathscr{A}$. A morphism from $f: X \to Y$ to $g: U \to V$ in $\mathscr{A}^{\to}$ is given by a pair of morphisms $m: X \to U$ and $n: Y \to V$ in $\mathscr{A}$ with $n \cdot f = g \cdot m$. Identities and composition are defined componentwise. If $\mathscr{A}$ has limits of some type, then also $\mathscr{A}^{\to}$ has these limits, and the two projection functors from $\mathscr{A}^{\to}$ to $\mathscr{A}$ mapping an arrow to its domain or codomain, respectively, preserve them.

Lemma 2.17. (1) *For every cofiltered diagram D in* $\mathbf{Set}_{\mathrm{f}}$ *with epic connecting maps, the limit cone of D in* **Stone** *is formed by epimorphisms.*

(2) *For every cofiltered diagram D in* $\mathbf{Stone}^{\to}$ *whose objects are epimorphisms in* **Stone**, *also* $\lim D$ *is epic.*

Proof. These properties follow easily from standard results about cofiltered limits in the category of compact Hausdorff spaces, see e.g. Ribes and Zalesskii [20, Sec. 1]. Here, we give an alternative proof using Stone duality, i.e. we verify that the category **BA** of boolean algebras satisfies the statements dual to (1) and (2).

The dual of (1) states that a filtered diagram of finite boolean algebras with monic connecting maps has a colimit in **BA** whose colimit maps are monic. This follows from the fact that filtered colimits in **BA** are created by the forgetful functor to **Set**, and that filtered colimits of monics in **Set** clearly have the desired property.

Similarly, the dual of (2) states that a filtered colimit of monomorphisms in $\mathbf{BA}^{\to}$ is a monomorphism, which follows from the corresponding property in $\mathbf{Set}^{\to}$. □

3 PSEUDOVARIETIES

In universal algebra, a pseudovariety of Σ-algebras is defined to be a class of finite algebras closed under finite products, subalgebras, and quotient algebras. In the present section, we introduce an abstract concept of pseudovariety in a given category $\mathscr{D}$ with a specified full subcategory $\mathscr{D}_{\mathrm{f}}$. The objects of $\mathscr{D}_{\mathrm{f}}$ are called "finite", but this is just terminology. Our approach follows the footsteps of Banaschewski and Herrlich [5] who introduced varieties of objects in a category $\mathscr{D}$, and proved that they are precisely the full subcategories of $\mathscr{D}$ presentable by an abstract notion of equation (see Definition 3.2). Here, we establish a similar result for pseudovarieties: they are precisely the full subcategories of $\mathscr{D}_{\mathrm{f}}$ that can be presented by pseudoequations (Proposition 3.8), which are shown to be equivalent to profinite equations in many examples (Theorem 3.23).

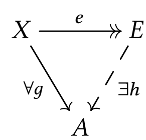
Assumption 3.1. For the rest of our paper, we fix a complete category $\mathscr{D}$ with a proper factorization system $(\mathcal{E}, \mathcal{M})$, that is, all morphisms in $\mathcal{E}$ are epic and all morphisms in $\mathcal{M}$ are monic. *Quotients* and *subobjects* in $\mathscr{D}$ are represented by morphisms in $\mathcal{E}$ and $\mathcal{M}$, respectively, and denoted by $\twoheadrightarrow$ and $\rightarrowtail$. Moreover, we fix a small full subcategory $\mathscr{D}_{\mathrm{f}}$ whose objects are called the *finite* objects of $\mathscr{D}$, and denote by $\mathcal{E}_{\mathrm{f}}$ and $\mathcal{M}_{\mathrm{f}}$ the morphisms of $\mathscr{D}_{\mathrm{f}}$ in $\mathcal{E}$ or $\mathcal{M}$, respectively. We assume that

(1) the category $\mathscr{D}_{\mathrm{f}}$ is closed under finite limits and subobjects, and
(2) every object of $\mathscr{D}_{\mathrm{f}}$ is a quotient of some projective object of $\mathscr{D}$.

Here, recall that an object X is called *projective* (more precisely, $\mathcal{E}$-*projective*) if for every quotient $e\colon P \twoheadrightarrow P'$ and every morphism $f\colon X \to P'$ there exists a morphism $g\colon X \to P$ with $e \cdot g = f$.

Definition 3.2 (Banaschewski and Herrlich [5]). (1) A *variety* is a full subcategory of $\mathscr{D}$ closed under products, subobjects, and quotients.

(2) An *equation* is a quotient $e\colon X \twoheadrightarrow E$ of a projective object X. An object A is said to *satisfy* the equation e provided that A is *injective* w.r.t. e, that is, if for every morphism $g\colon X \to A$ there exists a morphism $h\colon E \to A$ making the triangle below commute:



We note that Banaschewski and Herrlich worked with the factorization system of regular epimorphisms and monomorphisms. However, all their results and proofs apply to general proper factorization systems, as already pointed out in their paper [5].

Example 3.3. Let Σ be a one-sorted signature of operation symbols. If $\mathscr{D} = \Sigma$-Alg is the category of Σ-algebras with its usual factorization system ($\mathcal{E}$ = surjective homomorphisms and $\mathcal{M}$ = injective homomorphisms), then the above definition of a variety gives the usual concept in universal algebra: a class of Σ-algebras closed under product algebras, subalgebras, and homomorphic images. Moreover, equations in the above categorical sense are expressively equivalent to equations $t = t'$ between Σ-terms in the usual sense:

(1) Given a term equation $t = t'$, where $t, t' \in T_\Sigma X_0$ are taken from the free algebra of all Σ-terms in the set X_0 of variables, let $\sim$ denote the least congruence on $T_\Sigma X_0$ with $t \sim t'$. The corresponding quotient morphism $e\colon T_\Sigma X_0 \twoheadrightarrow T_\Sigma X_0/{\sim}$ is a categorical equation satisfied by precisely those Σ-algebras that satisfy $t = t'$ in the usual sense.

(2) Conversely, given a projective Σ-algebra X and a surjective homomorphism $e\colon X \twoheadrightarrow E$, then for any set X_0 of generators of X we have a split epimorphism $q\colon T_\Sigma X_0 \twoheadrightarrow X$ using the projectivity of X. Consider the set of term equations $t = t'$ where (t, t') ranges over the kernel of $e \cdot q\colon T_\Sigma X_0 \twoheadrightarrow E$. Then a Σ-algebra A satisfies all these equations iff it satisfies e in the categorical sense.

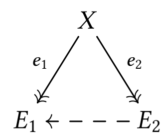
Recall that the category $\mathscr{D}$ is $\mathcal{E}$-*co-well-powered* if for every object X of $\mathscr{D}$ the quotients with domain X form a small set.

Theorem 3.4 (Banaschewski and Herrlich [5]). *Let $\mathscr{D}$ be a category with a proper factorization system $(\mathcal{E}, \mathcal{M})$. Suppose that $\mathscr{D}$ is complete, $\mathcal{E}$-co-well-powered, and has enough projectives, i.e. every object is a quotient of a projective one. Then, a full subcategory of $\mathscr{D}$ is a variety iff it can be presented by a class of equations. That is, it consists of precisely those objects satifying each of these equations.*

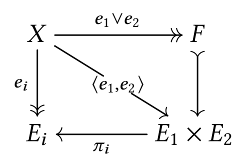
Note that the category of Σ-algebras satisfies all conditions of the theorem. Thus, in view of Example 3.3, Banaschewski and Herrlich's result subsumes Birkhoff's HSP theorem [6]. In the following, we are going to move from varieties in $\mathscr{D}$ to pseudovarieties in $\mathscr{D}_\mathsf{f}$.

Definition 3.5. A *pseudovariety* is a full subcategory of $\mathscr{D}_\mathsf{f}$ closed under finite products, subobjects, and quotients.

Remark 3.6. Quotients of an object X are ordered by factorization: given $\mathcal{E}$-quotients e_1, e_2, we put $e_1 \leq e_2$ if e_1 factorizes through e_2



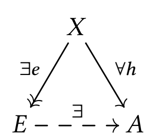
Every pair of quotients $e_i : X \twoheadrightarrow E_i$ has a least upper bound, or *join*, $e_1 \vee e_2$ obtained by $(\mathcal{E}, \mathcal{M})$-factorizing the mediating morphism $\langle e_1, e_2 \rangle : X \to E_1 \times E_2$ as follows:



(3.1)

A nonempty collection of quotients closed under joins is called a *semilattice of quotients*.

Definition 3.7. A *pseudoequation* is a semilattice ρ_X of quotients of a projective object X (of "variables"). A finite object A of $\mathscr{D}$ *satisfies* ρ_X if A is cone-injective w.r.t. ρ_X, that is, for every morphism $h : X \to A$, there exists a member $e : X \twoheadrightarrow E$ of ρ_X through which h factorizes:



Proposition 3.8. *A collection of finite objects of $\mathscr{D}$ forms a pseudovariety iff it can be presented by pseudoequations, i.e. it consists of precisely those finite objects that satisfy each of the given pseudoequations.*

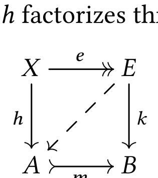
Proof. (1) We first prove the *if* direction. Since the intersection of a family of pseudovarieties is a pseudovariety, it suffices to prove that for every pseudoequation ρ_X over a projective object X, the class $\mathscr{V}$ of all finite objects satisfying ρ_X forms a pseudovariety, i.e. is closed under finite products, subobjects, and quotients.

(1a) *Finite products.* Let $A, B \in \mathscr{V}$. Since A and B satisfy ρ_X, for every morphism $\langle h, k \rangle : X \to A \times B$ there exists $e : X \twoheadrightarrow E$ in ρ_X such that both $h : X \to A$ and $k : X \to B$ factorize through e – this follows from the closedness of pseudoequations under binary joins. Given $h = e \cdot h'$ and $k = e \cdot k'$, then $\langle h', k' \rangle : X \to E_i$ is the desired factorization:

$$\langle h, k \rangle = e \cdot \langle h', k' \rangle.$$

Thus $A \times B \in \mathscr{V}$. Since the terminal object 1 clearly satisfies every pseudoequation, we also have $1 \in \mathscr{V}$.

(1b) *Subobjects.* Let $m : A \rightarrowtail B$ be a morphism in $\mathcal{M}_\mathrm{f}$ with $B \in \mathscr{V}$. Then for every morphism $h : X \to A$ we know that $m \cdot h$ factorizes as $e \cdot k$ for some $e : X \twoheadrightarrow E$ in ρ_X and some $k : E \to B$. The diagonal fill-in property then shows that h factorizes through e:



Thus, $A \in \mathscr{V}$.

(1c) *Quotients.* Let $q\colon B \twoheadrightarrow A$ be a morphism in $\mathcal{E}_\mathsf{f}$ with $B \in \mathscr{V}$. Every morphism $h\colon X \to A$ factorizes, since X is projective, as

$$h = q \cdot k \quad \text{for some } k\colon X \to B$$

Since k factorizes through some $e \in \rho_X$, so does h. Thus, $A \in \mathscr{V}$.

(2) For the "only if" direction, suppose that $\mathscr{V}$ is a pseudovariety. For every projective object X we form the pseudoequation ρ_X consisting of all quotients $e\colon X \twoheadrightarrow E$ with $E \in \mathscr{V}$. This is indeed a semilattice: given $e, f \in \rho_X$ we have $e \vee f \in \rho_X$ by (3.1), using that $\mathscr{V}$ is closed under finite products and subobjects. We claim that $\mathscr{V}$ is presented by the collection of all the above pseudoequations ρ_X.

(2a) Every object $A \in \mathscr{V}$ satisfies all ρ_X. Indeed, given a morphism $h\colon X \to A$, factorize it as $e\colon X \twoheadrightarrow E$ in $\mathcal{E}$ followed by $m\colon E \rightarrowtail A$ in $\mathcal{M}$. Then $E \in \mathscr{V}$ because $\mathscr{V}$ is closed under subobjects, so e is a member of ρ_X. Therefore $h = m \cdot e$ is the desired factorization of h, proving that A satisfies ρ_X.

(2b) Every finite object A satisfying all the pseudoequations ρ_X lies in $\mathscr{V}$. Indeed, by Assumption 3.1 there exists a quotient $q\colon X \twoheadrightarrow A$ for some projective object X. Since A satisfies ρ_X, there exists a factorization $q = h \cdot e$ for some $e\colon X \twoheadrightarrow E$ in ρ_X and some $h\colon E \to A$. We know that $E \in \mathscr{V}$, and from $q \in \mathcal{E}$ we deduce $h \in \mathcal{E}$. Thus A, being a quotient of an object of $\mathscr{V}$, lies in $\mathscr{V}$. □

Remark 3.9. (1) Proposition 3.8 would remain valid if we defined pseudoequations as semilattices of *finite* quotients of a projective object. This follows immediately from the above proof.

(2) Let us assume that a collection Var of projective objects of $\mathscr{D}$ is given such that every finite object is a quotient of an object of Var (cf. Assumption 3.1(2)). Then we could define pseudoequations as semilattices of quotients of members of Var with finite codomains. Again, from the above proof we see that Proposition 3.8 would remain true.

We would like to reduce pseudoequations to equations in the sense of Banaschewski and Herrlich. For that we need to move from the category $\mathscr{D}$ to the pro-completion of $\mathscr{D}_\mathsf{f}$.

Notation 3.10. Since $\mathscr{D}$ has (cofiltered) limits, the embedding $\mathscr{D}_\mathsf{f} \rightarrowtail \mathscr{D}$ extends to an essentially unique cofinitary functor

$$V\colon \mathsf{Pro}\,\mathscr{D}_\mathsf{f} \to \mathscr{D}.$$

Example 3.11. If $\mathscr{D}$ is a full subcategory of Σ-**Str** closed under cofiltered limits, we have seen that $\mathsf{Pro}\,\mathscr{D}_\mathsf{f}$ can be described as a full subcategory of **Stone**$\mathscr{D}$ by Corollary 2.12. The above functor

$$V\colon \mathsf{Pro}\,\mathscr{D}_\mathsf{f} \to \mathscr{D}$$

is the functor forgetting the topology. Indeed, the corresponding forgetful functor from **Stone**(Σ-**Str**) to Σ-**Str** is cofinitary, hence, so is V.

Remark 3.12. Recall, e.g. from Mac Lane [14], that the *right Kan extension* of a functor $F\colon \mathscr{A} \to \mathscr{C}$ along $K\colon \mathscr{A} \to \mathscr{B}$ is a functor $R = \mathsf{Ran}_K F\colon \mathscr{B} \to \mathscr{C}$ with a universal natural transformation $\varepsilon\colon RK \to F$, that is, for every functor $G\colon \mathscr{B} \to \mathscr{C}$ and every natural transformation $\gamma\colon GK \to F$ there exists a unique natural transformation $\gamma^\dagger\colon G \to R$ with $\gamma = \varepsilon \cdot \gamma^\dagger K$. If $\mathscr{A}$ is small and $\mathscr{C}$ is complete, then the right Kan extension exists [14, Theorem X.3.1, X.4.1], and the object RB ($B \in \mathscr{B}$) can be constructed as the limit

$$RB = \lim(B/K \xrightarrow{Q_B} \mathscr{A} \xrightarrow{F} \mathscr{C}),$$

where B/K denotes the slice category of all morphisms $f\colon B \to KA$ ($A \in \mathscr{A}$) and Q_B is the projection functor $f \mapsto A$. Equivalently, RB is given by the end

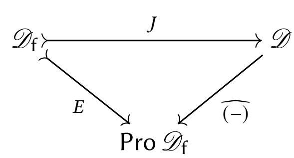
$$RB = \int_{A\in\mathscr{A}} \mathscr{B}(B, KA) \pitchfork FA,$$

with $S \pitchfork C$ denoting S-fold power of $C \in \mathscr{C}$.

Lemma 3.13. *The functor V has a left adjoint*

$$\widehat{(-)} = \mathsf{Ran}_J E\colon \mathscr{D} \to \mathsf{Pro}\,\mathscr{D}_\mathsf{f}$$

given by the right Kan extension of the embedding $E\colon \mathscr{D}_\mathsf{f} \rightarrowtail \mathsf{Pro}\,\mathscr{D}_\mathsf{f}$ along the embedding $J\colon \mathscr{D}_\mathsf{f} \rightarrowtail \mathscr{D}$ and making the following triangle commute up to isomorphism:



Proof. Recall that, up to equivalence, $\mathsf{Pro}\,\mathscr{D}_\mathsf{f}$ is the full subcategory of $[\mathscr{D}_\mathsf{f}, \mathbf{Set}]^{\mathrm{op}}$ on cofiltered limits of representables with $ED = \mathscr{D}_\mathsf{f}(D, -)$ for every $D \in \mathscr{D}_\mathsf{f}$ (see Remark A.6), and the functor V is given by

$$V = \mathsf{Ran}_E J\colon\ \mathsf{Pro}\,\mathscr{D}_\mathsf{f} \to \mathscr{D}.$$

Consider the following chain of isomorphisms natural in $D \in \mathscr{D}$ and $H \in \mathsf{Pro}\,\mathscr{D}_\mathsf{f}$:

$$
\begin{aligned}
\mathscr{D}(D, VH) &\cong \mathscr{D}(D, (\mathsf{Ran}_E J)\, H) \\
&\cong \mathscr{D}\left(D, \int_X \mathsf{Pro}\,\mathscr{D}_\mathsf{f}(H, EX) \pitchfork JX\right) && \text{by the end formula for Ran,} \\
&= \mathscr{D}\left(D, \int_X [\mathscr{D}_\mathsf{f}, \mathbf{Set}](EX, H) \pitchfork JX\right) && \mathsf{Pro}\,\mathscr{D}_\mathsf{f} \text{ full subcategory of } [\mathscr{D}_\mathsf{f}, \mathbf{Set}]^{\mathrm{op}}, \\
&\cong \mathscr{D}\left(D, \int_X HX \pitchfork JX\right) && \text{by the Yoneda lemma,} \\
&\cong \int_X \mathscr{D}(D, HX \pitchfork JX) && \mathscr{D}(D, -) \text{ preserves ends,} \\
&\cong \int_X \mathbf{Set}(HX, \mathscr{D}(D, JX)) && \text{by the universal property of power,} \\
&\cong [\mathscr{D}_\mathsf{f}, \mathbf{Set}](H, \mathscr{D}(D, J-)) && \text{the set of nat. trafo. as an end,} \\
&= \mathsf{Pro}\,\mathscr{D}_\mathsf{f}(\mathscr{D}(D, J-), H) && \mathsf{Pro}\,\mathscr{D}_\mathsf{f} \text{ full subcategory of } [\mathscr{D}_\mathsf{f}, \mathbf{Set}]^{\mathrm{op}}.
\end{aligned}
$$

Hence, the functor $\widehat{(-)}\colon D \mapsto \mathscr{D}(D, J-)$ is a left adjoint to V. Moreover, $\widehat{(-)}$ extends E: for each $D \in \mathscr{D}_\mathsf{f}$, we have

$$\widehat{D} = \mathscr{D}(JD, J-) = \mathscr{D}_\mathsf{f}(D, -) = ED,$$

and similarly on morphisms, since J is a full inclusion. It remains to verify that the functor $\widehat{(-)}$ coincides with $\mathsf{Ran}_J E$. This follows from the fact that every presheaf is a canonical colimit of representables expressed as a coend in $[\mathscr{D}_\mathsf{f}, \mathbf{Set}]$:

$$\mathscr{D}(D, J-) \cong \int^X \mathscr{D}(D, JX) \bullet EX,$$

with $\bullet$ denoting copowers. This corresponds to an end in $[\mathscr{D}_{\mathsf{f}}, \mathbf{Set}]^{\mathrm{op}}$:

$$\int_X \mathscr{D}(D, JX) \pitchfork EX = (\mathsf{Ran}_J E)D.$$

Thus $\widehat{(-)} = \mathsf{Ran}_J E$, as claimed. □

Construction 3.14. By expressing the right Kan extension

$$\widehat{(-)} = \mathsf{Ran}_J E \colon \mathscr{D} \to \mathsf{Pro}\,\mathscr{D}_{\mathsf{f}}$$

as a limit, the action $D \to \widehat{D}$ on objects, $f \mapsto \widehat{f}$ on morphisms, the unit, and the counit of the adjunction $\widehat{(-)} \dashv V$ are given as follows.

(1) For every object D of $\mathscr{D}$, the object $\widehat{D} \in \mathsf{Pro}\,\mathscr{D}_{\mathsf{f}}$ is a limit of the diagram

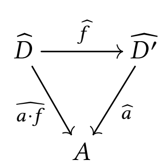
$$P_D \colon D/\mathscr{D}_{\mathsf{f}} \to \mathsf{Pro}\,\mathscr{D}_{\mathsf{f}}, \quad P_D(D \xrightarrow{a} A) = A.$$

We use the following notation for the limit cone of P_D:

$$\frac{D \xrightarrow{a} A}{\widehat{D} \xrightarrow{\widehat{a}} A}$$

where (A, a) ranges over $D/\mathscr{D}_{\mathsf{f}}$. For finite D we choose the trivial limit: $\widehat{D} = D$ and $\widehat{a} = a$.

(2) Given $f \colon D \to D'$ in $\mathscr{D}$, the morphisms $\widehat{a \cdot f}$ with a ranging over $D'/\mathscr{D}_{\mathsf{f}}$ form a cone over $P_{D'}$. Define $\widehat{f} \colon \widehat{D} \to \widehat{D'}$ to be the unique morphism such that the following triangles commute for all $a \colon D' \to A$ with $A \in \mathscr{D}_{\mathsf{f}}$:

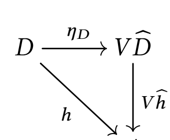


Note that overloading the notation $\widehat{(-)}$ causes no problem because if $\widehat{D'} = D' \in \mathscr{D}_{\mathsf{f}}$ then $\widehat{f}$ is a projection of the limit cone for P_D (see item (1)), since for $a = \mathsf{id}_{D'}$ we have $\widehat{a} = \mathsf{id}_{D'}$.

(3) The unit η at $D \in \mathscr{D}$ is given by the unique morphism

$$\eta_D \colon D \to V\widehat{D}$$

in $\mathscr{D}$ such that the following triangles commute for all $h \colon D \to A$ with $A \in \mathscr{D}_{\mathsf{f}}$:



Here one uses that V is cofinitary, and thus the morphisms $V\widehat{h}$ form a limit cone in $\mathscr{D}$.

(4) The counit ε at $D \in \mathsf{Pro}\,\mathscr{D}_{\mathsf{f}}$ is the unique morphism

$$\varepsilon_D \colon \widehat{VD} \to D$$

in $\widehat{\mathscr{D}}$ such that the following triangles commute, where $a\colon D \to A$ ranges over the slice category $D/\mathscr{D}_\mathsf{f}$:

$$\begin{array}{ccc} \widehat{VD} & \xrightarrow{\varepsilon_D} & D \\ & {\scriptstyle \widehat{Va}}\searrow \quad \swarrow{\scriptstyle a} & \\ & A & \end{array}$$

Notation 3.15. Recall that $\mathcal{E}_\mathsf{f}$ and $\mathcal{M}_\mathsf{f}$ are the morphisms of $\mathscr{D}_\mathsf{f}$ in $\mathcal{E}$ and $\mathcal{M}$, respectively. We denote by

$$\widehat{\mathcal{E}} \qquad \text{and} \qquad \widehat{\mathcal{M}}$$

the collection of all morphisms of $\mathsf{Pro}\,\mathscr{D}_\mathsf{f}$ that are cofiltered limits of members of $\mathcal{E}_\mathsf{f}$ or $\mathcal{M}_\mathsf{f}$ in the arrow category $(\mathsf{Pro}\,\mathscr{D}_\mathsf{f})^\to$, respectively.

Remark 3.16. (1) For every finite $\mathcal{E}$-quotient $e\colon X \twoheadrightarrow E$ in $\mathscr{D}$, the corresponding limit projection $\widehat{e}\colon \widehat{X} \twoheadrightarrow E$ lies in $\widehat{\mathcal{E}}$. Indeed, since E is finitely copresentable in $\mathsf{Pro}\,\mathscr{D}_\mathsf{f}$, the morphism $\widehat{e}$ factorizes through $\widehat{h}$ for some h in $X/\mathscr{D}_\mathsf{f}$, which can be assumed to be a quotient in $\mathcal{E}$. Otherwise, take the $(\mathcal{E}, \mathcal{M})$-factorization $h = m \cdot q$ of h and replace h by q.

Thus, we obtain an initial subdiagram $P'_X\colon \mathscr{I} \to \mathsf{Pro}\,\mathscr{D}_\mathsf{f}$ of $P_X\colon X/\mathscr{D}_\mathsf{f} \to \mathsf{Pro}\,\mathscr{D}_\mathsf{f}$ by restricting P_X to the full subcategory of finite quotients $h\colon X \twoheadrightarrow A$ in $X/\mathscr{D}_\mathsf{f}$ through which e factorizes, i.e. where $e = e_h \cdot h$ for some $e_h\colon A \to E$. Note that $e_h \in \mathcal{E}$ because $e, h \in \mathcal{E}$. The quotients e_h ($h \in \mathscr{I}$) form a cofiltered diagram in $(\mathsf{Pro}\,\mathscr{D}_\mathsf{f})^\to$ with limit cone $(\widehat{h}, \mathsf{id}_E)$:

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\widehat{e}} & E \\ {\scriptstyle \widehat{h}}\downarrow & & \downarrow{\scriptstyle \mathsf{id}_e} \\ A & \underset{e_h}{\twoheadrightarrow} & E \end{array}$$

Thus, $\widehat{e} \in \widehat{\mathcal{E}}$.

(2) For every cofiltered diagram $B\colon I \to \mathscr{D}_\mathsf{f}$ with connecting morphisms in $\mathcal{E}$, the limit projections in $\mathsf{Pro}\,\mathscr{D}_\mathsf{f}$ lie in $\widehat{\mathcal{E}}$. Indeed, let $b_i\colon X \to B_i$ ($i \in I$) denote the limit cone. Given $j \in I$, we are to show $b_j \in \widehat{\mathcal{E}}$. Form the diagram in $\mathscr{D}_\mathsf{f}^\to$ whose objects are all connecting morphisms of B with codomain B_j and whose morphisms from $h\colon B_i \twoheadrightarrow B_j$ to $h'\colon B_{i'} \twoheadrightarrow B_j$ are all connecting maps $k\colon B_i \twoheadrightarrow B_{i'}$ of B. This is a cofiltered diagram in $\mathsf{Pro}\,\mathscr{D}_\mathsf{f}$ with limit b_j and the following limit cone:

$$\begin{array}{ccc} X & \xrightarrow{b_j} & B_j \\ {\scriptstyle b_i}\downarrow & & \downarrow{\scriptstyle \mathsf{id}} \\ B_i & \underset{h}{\twoheadrightarrow} & B_j \end{array}$$

Since each h lies in $\mathcal{E}_\mathsf{f}$, this proves $b_j \in \widehat{\mathcal{E}}$.

(3) For every cone $p_i\colon P \to B_i$ of the diagram B in (2) with $p_i \in \widehat{\mathcal{E}}$ for all $i \in I$, the unique factorization $p\colon P \to X$ through the limit of B lies in $\widehat{\mathcal{E}}$. Indeed, p is the limit of p_i, $i \in I$, with the

following limit cone:

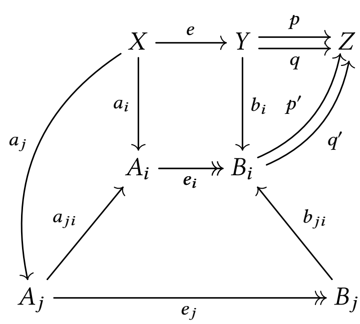
$$\begin{array}{ccc} P & \xrightarrow{p} & X \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle b_i} \\ P & \xrightarrow[p_i]{} & B_i \end{array}$$

Proposition 3.17. *The pair $(\widehat{\mathcal{E}}, \widehat{\mathcal{M}})$ is a proper factorization system of* Pro $\mathscr{D}_\mathrm{f}$.

Proof. (1) All morphisms of $\widehat{\mathcal{E}}$ are epic. This follows from the dual of [3, Prop. 1.62]; however, we give a direct proof. Given $e\colon X \to Y$ in $\widehat{\mathcal{E}}$, we have a limit cone of a cofiltered diagram D in $(\mathrm{Pro}\,\mathscr{D}_\mathrm{f})^\to$ as follows:

$$\begin{array}{ccc} X & \xrightarrow{e} & Y \\ {\scriptstyle a_i}\downarrow & & \downarrow{\scriptstyle b_i} \\ A_i & \xrightarrow[e_i]{} \!\!\!\!\rightarrow & B_i \end{array} \qquad (i \in I)$$

where $e_i \in \mathcal{E}_\mathrm{f}$ for each $i \in I$. Let $p, q\colon Y \to Z$ be two morphisms with $p \cdot e = q \cdot e$; we need to show $p = q$. Without loss of generality we can assume that the object Z is finite because $\mathscr{D}_\mathrm{f}$ is limit-dense in Pro $\mathscr{D}_\mathrm{f}$. Since (b_i) is a cofiltered limit cone in Pro $\mathscr{D}_\mathrm{f}$, there exists $i \in I$ such that p and q factorize through b_i, i.e. there exists morphisms p', q' with $p' \cdot b_i = p$ and $q' \cdot b_i = q$. The limit projection a_i of the cofiltered limit $X = \lim A_i$ in Pro $\mathscr{D}_\mathrm{f}$ merges $p' \cdot e_i$ and $q' \cdot e_i$ (since e merges p and q). Since Z is finite, there exists a connecting morphism (a_{ji}, b_{ji}) of D such that $p' \cdot e_i$ and $q' \cdot e_i$ are merged by a_{ji}.



Therefore

$$p' \cdot b_{ji} \cdot e_j = p' \cdot e_i \cdot a_{ji} = q' \cdot e_i \cdot a_{ji} = q' \cdot b_{ji} \cdot e_j.$$

Since e_j is an epimorphism in $\mathscr{D}_\mathrm{f}$, this implies $p' \cdot b_{ji} = q' \cdot b_{ji}$. Thus

$$p = p' \cdot b_i = p' \cdot b_{ji} \cdot b_j = q' \cdot b_{ji} \cdot b_j = q' \cdot b_i = q.$$

(2) All morphisms of $\widehat{\mathcal{M}}$ are monic. Indeed, given $m\colon X \to Y$ in $\widehat{\mathcal{M}}$, we have a limit cone of a cofiltered diagram D in $(\mathrm{Pro}\,\mathscr{D}_\mathrm{f})^\to$ as follows:

$$\begin{array}{ccc} X & \xrightarrow{m} & Y \\ {\scriptstyle a_i}\downarrow & & \downarrow{\scriptstyle b_i} \\ A_i & \rightarrowtail_{m_i} & B_i \end{array} \qquad (i \in I)$$

where $m_i \in \mathcal{M}_\mathrm{f}$ for each $i \in I$. Suppose that $f, g\colon Z \to X$ with $m \cdot f = m \cdot g$ are given. Express Z as a cofiltered limit $z_j\colon Z \twoheadrightarrow Z_j$ $(j \in J)$ of finite objects with epimorphic limit projections z_j. For each

$i \in I$, since A_i is finitely copresentable, we obtain a factorization of $a_i \cdot f$ and $a_i \cdot g$ through some z_{j_i}, say $f_i \cdot z_{j_i} = f$ and $g_i \cdot z_{j_i} = g$.

$$\begin{array}{ccccc} Z & \overset{f}{\underset{g}{\rightrightarrows}} & X & \xrightarrow{m} & Y \\ {\scriptstyle z_{j_i}}\big\downarrow & & \big\downarrow{\scriptstyle a_i} & & \big\downarrow{\scriptstyle b_i} \\ Z_{j_i} & \overset{f_i}{\underset{g_i}{\rightrightarrows}} & A_i & \underset{m_i}{\rightarrowtail} & B_i \end{array}$$

From $m \cdot f = m \cdot g$ it follows that $m_i \cdot f_i \cdot z_{j_i} = m_i \cdot g_i \cdot z_{j_i}$ for each i. This implies $m_i \cdot f_i = m_i \cdot g_i$ because z_{j_i} is epic, and thus $f_i = g_i$ because m_i is monic in $\mathscr{D}_\mathrm{f}$. Therefore, $a_i \cdot f = a_i \cdot g$ for each i, thus $f = g$ because the limit projections a_i are collectively monic.

(3) Every morphism $g\colon X \to Y$ of $\mathrm{Pro}\,\mathscr{D}_\mathrm{f}$ has an $(\widehat{\mathcal{E}}, \widehat{\mathcal{M}})$-factorization. Indeed, $(\mathrm{Pro}\,\mathscr{D}_\mathrm{f})^\to$ is the pro-completion of $\mathscr{D}_\mathrm{f}^\to$; see [3, Cor. 1.54] for the dual statement. Thus, there exists a cofiltered diagram $R\colon I \to \mathscr{D}_\mathrm{f}^\to$ with limit g. Let the following morphisms

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ {\scriptstyle a_i}\big\downarrow & & \big\downarrow{\scriptstyle b_i} \\ A_i & \xrightarrow[g_i]{} & B_i \end{array} \qquad (i \in I)$$

form the limit cone. Factorize g_i into an $\mathcal{E}$-morphism $e_i\colon A_i \twoheadrightarrow C_i$ and followed by an $\mathcal{M}$-morphism $m_i\colon C_i \rightarrowtail B_i$. Since $\mathscr{D}_\mathrm{f}$ is closed under subobjects, we have $e_i \in \mathcal{E}_\mathrm{f}$ and $m_i \in \mathcal{M}_\mathrm{f}$. Diagonal fill-in yields a diagram $\overline{R}\colon I \to \mathscr{D}_\mathrm{f}$ with objects C_i, $i \in I$, and connecting morphisms derived from those of R. Let $Z \in \mathrm{Pro}\,\mathscr{D}_\mathrm{f}$ be a limit of $\overline{R}$ with the limit cone

$$c_i\colon Z \to C_i \quad (i \in I).$$

Then there are unique morphisms $e = \lim e_i \in \widehat{\mathcal{E}}$, and $m = \lim m_i \in \widehat{\mathcal{M}}$ such that the following diagrams commute for all $i \in I$:

$$\begin{array}{ccccc} X & & \xrightarrow{\quad g \quad} & & Y \\ & \searrow^{e} & & \nearrow^{m} & \\ {\scriptstyle a_i}\big\downarrow & & Z & & \big\downarrow{\scriptstyle b_i} \\ & & \big\downarrow{\scriptstyle c_i} & & \\ A_i & \underset{e_i}{\twoheadrightarrow} & C_i & \underset{m_i}{\rightarrowtail} & B_i \end{array}$$

(4) We verify the diagonal fill-in property. Let a commutative square

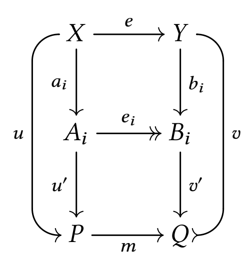
$$\begin{array}{ccc} X & \xrightarrow{e} & Y \\ {\scriptstyle u}\big\downarrow & & \big\downarrow{\scriptstyle v} \\ P & \xrightarrow[m]{} & Q \end{array}$$

with $e \in \widehat{\mathcal{E}}$ and $m \in \widehat{\mathcal{M}}$ be given.

(4a) Assume first that $m \in \mathcal{M}_\mathrm{f}$. Express e as a cofiltered limit of objects $e_i \in \mathcal{E}_\mathrm{f}$ with the following limit cone:

$$\begin{CD} X @>{e}>> Y \\ @V{a_i}VV @VV{b_i}V \\ A_i @>>{e_i}> B_i \end{CD}$$

Since P is finite and $X = \lim A_i$ is a cofiltered limit, u factorizes through some a_i. Analogously for v and some b_i; the index i can be chosen to be the same since the diagram is cofiltered. Thus we have morphisms u', v' such that in the following diagram the left-hand triangle and the right-hand one commute:



Without loss of generality, we can assume that the lower part also commutes. Indeed, Q is finite and the limit map a_i merges the lower part:

$$(m \cdot u') \cdot a_i = m \cdot u = v \cdot e = v' \cdot b_i \cdot e = (v' \cdot e_i) \cdot a_i.$$

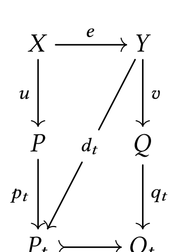
Since our diagram is cofiltered, some connecting morphism from A_j to A_i also merges the lower part. Hence, by choosing j instead of i we could get the lower part commutative.

Since $e_i \in \mathcal{E}$ and $m \in \mathcal{M}$, we use diagonal fill-in to get a morphism $d \colon B_i \longrightarrow B$ with $d \cdot e_i = u'$ and $m \cdot d = v'$. Then $d \cdot b_i \colon Y \longrightarrow P$ is the desired diagonal in the original square.

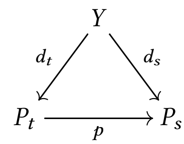
(4b) Now suppose that $m \in \widehat{\mathcal{M}}$ is arbitrary, i.e. a cofiltered limit a diagram D whose objects are morphisms m_t of $\mathcal{M}_\mathrm{f}$ with a limit cone as follows:

$$\begin{CD} P @>{m}>> Q \\ @V{p_t}VV @VV{q_t}V \\ P_t @>>{m_t}> Q_t \end{CD} \qquad (t \in T)$$

For each t we have, due to item (4a) above, a diagonal fill-in



Given a connecting morphism $(p, q)\colon m_t \longrightarrow m_s$ $(t, s \in T)$ of the diagram D, the following triangle



commutes, that is, all d_t form a cone of the diagram $D_0 \cdot D$, where $D_0\colon \mathscr{D}_{\mathrm{f}}^{\rightarrow} \longrightarrow \mathscr{D}_{\mathrm{f}}$ is the domain functor, with limit $p_t\colon P \longrightarrow P_t$ $(t \in T)$. Indeed, e is epic by item (1), and from the fact that $p_s = p \cdot p_t$ we obtain

$$(p \cdot d_t) \cdot e = p \cdot p_t \cdot u = p_s \cdot u = d_s \cdot e.$$

Thus, there exists a unique $d\colon Y \longrightarrow P$ with $d_t = p_t \cdot d$ for all $t \in T$. This is the desired diagonal: $u = d \cdot e$ follows from $(p_t)_{t \in T}$ being collectively monic, since

$$p_t \cdot u = d_t \cdot e = p_t \cdot d \cdot e.$$

□

Proposition 3.18. *Let $\mathscr{D}$ be a full subcategory of* Σ-**Str** *closed under products and subobjects. Then in* Pro $\mathscr{D}_{\mathrm{f}} \subseteq$ **Stone**$\mathscr{D}$ *we have*

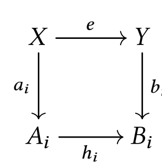
$$\widehat{\mathcal{E}} = \textit{surjective morphisms, and}$$

$$\widehat{\mathcal{M}} = \textit{relation-reflecting injective morphisms,}$$

cf. Remark 2.15(2).

Proof. (1) Let $e\colon X \longrightarrow Y$ be a surjective morphism of Pro $\mathscr{D}_{\mathrm{f}}$. We shall prove that $e \in \widehat{\mathcal{E}}$ by expressing it as a cofiltered limit of a diagram of quotients in $\mathscr{D}_{\mathrm{f}}^{\rightarrow}$. In **Stone**$\mathscr{D}$ we have the factorization system $(\mathcal{E}_0, \mathcal{M}_0)$ where $\mathcal{E}_0$ = surjective homomorphisms, and $\mathcal{M}_0$ = injective relation-reflecting homomorphisms. This follows from Remark 2.15 and the fact that $\mathscr{D}$, being closed under subobjects in Σ-**Str**, inherits the factorization system Σ-**Str**.

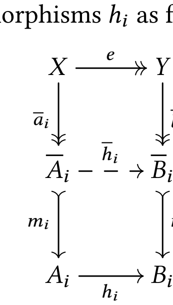
Since Pro $\mathscr{D}_{\mathrm{f}}$ is the closure of $\mathscr{D}_{\mathrm{f}}$ under cofiltered limits in **Stone**$(\mathscr{D})$ by Corollary 2.12, also $(\text{Pro}\,\mathscr{D}_{\mathrm{f}})^{\rightarrow} = \text{Pro}(\mathscr{D}_{\mathrm{f}}^{\rightarrow})$ is the closure of $\mathscr{D}_{\mathrm{f}}^{\rightarrow}$ under cofiltered limits in $(\mathbf{Stone}\mathscr{D})^{\rightarrow}$. Thus for e there exists a cofiltered diagram D in $\mathscr{D}_{\mathrm{f}}^{\rightarrow}$ of morphisms $h_i\colon A_i \longrightarrow B_i$ $(i \in I)$ of $\mathscr{D}_{\mathrm{f}}$ with a limit cone in $(\mathbf{Stone}\mathscr{D})^{\rightarrow}$ as follows:



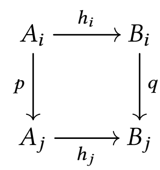
Using the factorization system $(\mathcal{E}_0, \mathcal{M}_0)$ we factorize

$$a_i = m_i \cdot \overline{a}_i \quad \text{and} \quad b_i = n_i \cdot \overline{b}_i \quad \text{for } i \in I,$$

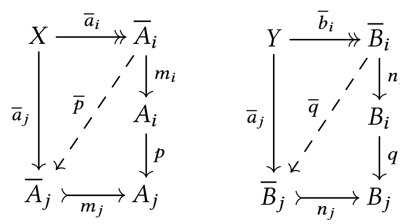
and use the diagonal fill-in to define morphisms $\overline{h}_i$ as follows:



We obtain a diagram $\overline{D}$ with objects $\overline{h}_i : \overline{A}_i \to \overline{B}_i$ ($i \in I$) in $(\mathbf{Stone}\mathscr{D})^{\to}$. Connecting morphisms are derived from those of D: given $(p, q): h_i \to h_j$ in D



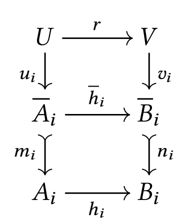
the diagonal fill-in property yields morphisms $\overline{p}$ and $\overline{q}$ as follows:



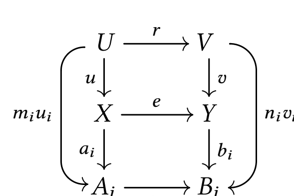
It is easy to see that $(\overline{p}, \overline{q})$ is a morphism from $\overline{h}_i$ to $\overline{h}_j$ in $(\mathbf{Stone}\mathscr{D})^{\to}$. This yields a cofiltered diagram $\overline{D}$. Since $\overline{h}_i \cdot \overline{a}_i = \overline{b}_i \cdot e$ is surjective, it follows that $\overline{h}_i$ is also surjective. We claim that the morphisms

$$(\overline{a}_i, \overline{b}_i) : e \to \overline{h}_i \quad (i \in I)$$

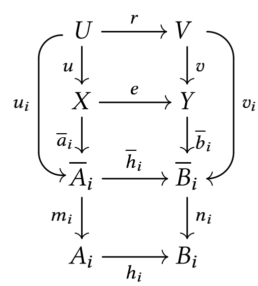
form a limit cone of $\overline{D}$. To see this, note first that since the morphisms $(a_i, b_i): e \to h_i$, $i \in I$, form a cone of D and all m_i and n_i are monic, the morphisms $(\overline{a}_i, \overline{b}_i)$, $i \in I$, form a cone of $\overline{D}$. Now let another cone be given with domain $r: U \to V$ as follows:



Then we get a cone of D for all $i \in I$ by the morphisms $(m_i u_i, n_i v_i): r \to h_i$. The unique factorization (u, v) through the limit cone of D:



is a factorization of (u_i, v_i) through the cone $(\overline{a}_i, \overline{b}_i)$. Indeed, in the following diagram

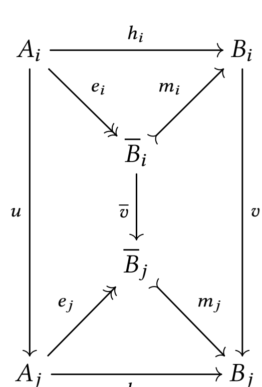


the desired equality $v_i = \overline{b}_i v$ follows since n_i is monic; analogously for $u_i = \overline{a}_i u$. The uniqueness of the factorization (u, v) also follows from the last diagram: if the upper left-hand and right-hand parts commute, then (u, v) is a factorization of the cone $(m_i u_i, n_i v_i)$ through the limit cone of D. Thus, it is unique.

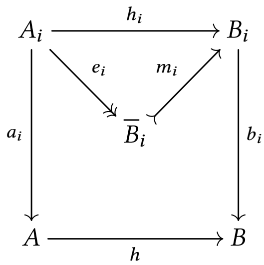
(2) Conversely, every cofiltered limit of quotients in $\mathscr{D}_\mathrm{f}^{\to}$ is surjective in Pro $\mathscr{D}_\mathrm{f}$. Indeed, cofiltered limits in Pro $\mathscr{D}_\mathrm{f}$ are formed in **Stone**(Σ-**Str**) by Corollary 2.12, and the forgetful functor into **Stone** thus preserves them. Hence the same is true about the forgetful functor from $(\mathrm{Pro}\, \mathscr{D}_\mathrm{f})^{\to}$ to $\mathbf{Stone}^{\to}$. Thus, the claim follows from Lemma 2.17.

(3) We show that every morphism of Pro $\mathscr{D}_\mathrm{f}$ which is monic and reflects relations is an element of $\widehat{\mathcal{M}}$.

(3a) We first prove a property of filtered colimits in $\mathbf{BA}^{\to}$. Let D be a filtered diagram with objects $h_i \colon A_i \to B_i$ $(i \in I)$ in $\mathbf{BA}^{\to}$. Let $h_i = m_i \cdot e_i$ be the factorization of h_i into an epimorphism $e_i \colon A_i \twoheadrightarrow \overline{B}_i$ followed by a monomorphism $m_i \colon \overline{B}_i \rightarrowtail B_i$ in **BA**. Using diagonal fill-in we get a filtered diagram $\overline{D}$ with objects e_i $(i \in I)$ and with connecting morphisms $(u, \overline{v}) \colon e_i \to e_j$ derived from the connecting morphisms $(u, v) \colon h_i \to h_j$ of D using diagonal fill-in:



Our claim is that if the colimit $h = \operatorname{colim} h_i$ in $\mathbf{BA}^{\to}$ is an epimorphism of $\mathbf{BA}$, then one has $h = \operatorname{colim} e_i$. To see this, suppose that a colimit cocone of D is given as follows:



Then we prove that $\overline{D}$ has the colimit cocone $(a_i, b_i \cdot m_i)$, $i \in I$. Indeed, since $A = \operatorname{colim} A_i$ with colimit cocone (a_i), all we need to verify is that $B = \operatorname{colim} \overline{B}_i$ with cocone $(b_i \cdot m_i)$. This cocone is collectively epic because every element x of B has the form $x = h(y)$ for some $y \in A$, using that h is epic by hypothesis, and that the cocone (a_i) is collectively epic. The diagram $\overline{D}$ is filtered, thus, to prove that $B = \operatorname{colim} \overline{B}_i$, we only need to verify that whenever a pair $x_1, x_2 \in \overline{B}_i$ (for some $i \in I$) is merged by $b_i \cdot m_i$, there exists a connecting morphism $\overline{v}\colon \overline{B}_i \to \overline{B}_j$ merging x_1, x_2. Since m_i is monic and $B = \operatorname{colim} B_i$, some connecting morphism $v\colon B_i \to B_j$ merges $m_i(x_1)$ and $m_i(x_2)$. Then

$$m_j \cdot \overline{v}(x_1) = v \cdot m_i(x_1) = v \cdot m_i(x_2) = m_j \cdot \overline{v}(x_2),$$

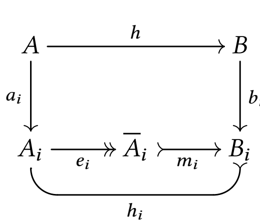
whence $\overline{v}(x_1) = \overline{v}(x_2)$ because m_j is monic.

(3b) Denote by $W\colon \mathbf{Stone}(\Sigma\text{-}\mathbf{Str}) \to \mathbf{Set}^{\mathcal{S}}$ the forgetful functor mapping a Stone-topological Σ-structure to its underlying sorted set. Moreover, letting $\Sigma_{\mathsf{rel}} \subseteq \Sigma$ denote the set of all relation symbols in Σ, we have the forgetful functors

$$W_r\colon \mathbf{Stone}(\Sigma\text{-}\mathbf{Str}) \to \mathbf{Set} \quad (r \in \Sigma_{\mathsf{rel}})$$

assigning to every object A the corresponding subset $r_A \subseteq A^{s_1} \times \cdots \times A^{s_n}$. From the description of limits in Σ-**Str** in Remark 2.6, it follows that the functors W and W_r ($r \in \Sigma_{\mathsf{rel}}$) collectively preserve and reflect limits. That is, given a diagram D in $\mathbf{Stone}(\Sigma\text{-}\mathbf{Str})$, a cocone of D is a limit cone if and only if its image under W is a limit cone of $W \cdot D$ and its image under W_r is a limit cone of $W_r \cdot D$ for all $r \in \Sigma_{\mathsf{rel}}$.

(3c) We are ready to prove that if $h\colon A \to B$ in $\operatorname{Pro} \mathscr{D}_{\mathsf{f}}$ is a relation-reflecting monomorphism, then $h \in \widehat{\mathcal{M}}$. We have a cofiltered diagram D in $\mathscr{D}_{\mathsf{f}}^{\to}$ with objects $h_i\colon A_i \to B_i$ and a limit cone $(a_i, b_i)\colon h_i \to h$ ($i \in I$). Let $h_i = m_i \cdot e_i$ be the image factorization in Σ-**Str**.



It is our goal to prove that $h = \lim_{i \in I} m_i$. More precisely: we have m_i in $\mathscr{D}_{\mathsf{f}}^{\to}$ and diagonal fill-in yields a cofiltered diagram $\overline{D}$ of these objects in $\mathscr{D}_{\mathsf{f}}^{\to}$. We will prove that $(e_i \cdot a_i, b_i)\colon h \to m_i$ ($i \in I$) is a limit cone. By part (3b) above it suffices to show that the images of that cone under $W^{\to}$ and $W_r^{\to}$ ($r \in \Sigma_{\mathsf{rel}}$) are limit cones.

For $W^{\rightarrow}$ just dualize (3a): from the fact that $Wh = \lim Wh_i$ we derive $Wh = \lim Wm_i$. Given $r\colon s_1, \dots, s_n$ in Σ_{rel}, we know that r_A consists of the n-tuples $(x_1, \dots, x_n)$ with $(a_i(x_1), \dots, a_i(x_n)) \in r_{A_i}$ for every $i \in I$ (see Remark 2.6). In particular, for $(x_1, \dots, x_n) \in r_A$ we have $(e_i \cdot a_i(x_1), \dots, e_i \cdot a_i(x_n)) \in r_{\overline{A_i}}$. Conversely, given $(x_1, \dots, x_n)$ with the latter property, then $(m_i \cdot e_i \cdot a_i(x_1), \dots, m_i \cdot e_i \cdot a_i(x_n)) \in r_{B_i}$, i.e. $(b_i \cdot h(x_1), \dots, b_i \cdot h(x_n)) \in r_{B_i}$ for all $i \in I$. Since $B = \lim B_i$, this implies $(h(x_1), \dots, h(x_n)) \in r_B$, whence $(x_1, \dots, x_n) \in r_A$ because h is relation-reflecting.

(4) It remains to prove that every morphism $m \in \widehat{\mathcal{M}}$ is a relation-reflecting monomorphism. Let a cofiltered limit cone be given as follows:

$$\begin{array}{ccc} A & \xrightarrow{\ m\ } & B \\ {\scriptstyle a_i}\downarrow & & \downarrow{\scriptstyle b_i} \\ A_i & \underset{m_i}{\rightarrowtail} & B_i \end{array} \qquad (i \in I)$$

where each m_i lies in $\mathcal{M}_{\mathsf{f}}$, i.e. is a relation-reflecting monomorphism in $\mathscr{D}_{\mathsf{f}}$. Then m is monic: given $x \neq y$ in A, there exists $i \in I$ with $a_i(x) \neq a_i(y)$ because the limit projections a_i are collectively monic. Since m_i is monic, this implies $b_i \cdot m(x) \neq b_i \cdot m(y)$, whence $m(x) \neq m(y)$.

Moreover, for every relation symbol $r\colon s_1, \dots, s_n$ in Σ and $(x_1, \dots, x_n) \in A^{s_1} \times \cdots \times A^{s_n}$, we have that

$$(x_1, \dots, x_n) \in r_A \quad \text{iff} \quad (m(x_1), \dots, m(x_n)) \in r_B.$$

Indeed, the *only if* direction follows from the fact that the maps $m_i \cdot a_i$ preserve relations and the maps b_i collectively reflect them. For the *if* direction, suppose that $(m(x_1), \dots m(x_n)) \in r_B$. Since for every $i \in I$ the morphism b_i preserves relations and m_i reflects them, we get $(a_i(x_1), \dots, a_i(x_n)) \in r_{A_i}$ for every i. Since the maps a_i collectively reflect relations, this implies $(x_1, \dots, x_n) \in r_A$. □

Definition 3.19. The factorization system $(\mathcal{E}, \mathcal{M})$ of $\mathscr{D}$ is called *profinite* if $\mathcal{E}$ is closed in $\mathscr{D}^{\rightarrow}$ under cofiltered limits of finite quotients; that is, for every cofiltered diagram D in $\mathscr{D}^{\rightarrow}$ whose objects are elements of $\mathcal{E}_{\mathsf{f}}$, the limit of D in $\mathscr{D}^{\rightarrow}$ lies in $\mathcal{E}$.

Example 3.20. For every full subcategory $\mathscr{D} \subseteq \Sigma$-**Str** closed under limits and subobjects, the factorization system of surjective morphisms and relation-reflecting injective morphisms is profinite. This follows from Lemma 2.17 and the fact that limits in $\mathscr{D}$ are formed at the level of underlying sets (see Remark 2.6).

Proposition 3.21. *If the factorization system $(\mathcal{E}, \mathcal{M})$ of $\mathscr{D}$ is profinite, the following holds:*

(1) *The forgetful functor $V\colon \operatorname{Pro} \mathscr{D}_{\mathsf{f}} \to \mathscr{D}$ is faithful and satisfies $V(\widehat{\mathcal{E}}) \subseteq \mathcal{E}$.*

(2) *For every $\mathcal{E}$-projective object $X \in \mathscr{D}$, the object $\widehat{X} \in \widehat{\mathscr{D}}$ is $\widehat{\mathcal{E}}$-projective.*

(3) *Every object of $\mathscr{D}_{\mathsf{f}}$ is an $\widehat{\mathcal{E}}$-quotient of some $\widehat{\mathcal{E}}$-projective object in* $\operatorname{Pro} \mathscr{D}_{\mathsf{f}}$.

Proof. (1) $V(\widehat{\mathcal{E}}) \subseteq \mathcal{E}$ is clear: given $e \in \widehat{\mathcal{E}}$ expressed as a cofiltered limit of finite quotients e_i, $i \in I$, in $(\operatorname{Pro} \mathscr{D}_{\mathsf{f}})^{\rightarrow}$, then since V is cofinitary, we see that Ve is a cofiltered limit of $Ve_i = e_i$ in $\mathscr{D}^{\rightarrow}$, thus $Ve \in \mathcal{E}$ by the definition of a profinite factorization system.

To prove that V is faithful, recall that a right adjoint is faithful if and only if each component of its counit is epic. Thus, it suffices to prove that $\varepsilon_D \in \widehat{\mathcal{E}}$ (and use that by Proposition 3.17 every $\widehat{\mathcal{E}}$-morphism is epic). The triangles defining ε_D in Construction 3.14(4) can be restricted to those with $a \in \widehat{\mathcal{E}}$. Indeed, in the slice category $D/\mathscr{D}_{\mathsf{f}}$ all objects $a\colon D \to A$ in $\widehat{\mathcal{E}}$ form an initial subcategory.

Now given such a triangle with $a \in \widehat{\mathcal{E}}$ we know that $Va \in \mathcal{E}$. Thus all those objects A form a cofiltered diagram with connecting morphisms in $\mathcal{E}$. Moreover, $\widehat{Va} \in \widehat{\mathcal{E}}$ by Remark 3.16(1). This implies $\varepsilon_D \in \widehat{\mathcal{E}}$ by Remark 3.16(3) .

(2) Let X be an $\mathcal{E}$-projective object. To show that $\widehat{X}$ is $\widehat{\mathcal{E}}$-projective, suppose that a quotient $e\colon A \twoheadrightarrow B$ in $\widehat{\mathcal{E}}$ and a morphism $f\colon \widehat{X} \to B$ are given. Since $\widehat{(-)}$ is left adjoint to V and $V(\widehat{\mathcal{E}}) \subseteq \mathcal{E}$, the morphism f has an adjoint transpose $f^*\colon X \to VB$ that factorizes through VE via g^* for some $g\colon \widehat{X} \to A$. Then $e \cdot g = f$, which proves that $\widehat{X}$ is projective.

$$
\begin{array}{ccc}
 & X & \\
{}^{g^*}\swarrow & & \searrow^{f^*} \\
VA & \xrightarrow[Ve]{} & VB
\end{array}
\qquad \text{iff} \qquad
\begin{array}{ccc}
 & \widehat{X} & \\
{}^{g}\swarrow & & \searrow^{f} \\
A & \xrightarrow[e]{} & B
\end{array}
$$

(3) Given $A \in \mathscr{D}_{\mathrm{f}}$, by Assumption 3.1 there exists an $\mathcal{E}$-projective object $X \in \mathscr{D}$ and a quotient $e\colon X \twoheadrightarrow A$. The limit projection $\widehat{e}\colon \widehat{X} \twoheadrightarrow A$ lies in $\widehat{\mathcal{E}}$ by Remark 3.16(1), and item (2) above shows that $\widehat{X}$ is $\widehat{\mathcal{E}}$-projective. □

We are ready to prove the following general form of the Reiterman Theorem: given the factorization system $(\widehat{\mathcal{E}}, \widehat{\mathcal{M}})$ on the pro-completion of $\mathscr{D}_{\mathrm{f}}$, we have the concept of an equation in Pro $\mathscr{D}_{\mathrm{f}}$. We call it a profinite equation for $\mathscr{D}$, and prove that pseudovarieties in $\mathscr{D}$ are precisely the classes in $\mathscr{D}_{\mathrm{f}}$ that can be presented by profinite equations.

Definition 3.22. A *profinite equation* is an equation in Pro $\mathscr{D}_{\mathrm{f}}$, i.e. a morphism $e\colon X \twoheadrightarrow E$ in $\widehat{\mathcal{E}}$ whose domain X is $\widehat{\mathcal{E}}$-projective. It is *satisfied* by a finite object D provided that D is injective w.r.t. e.

Theorem 3.23 (Generalized Reiterman Theorem)**.** *Given a profinite factorization system on $\mathscr{D}$, a class of finite objects is a pseudovariety iff it can be presented by profinite equations.*

Proof. Every class $\mathcal{V} \subseteq \mathscr{D}_{\mathrm{f}}$ presented by profinite equations is a pseudovariety: this is proved precisely as (1) in Proposition 3.8.

Conversely, every pseudovariety can be presented by profinite equations. Indeed, following the same proposition, it suffices to construct, for every pseudoequation $e_i\colon X \twoheadrightarrow E_i$ $(i \in I)$, a profinite equation satisfied by the same finite objects.

For every $i \in I$, we have the corresponding limit projection

$$\widehat{e_i}\colon \widehat{X} \twoheadrightarrow E_i \quad \text{with } e_i = V\widehat{e_i} \cdot \eta_X.$$

Let R be the diagram in $\mathscr{D}_{\mathrm{f}}$ of objects E_i. The connecting morphism $k\colon E_i \to E_j$ are given by the factorization

$$
\begin{array}{ccc}
 & X & \\
{}^{e_i}\swarrow & & \searrow^{e_j} \\
E_i & \xrightarrow[k]{} & E_j
\end{array}
$$

iff $e_j \leq e_i$. Since the pseudoequation is closed under finite joins, R is cofiltered. Form the limit of R in Pro $\mathscr{D}_{\mathrm{f}}$ with the limit cone

$$p_i\colon E \twoheadrightarrow E_i \quad (i \in I).$$

The morphisms $\widehat{e_i}$ above form a cone of R: given $e_j = k \cdot e_i$, then $V\widehat{e_j} \cdot \eta_X = V(\widehat{k \cdot e_i}) \cdot \eta_X = k \cdot V\widehat{e_i} \cdot \eta_X$ implies $\widehat{e_j} = k \cdot \widehat{e_i}$ by the universal property of η_X. Thus we have a unique morphism $e\colon \widehat{X} \twoheadrightarrow E$

making the following triangles commutative:

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{e} & E \\ & \searrow^{\widehat{e_i}} & \downarrow{\scriptstyle p_i} \\ & & E_i \end{array} \qquad (i \in I)$$

The connecting morphisms of R lie in $\mathcal{E}$ (since $k \cdot e_i \in \mathcal{E}$ implies $k \in \mathcal{E}$). Thus each $\widehat{e_i}$ lies in $\widehat{\mathcal{E}}$ since $e_i \in \mathcal{E}$, see Remark 3.16(2). Therefore, $e \in \widehat{\mathcal{E}}$ by Remark 3.16(3). Since $\widehat{X}$ is $\widehat{\mathcal{E}}$-projective by Proposition 3.21, we have thus obtained a profinite equation $e\colon \widehat{X} \twoheadrightarrow E$.

We are going to prove that a finite object A satisfies the pseudoequation $(e_i)_{i\in I}$ iff it satisfies the profinite equation e.

(1) Let A satisfy the pseudoequation (e_i). For every morphism $f\colon \widehat{X} \to A$ we present a factorization through e. The morphism $Vf \cdot \eta_X$ factorizes through some e_j, $j \in I$:

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & V\widehat{X} \\ {\scriptstyle e_j}\downarrow & & \downarrow{\scriptstyle Vf} \\ E_j & \xrightarrow[g]{} & A \end{array}$$

Since $e_j = V\widehat{e_j} \cdot \eta_X$, we get $V(g \cdot \widehat{e_j}) \cdot \eta_X = Vf \cdot \eta_X$. By the universal property of η_X this implies

$$g \cdot \widehat{e_j} = f.$$

The desired factorization is $g \cdot p_j$:

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{e} & E \\ {\scriptstyle f}\downarrow & \searrow^{\widehat{e_j}} & \downarrow{\scriptstyle p_j} \\ A & \xleftarrow[g]{} & E_j \end{array}$$

(2) Let A satisfy the profinite equation e. For every morphism $h\colon X \to A$ we find a factorization through some e_j. The morphism $\widehat{h}\colon \widehat{X} \to A$ factorizes through e:

$$\widehat{h} = u \cdot e \quad \text{with } u\colon E \to A.$$

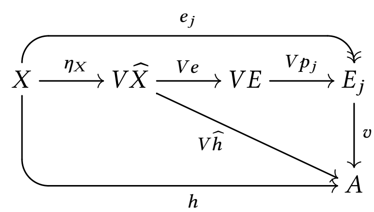
The codomain of u is finite, thus, u factorizes through one of the limit projection of E, i.e.

$$u = v \cdot p_j \quad \text{with } j \in I \text{ and } v\colon E_j \to A.$$

This gives the following commutative diagram:

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{e} & E \\ {\scriptstyle \widehat{h}}\downarrow & \swarrow_{u} & \downarrow{\scriptstyle p_j} \\ A & \xleftarrow[v]{} & E_j \end{array} \tag{3.2}$$

That v is the desired factorization of h is now shown using the following diagram:



Indeed, the upper part commutes since since

$$e_j = V\widehat{e}_j \cdot \eta_X = Vp_j \cdot Ve \cdot \eta_X,$$

the lower left-hand part commutes since $h = V\widehat{h} \cdot \eta_X$, and for the remaining lower right-hand part apply V to (3.2) and use that $Vv = v$ since v lies in $\mathscr{D}_\mathsf{f}$. □

4 PROFINITE MONAD

In the present section we establish the main result of our paper: a generalization of Reiterman's theorem from algebras over a signature to algebras for a given monad $\mathbf{T}$ in a category $\mathscr{D}$ (Theorem 4.20). To this end, we introduce and investigate the *profinite monad* $\widehat{\mathbf{T}}$ associated to the monad $\mathbf{T}$. It provides an abstract perspective on the formation of spaces of profinite words or profinite terms and serves as key technical tool for our categorical approach to profinite algebras.

Assumption 4.1. Throughout this section, $\mathscr{D}$ is a category satisfying Assumption 3.1, and $\mathbf{T} = (T, \mu, \eta)$ is a monad on $\mathscr{D}$ preserving quotients, i.e. $T(\mathcal{E}) \subseteq \mathcal{E}$.

We denote by $\mathscr{D}^\mathbf{T}$ the category of $\mathbf{T}$-algebras and $\mathbf{T}$-homomorphisms, and by $\mathscr{D}_\mathsf{f}^\mathbf{T}$ the full subcategory of all *finite algebras*, i.e. $\mathbf{T}$-algebras whose underlying object lies in $\mathscr{D}_\mathsf{f}$.

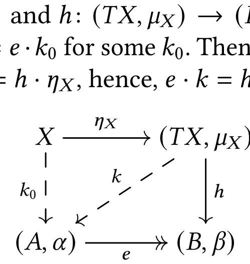
Remark 4.2. The category $\mathscr{D}^\mathbf{T}$ satisfies Assumption 3.1. More precisely:

(1) Since $\mathbf{T}$ preserves quotients, the factorization system of $\mathscr{D}$ lifts to $\mathscr{D}^\mathbf{T}$: every homomorphism in $\mathscr{D}^\mathbf{T}$ factorizes as a homomorphism in $\mathcal{E}$ followed by one in $\mathcal{M}$. When speaking about *quotient algebras* and *subalgebras* of $\mathbf{T}$-algebras, we refer to this lifted factorization system $(\mathcal{E}^\mathbf{T}, \mathcal{M}^\mathbf{T})$.

(2) Since $\mathscr{D}$ is complete, so is $\mathscr{D}^\mathbf{T}$ with limits created by the forgetful functor into $\mathscr{D}$.

(3) The category $\mathscr{D}_\mathsf{f}^\mathbf{T}$ is closed under finite products and subalgebras, since $\mathscr{D}_\mathsf{f}$ is closed under finite products and subobjects.

(4) For every $\mathcal{E}$-projective object X, the free algebra (TX, μ_X) is $\mathcal{E}^\mathbf{T}$-projective. Indeed, given $\mathbf{T}$-homomorphisms $e\colon (A, \alpha) \twoheadrightarrow (B, \beta)$ and $h\colon (TX, \mu_X) \to (B, \beta)$ with $e \in \mathcal{E}$, then $h \cdot \eta_X \colon X \to B$ factorizes through e in $\mathscr{D}$, i.e. $h \cdot \eta_X = e \cdot k_0$ for some k_0. Then the $\mathbf{T}$-homomorphism $k\colon (TX, \mu_X) \to (A, \alpha)$ extending k_0 fulfils $e \cdot k \cdot \eta_X = h \cdot \eta_X$, hence, $e \cdot k = h$ by the universal property of η_X.



It follows that every finite algebra is a quotient of an $\mathcal{E}^\mathbf{T}$-projective $\mathbf{T}$-algebra.

Notation 4.3. The forgetful functor of $\mathscr{D}_\mathsf{f}^\mathbf{T}$ into $\operatorname{Pro} \mathscr{D}_\mathsf{f}$ is denoted by

$$K\colon \mathscr{D}_\mathsf{f}^\mathbf{T} \to \operatorname{Pro} \mathscr{D}_\mathsf{f}$$

For example, if $\mathscr{D} = \Sigma$-**Str**, then K assigns to every finite $\mathbf{T}$-algebra its underlying Σ-structure, equipped with the discrete topology.

Remark 4.4. For any functor $K\colon \mathscr{A} \to \mathscr{C}$, the right Kan extension

$$R = \mathsf{Ran}_K K\colon \mathscr{C} \to \mathscr{C}$$

can be naturally equipped with the structure of a monad. Its unit and multiplication are given by

$$\widehat{\eta} = (\mathsf{id}_K)^\dagger\colon \mathsf{Id} \to R \qquad \text{and} \qquad \widehat{\mu} = (\varepsilon \cdot R\varepsilon)^\dagger\colon RR \to R,$$

where $\varepsilon\colon RK \to K$ denotes the universal natural transformation and $(-)^\dagger$ is defined as in Remark 3.12. The monad $(R, \widehat{\eta}, \widehat{\mu})$ is called the *codensity monad* of K, see e.g. Linton [13].

Definition 4.5. The *profinite monad*

$$\widehat{\mathbf{T}} = (\widehat{T}, \widehat{\mu}, \widehat{\eta})$$

of the monad $\mathbf{T}$ is the codensity monad of the forgetful functor $K\colon \mathscr{D}_\mathsf{f}^\mathbf{T} \to \mathsf{Pro}\,\mathscr{D}_\mathsf{f}$.

Construction 4.6. Since $\mathsf{Pro}\,\mathscr{D}_\mathsf{f}$ is complete and $\mathscr{D}_\mathsf{f}^\mathbf{T}$ is small, the limit formula for right Kan extensions (see Remark 3.12) yields the following concrete description of the profinite monad:

(1) To define the action of $\widehat{T}$ on an object X, form the coslice category X/K of all morphisms $a\colon X \to K(A,\alpha)$ with $(A,\alpha) \in \mathscr{D}_\mathsf{f}^\mathbf{T}$. The projection functor $Q_X\colon X/K \to \mathsf{Pro}\,\mathscr{D}_\mathsf{f}$, mapping a to A, has a limit

$$\widehat{T}X = \lim Q_X.$$

The limit cone is denoted as follows:

$$\frac{X \xrightarrow{a} K(A,\alpha)}{\widehat{T}X \xrightarrow{\alpha_a^+} A}$$

For every finite $\mathbf{T}$-algebra (A,α), we write

$$\alpha^+\colon \widehat{T}A \to A$$

instead of $\alpha^+_{\mathsf{id}_A}$.

(2) The action of $\widehat{T}$ on morphisms $f\colon Y \to X$ is given by the following commutative triangles

$$\begin{array}{ccc} \widehat{T}Y & \xrightarrow{\widehat{T}f} & \widehat{T}X \\ & {\scriptstyle \alpha^+_{af}}\searrow \quad \swarrow{\scriptstyle \alpha^+_a} & \\ & A & \end{array} \qquad \text{for all } a\colon X \to K(A,\alpha).$$

(3) The unit $\widehat{\eta}\colon \mathsf{Id} \to \widehat{T}$ is given by the following commutative triangles

$$\begin{array}{ccc} X & \xrightarrow{\widehat{\eta}_X} & \widehat{T}X \\ & {\scriptstyle a}\searrow \quad \swarrow{\scriptstyle \alpha^+_a} & \\ & A & \end{array} \qquad \text{for all } a\colon X \to K(A,\alpha).$$

and the multiplication by the following commutative squares

$$\begin{array}{ccc} \widehat{T}\widehat{T}X & \xrightarrow{\widehat{\mu}_X} & \widehat{T}X \\ {\scriptstyle \widehat{T}\alpha^+_a}\downarrow & & \downarrow{\scriptstyle \alpha^+_a} \\ \widehat{T}A & \xrightarrow[\alpha^+]{} & A \end{array} \qquad \text{for all } a\colon X \to K(A,\alpha).$$

Remark 4.7. A concept related to the profinite monad was studied by Bojańczyk [7] who associates to every monad $\mathbf{T}$ on $\mathbf{Set}$ a monad $\overline{\mathbf{T}}$ on $\mathbf{Set}$ (rather than on $\mathsf{Pro}\,\mathbf{Set}_\mathsf{f} = \mathbf{Stone}$ as in our setting). Specifically, $\overline{\mathbf{T}}$ is the monad induced by the composite right adjoint $\mathbf{Stone}^{\widehat{\mathbf{T}}} \to \mathbf{Stone} \xrightarrow{V} \mathbf{Set}$. Its construction also appears in the work of Kennison and Gildenhuys [11] who investigated codensity monads for $\mathbf{Set}$-valued functors and their connection with profinite algebras.

Remark 4.8. (1) Every finite $\mathbf{T}$-algebra (A, α) yields a finite $\widehat{\mathbf{T}}$-algebra (A, α^+). Indeed, the unit law and the associative law for α^+ follow from Construction 4.6(3) with $X = A$ and $a = \mathsf{id}_A$.

(2) The monad $\widehat{T}$ is cofinitary. To see this, let $x_i \colon X \to X_i$ $(i \in I)$ be a cofiltered limit cone in $\mathsf{Pro}\,\mathscr{D}_\mathsf{f}$. For each object of Q_X given by an algebra (A, α) and morphism $a \colon X \to A$, due to $A \in \mathscr{D}_\mathsf{f}$ there exists $i \in I$ and a morphism $b \colon X_i \to A$ with $a = b \cdot x_i$. From the definition of $\widehat{T}$ on morphisms we get

$$\alpha_a^+ = (\,\widehat{T}X_i \xrightarrow{\widehat{T}x_i} \widehat{T}A \xrightarrow{\alpha_b^+} B\,).$$

To prove that $\widehat{T}x_i \colon \widehat{T}X \to \widehat{T}X_i$ $(i \in I)$ forms a limit cone, suppose that any cone $c_i \colon C \to \widehat{T}X_i$ $(i \in I)$ is given. It is easy to verify that then the cone of Q_X (see Construction 4.6(1)) assigning to the above a the morphism $\alpha_b^+ \cdot c_i$ is well-defined, i.e. independent of the choice of i and b and compatible with Q_X. The unique morphism $c \colon C \to \widehat{T}X$ factorizing that cone fulfils $c_i = \widehat{T}x_i \cdot c$ because this equation holds when postcomposed with the members of the limit cone of Q_{X_i}. This proves the claim.

(3) The free $\widehat{\mathbf{T}}$-algebra $(\widehat{T}X, \widehat{\mu}_X)$ on an object X of $\mathsf{Pro}\,\mathscr{D}_\mathsf{f}$ is a cofiltered limit of finite $\widehat{\mathbf{T}}$-algebras. In fact, for the squares in Construction 4.6(3) defining $\widehat{\mu}_X$ we have the limit cone (α_a^+) in $\mathsf{Pro}\,\mathscr{D}_\mathsf{f}$, and since all α_a^+ are homomorphisms of $\widehat{\mathbf{T}}$-algebras and the forgetful functor from $(\mathsf{Pro}\,\mathscr{D}_\mathsf{f})^{\widehat{\mathbf{T}}}$ to $\mathsf{Pro}\,\mathscr{D}_\mathsf{f}$ reflects limits, it follows that $(\widehat{T}X, \widehat{\mu}_X)$ is a limit of the algebras (A, α^+).

(4) For "free" objects of $\mathsf{Pro}\,\mathscr{D}_\mathsf{f}$, i.e. those of the form $\hat{X}$ for $X \in \mathscr{D}$ (cf. Lemma 3.13), the definition of $\widehat{T}\widehat{X}$ can be stated in a more convenient form: $\widehat{T}\widehat{X}$ is the cofiltered limit of all finite quotient algebras of the free $\mathbf{T}$-algebra (TX, μ_X). More precisely, let $(TX, \mu_X)\!\downarrow\!\mathscr{D}_\mathsf{f}^\mathbf{T}$ denote the full subcategory of the slice category $(TX, \mu_X)/\mathscr{D}_\mathsf{f}^\mathbf{T}$ on all finite quotient algebras of (TX, μ_X), and consider the diagram

$$D_X \colon (TX, \mu_X)\!\downarrow\!\mathscr{D}_\mathsf{f}^\mathbf{T} \to \mathsf{Pro}\,\mathscr{D}_\mathsf{f}$$

that maps $e \colon (TX, \mu_X) \twoheadrightarrow (A, \alpha)$ to A. Then we have the following

Lemma 4.9. *For every object X of $\mathscr{D}$, one has $\widehat{T}\widehat{X} = \lim D_X$.*

Proof. The diagram D_X is the composite

$$(TX, \mu_X)\!\downarrow\!\mathscr{D}_\mathsf{f}^\mathbf{T} \rightarrowtail (TX, \mu_X)/\mathscr{D}_\mathsf{f}^\mathbf{T} \cong \widehat{X}/K \xrightarrow{Q_{\hat{X}}} \mathsf{Pro}\,\mathscr{D}_\mathsf{f},$$

where the isomorphism $(TX, \mu_X)/\mathscr{D}_\mathsf{f}^\mathbf{T} \cong \widehat{X}/K$ maps $e \colon (TX, \mu_X) \to (A, \alpha)$ to $\widehat{e \cdot \eta_X} \colon \widehat{X} \to A$. Since every $\mathbf{T}$-homomorphism has an $(\mathcal{E}^\mathbf{T}, \mathcal{M}^\mathbf{T})$-factorization, $(TX, \mu_X)\!\downarrow\!\mathscr{D}_\mathsf{f}^\mathbf{T}$ is an initial subcategory of $(TX, \mu_X)/\mathscr{D}_\mathsf{f}^\mathbf{T}$. Thus, $\widehat{T}X = \lim Q_{\hat{X}} = \lim D$. □

Notation 4.10. The above proof gives, for every object $X \in \mathscr{D}$, the limit cone $\alpha^+_{\widehat{e \cdot \eta_X}} \colon \widehat{T}\widehat{X} \twoheadrightarrow A$ with $e \colon (TX, \mu_X) \twoheadrightarrow (A, \alpha)$ ranging over $(TX, \mu_X)\!\downarrow\!\mathscr{D}_\mathsf{f}^\mathbf{T}$. In the following, we abuse notation and simply write α_e^+ for $\alpha^+_{\widehat{e \cdot \eta_X}}$.

Example 4.11. Given the monad $TX = X^*$ of monoids on $\mathscr{D} = \mathbf{Set}$, the profinite monad is the monad of monoids in **Stone**

$$\widehat{T}X = \text{free monoid in } \mathbf{Stone} \text{ on the space } X.$$

For a finite set X, the elements of $\widehat{T}X$ are called the *profinite words* over X. A profinite word is a compatible choice of a congruence class of $X^*/{\sim}$ for every congruence $\sim$ of finite rank. Compatibility means that given another congruence $\approx$ containing $\sim$, the class chosen for $\approx$ contains the above class as a subset.

Lemma 4.12. *The monad $\widehat{T}$ preserves quotients, i.e. $\widehat{T}(\widehat{\mathcal{E}}) \subseteq \widehat{\mathcal{E}}$.*

Proof. Suppose that $e\colon X \to Y$ is a morphism im $\widehat{\mathcal{E}}$. This means that it can be expressed as a cofiltered limit in $\widehat{\mathscr{D}}^{\to}$ of morphisms $e_i \in \mathcal{E}_\mathsf{f}$ $(i \in I)$:

$$\begin{array}{ccc} X & \xrightarrow{\;e\;} & Y \\ {\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle q_i} \\ X_i & \xrightarrow[e_i]{} & Y_i \end{array}$$

Since $\widehat{T}$ is cofinitary by Remark 4.8(2), it follows that $\widehat{T}e$ is the limit of $\widehat{T}e_i = Te_i$ $(i \in I)$ in $\widehat{\mathscr{D}}^{\to}$. Since T preserves $\mathcal{E}$, we have $Te_i \in \mathcal{E}$ for all $i \in I$, which proves that $\widehat{T}e \in \widehat{\mathcal{E}}$. □

It follows that the factorization system $(\widehat{\mathcal{E}}, \widehat{\mathcal{M}})$ of $\mathsf{Pro}\,\mathscr{D}_\mathsf{f}$ lifts to the category $(\mathsf{Pro}\,\mathscr{D}_\mathsf{f})^{\widehat{\mathbf{T}}}$. Moreover, this category with the choice

$$(\mathsf{Pro}\,\mathscr{D}_\mathsf{f})^{\widehat{\mathbf{T}}}_\mathsf{f} = \text{all } \widehat{\mathbf{T}}\text{-algebras } (A, \alpha) \text{ with } A \in \mathscr{D}_\mathsf{f}$$

satisfies all the requirements of Assumption 3.1; this is analogous to the corresponding observations for $\mathscr{D}^{\mathbf{T}}$ in Remark 4.2. Note that we are ultimately interested in finite $\mathbf{T}$-algebras, not finite $\widehat{\mathbf{T}}$-algebras. However, there is no clash: we shall prove in Proposition 4.16 that they coincide.

Notation 4.13. Recall from Construction 4.6 the definition of $\widehat{T}X$ as a cofiltered limit $\alpha_a^+\colon \widehat{T}X \to A$ of $Q_X\colon X/K \to \mathsf{Pro}\,\mathscr{D}_\mathsf{f}$. Since the functor $V\colon\ \mathsf{Pro}\,\mathscr{D}_\mathsf{f} \to \mathscr{D}$ (see Notation 3.10) preserves that limit, and since all morphisms

$$TVX \xrightarrow{TVa} TA \xrightarrow{\alpha} A$$

form a cone of $V \cdot Q_X$, there is a unique morphism φ_X such the squares below commute for for every finite $\mathbf{T}$-algebra (A, α):

$$\begin{array}{ccc} TVX & \overset{\varphi_X}{- - \to} & V\widehat{T}X \\ {\scriptstyle TVa}\downarrow & & \downarrow{\scriptstyle V\alpha_a^+} \\ TA & \xrightarrow[\alpha]{} & A \end{array} \tag{4.1}$$

Example 4.14. For the monoid monad $TX = X^*$ on **Set**, the map

$$\varphi_X\colon (VX)^* \to V\widehat{T}X$$

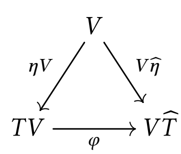
is the embedding of finite words into profinite words. More precisely, by representing elements of $\widehat{T}X$ as compatible choices of congruences classes (see Example 4.11), φ_X maps $w \in X^*$ to the compatible family of all congruence classes $[w]_\sim$ of w, where $\sim$ ranges over all congruences on X^* of finite rank.

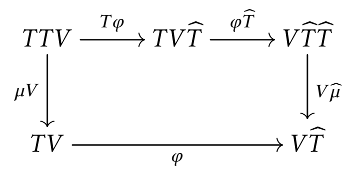
We now prove that the morphisms φ_X are the components of a monad morphism from $\mathbf{T}$ to $\widehat{\mathbf{T}}$ in the sense of Street [23].

Lemma 4.15. *The morphisms φ_X form a natural transformation*

$$\varphi\colon TV \to V\widehat{T}$$

such that the following diagrams commute:

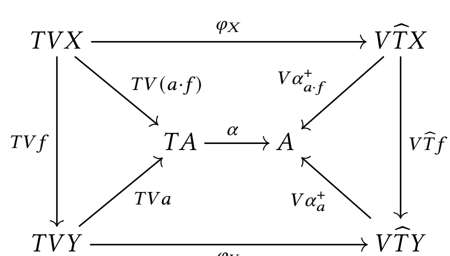




Proof. (1) We first prove that φ is natural. Given a morphism $f\colon X \to Y$ in Pro $\mathscr{D}_\mathsf{f}$, consider an arbitrary object $a\colon Y \to K(A,\alpha)$ of Q_Y (see Construction 4.6(1)) and recall that by the definition of $\widehat{T}$ on the morphism f we have

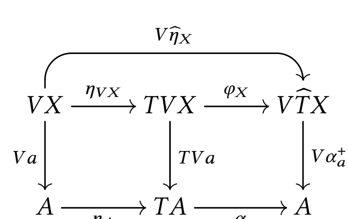
$$\alpha_a^+ \cdot \widehat{T}f = \alpha_{a\cdot f}^+.$$

Consider the following diagram:



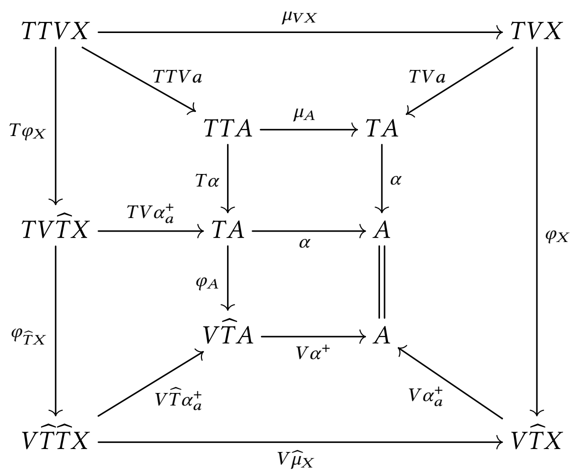
Since all inner parts commute by definition, and the morphisms $V\alpha_a^+$ form a collectively monic cone using that V is cofinitary, we see that the outside commutes, i.e. φ is natural.

(2) To prove $V\widehat{\eta}_X = \varphi_X \cdot \eta_{VX}$, use the collectively monic cone $V\alpha_a^+\colon V\widehat{T}X \to VA$, where $a\colon X \to K(A,\alpha)$ ranges over Q_X. Using the triangle in Construction 4.6(3), we see that the following diagram



has the desired upper part commutative, since it commutes when post-composed by every $V\alpha_a^+$, which follows from the fact that the two lower squares and the outside clearly commute.

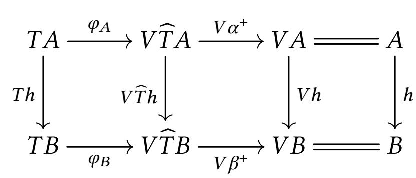
(3) To prove $V\widehat{\mu}_X \cdot \varphi_{\widehat{T}X} \cdot T\varphi_X = \varphi_X \cdot \mu_{VX}$, we again use the collectively monic cone $V\alpha_a^+$. The square in Construction 4.6(3) makes it clear that in the following diagram



the outside commutes, since it does when post-composed by all $V\alpha_a^+$. □

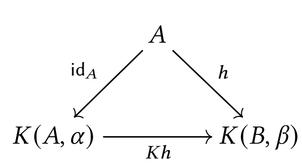
Proposition 4.16. *The categories of finite* **T***-algebras and finite* $\widehat{\mathbf{T}}$*-algebras are isomorphic: the functor taking* (A, α) *to* (A, α^+) *and being the identity map on morphisms is an isomorphism.*

Proof. (1) We first prove that, given finite **T**-algebras (A, α) and (B, β), a morphism $h\colon A \to B$ is a homomorphism for **T** iff $h\colon (A, \alpha^+) \to (B, \beta^+)$ is a homomorphism for $\widehat{\mathbf{T}}$. If the latter holds, then the naturality of φ yields a commutative diagram as follows

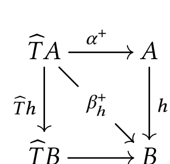


Thus h is a homomorphism for **T**, since the horizontal morphisms are α and β, respectively.

Conversely, if h is a homomorphism for **T**, then the diagram Q_A of Construction 4.6(1) has the following connecting morphism



This implies $h \cdot \alpha^+ = \beta_h^+$. The definition of $\widehat{T}h$ yields $\beta^+ \cdot \widehat{T}h = \beta_h^+$ (see Construction 4.6(1) again). Thus, h is a homomorphism for $\widehat{\mathbf{T}}$:



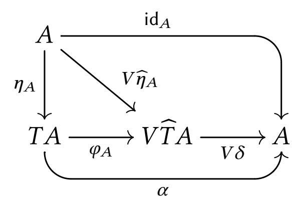
Note that the *only if* part implies that the object assignment $(A, \alpha) \mapsto (A, \alpha^+)$ is indeed functorial.

(2) For every finite $\widehat{\mathbf{T}}$-algebra (A, δ) we prove that the composite

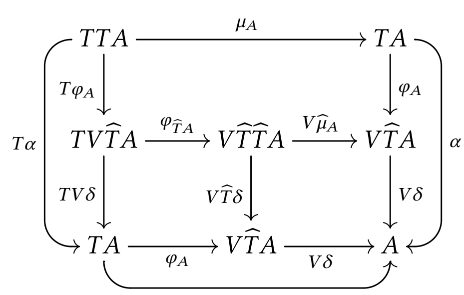
$$\alpha = TA \xrightarrow{\varphi_A} V\widehat{T}A \xrightarrow{V\delta} VA = A \tag{4.2}$$

defines a $\mathbf{T}$-algebra with $\alpha^+ = \delta$.

The unit law follows from that of δ, $\delta \cdot \widehat{\eta}_A = \mathrm{id}$ and from $\varphi_A \cdot \eta_A = V\widehat{\eta}_A$ (see Lemma 4.15):



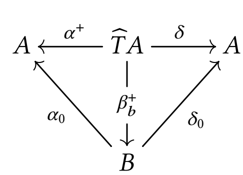
The associative law follows from that of δ, $\delta \cdot \widehat{\mu}_A = \delta \cdot \widehat{T}\delta$ and from $\varphi_A \cdot \mu_A = V\widehat{\mu}_A \cdot \varphi_{\widehat{T}A} \cdot T\varphi_A$ (see Lemma 4.15):



To prove that

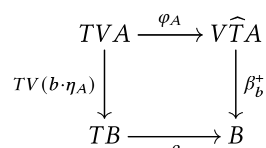
$$\alpha^+ = \delta,$$

recall from Lemma 4.9 and Notation 4.10 that $\widehat{T}A$ is a cofiltered limit of all finite quotients $b\colon (TA, \mu_A) \twoheadrightarrow (B, \beta)$ in $\mathscr{D}^{\mathbf{T}}$ with the limit cone $\beta_b^+\colon \widehat{T}A \twoheadrightarrow B$. Since A is finite, both α^+ and δ factorize through one of the limit projections β_b^+, i.e. we have commutative triangles as follows:



(4.3)

Recall from Notation 4.10 that β_b^+ denotes $\beta^+_{\widehat{b\cdot\eta_A}}$, and by Lemma 3.13 we have $\widehat{b \cdot \eta_A} = b \cdot \eta_A\colon A \to B$ since this morphism lies in $\mathscr{D}_\mathsf{f}$. Combining this with the definition (4.1) of φ_A we have a commutative square



(4.4)

Now we compute

$$\begin{aligned}
\delta_0 \cdot \beta \cdot T(b \cdot \eta_A) &= \delta_0 \cdot \beta \cdot TV(b \cdot \eta_A) && \text{since } b \cdot \eta_A \text{ lies in } \mathscr{D}_\mathrm{f} \\
&= \delta_0 \cdot V\beta_b^+ \cdot \varphi_A && \text{by (4.4)} \\
&= V\delta_0 \cdot V\beta_b^+ \cdot \varphi_A && \text{since } \delta_0 \text{ lies in } \mathscr{D}_\mathrm{f} \\
&= V\delta \cdot \varphi_A && \text{by (4.3).}
\end{aligned}$$

Analogously, we obtain

$$V\alpha^+ \cdot \varphi_A = \alpha_0 \cdot \beta \cdot T(b \cdot \eta_A). \tag{4.5}$$

From the definition (4.1) of φ_A, we also get

$$V\alpha^+ \cdot \varphi_A = V\alpha_{id}^+ \cdot \varphi_A = \alpha \cdot TV\mathrm{id}_A = \alpha = V\delta \cdot \varphi_A, \tag{4.6}$$

where we use (4.2) in the last step. Therefore, we can compute

$$\begin{aligned}
\delta_0 \cdot b &= \delta_0 \cdot b \cdot \mu_A \cdot T\eta_A && \text{since } \mu_A \cdot T\eta_A = \mathrm{id} \\
&= \delta_0 \cdot \beta \cdot Tb \cdot T\eta_A && \text{since } b \text{ is a } \mathbf{T}\text{-homomorphism} \\
&= V\delta \cdot \varphi_A && \text{shown previously} \\
&= V\alpha \cdot \varphi_A && \text{by (4.6)} \\
&= \alpha_0 \cdot \beta \cdot Tb \cdot T\eta_A && \text{by (4.5)} \\
&= \alpha_0 \cdot b \cdot \mu_A \cdot T\eta_A && \text{since } b \text{ is a } \mathbf{T}\text{-homomorphism} \\
&= \alpha_0 \cdot b. && \text{since } \mu_A \cdot T\eta_A = \mathrm{id}.
\end{aligned}$$

Since b is epic, this implies $\alpha_0 = \delta_0$, whence $\alpha^+ = \delta$.

(3) Uniqueness of α. Let (A, α) be a finite $\mathbf{T}$-algebra with $\alpha^+ = \delta$. By the definition of φ_A this implies

$$\alpha = V\alpha^+ \cdot \varphi_A = V\delta \cdot \varphi_A,$$

so α is unique. □

From now on, we identify finite algebras for $\mathbf{T}$ and for $\widehat{\mathbf{T}}$.

Proposition 4.17. *The pro-completion of the category $\mathscr{D}_\mathrm{f}^\mathbf{T}$ of finite $\mathbf{T}$-algebras is the full subcategory of the category of $\widehat{\mathbf{T}}$-algebras given by all cofiltered limits of finite $\mathbf{T}$-algebras.*

Proof. Let $\mathscr{L}$ denote the full subcategory of $(\mathrm{Pro}\,\mathscr{D}_\mathrm{f})^{\widehat{\mathbf{T}}}$ given by all cofiltered limits of finite $\mathbf{T}$-algebras. To show that $\mathscr{L}$ forms the pro-completion of $\mathscr{D}_\mathrm{f}^\mathbf{T}$, we verify the three conditions of Corollary A.5. By definition $\mathscr{L}$ satisfies condition (2), and condition (1) follows from the fact that since $\mathrm{Pro}\,\mathscr{D}_\mathrm{f}$ has cofiltered limits, so does $(\mathrm{Pro}\,\mathscr{D}_\mathrm{f})^{\widehat{\mathbf{T}}}$. Thus, it only remains to prove condition (3): every algebra (A, α^+) with $(A, \alpha) \in \mathscr{D}_\mathrm{f}^\mathbf{T}$ is finitely copresentable in $\mathscr{L}$. Let $b_i \colon (B, \beta) \to (B_i, \beta_i)$, $(i \in I)$, be a limit cone of a cofiltered diagram D in $\mathscr{L}$. Our task is to prove for every morphism $f \colon (B, \beta) \to (A, \alpha^+)$ that

(a) a factorization through a limit projection exists, i.e. $f = f' \cdot b_i$ for some $i \in I$ and $f' \colon (B_i, \beta_i) \to (A, \alpha^+)$, and

(b) given another factorization $f = f'' \cdot b_i$ in $\mathscr{L}$, then f' and f'' are merged by a connecting morphism $b_{ji} \colon (B_j, \beta_j) \to (B_i, \beta_i)$ of D (for some $j \in I$).

Ad (a), since $b_i \colon B \to B_i$ is a limit of a cofiltered diagram in $\operatorname{Pro} \mathscr{D}_{\mathrm{f}}$ and A is as an object of $\mathscr{D}_{\mathrm{f}}$ finitely copresentable in $\operatorname{Pro} \mathscr{D}_{\mathrm{f}}$, we have $i \in I$ and a factorization $f = f' \cdot b_i$, for some $f' \colon B_i \to A$ in $\operatorname{Pro} \mathscr{D}_{\mathrm{f}}$. If f' is a $\mathbf{T}$-homomorphism, i.e. if the following diagram

$$\widehat{T}f \left(\begin{array}{ccc} \widehat{T}B & \xrightarrow{\ \beta\ } & B \\ {\scriptstyle \widehat{T}b_i}\big\downarrow & & \big\downarrow{\scriptstyle b_i} \\ \widehat{T}B_i & \xrightarrow{\ \beta_i\ } & B_i \\ {\scriptstyle \widehat{T}f'}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ \widehat{T}A & \xrightarrow[\ \alpha^+\ ]{} & A \end{array} \right) f \tag{4.7}$$

commutes, we are done. In general, we have to change the choice of i: from Construction 4.6(2) recall that $\widehat{T}$ is cofinitary, thus $(\widehat{T}b_i)_{i \in I}$ is a limit cone. The parallel pair

$$f' \cdot \beta_i, \alpha^+ \cdot \widehat{T}f' \colon \widehat{T}B_i \to A$$

has a finitely copresentable codomain (in $\operatorname{Pro} \mathscr{D}_{\mathrm{f}}$) and is merged by $\widehat{T}b_i$. Indeed, the outside of the above diagram (4.7) commutes since $f = f' \cdot b_i$ is a homomorphism. Consequently, that parallel pair is also merged by $\widehat{T}b_{ji}$ for some connecting morphism $b_{ji} \colon (B_j, \beta_j) \to (B_i, \beta_i)$ of the diagram D:

$$(\alpha^+ \cdot \widehat{T}f') \cdot \widehat{T}b_{ji} = (f' \cdot \beta_i) \cdot \widehat{T}b_{ji}.$$

From $b_i = b_{ji} \cdot b_j$ we get another factorization of f:

$$f = (f' \cdot b_{ji}) \cdot b_j$$

and this tells us that the factorization morphism $\overline{f} = f' \cdot b_{ji}$ is a homomorphism as desired:

$$\begin{array}{ccccc} & \widehat{T}B & \xrightarrow{\ \beta\ } & B & \\ & {\scriptstyle \widehat{T}b_j}\big\downarrow & & \big\downarrow{\scriptstyle b_j} & \\ & \widehat{T}B_j & \xrightarrow{\ \beta_j\ } & B_j & \\ & {\scriptstyle \widehat{T}b_{ji}}\big\downarrow & & \big\downarrow{\scriptstyle b_{ji}} & \\ \widehat{T}\overline{f} & \widehat{T}B_i & \xrightarrow{\ \beta_i\ } & B_i & \overline{f} \\ & {\scriptstyle \widehat{T}f'}\big\downarrow & & \big\downarrow{\scriptstyle f'} & \\ & \widehat{T}A & \xrightarrow[\ \alpha^+\ ]{} & A & \end{array}$$

Ad (b), suppose that $f', f'' \colon (B_i, \beta_i) \to (A, \alpha)$ are homomorphisms satisfying $f = f' \cdot b_i = f'' \cdot b_i$. Since $B = \lim B_i$ is a cofiltered limit in $\operatorname{Pro} \mathscr{D}_{\mathrm{f}}$ and the limit projection b_i merges $f', f'' \colon B_i \to A$, it follows that some connecting morphism $b_{ji} \colon (B_j, \beta_j) \to (B_i, \beta_i)$ also merges f', f'', as desired. □

Remark 4.18. If $(\mathcal{E}, \mathcal{M})$ is a profinite factorization system on $\mathscr{D}$, then $(\mathcal{E}^{\mathbf{T}}, \mathcal{M}^{\mathbf{T}})$ is a profinite factorization system on $\mathscr{D}^{\mathbf{T}}$. Indeed, since $\mathcal{E}$ is closed in $\mathscr{D}^{\to}$ under cofiltered limits of finite quotients, and since the forgetful functor from $(\mathscr{D}^{\mathbf{T}})^{\to}$ to $\mathscr{D}^{\to}$ creates limits, it follows that $\mathcal{E}^{\mathbf{T}}$ is also closed under cofiltered limits of finite quotients.

Definition 4.19. A $\widehat{\mathbf{T}}$-*equation* is an equation in the category of $\widehat{\mathbf{T}}$-algebras, i.e. a $\widehat{\mathbf{T}}$-homomorphism e in $\widehat{\mathcal{E}}^{\widehat{\mathbf{T}}}$ with $\widehat{\mathcal{E}}^{\widehat{\mathbf{T}}}$-projective domain. A finite $\mathbf{T}$-algebra *satisfies* e if it is injective with respect to e in $(\mathsf{Pro}\,\mathscr{D}_{\mathsf{f}})^{\widehat{\mathbf{T}}}$.

Theorem 4.20 (Generalized Reiterman Theorem for Monads)**.** *Let $\mathscr{D}$ be a category with a profinite factorization system $(\mathcal{E}, \mathcal{M})$, and suppose that $\mathbf{T}$ is a monad preserving quotients. Then a class of finite $\mathbf{T}$-algebras is a pseudovariety in $\mathscr{D}_{\mathsf{f}}^{\mathbf{T}}$ iff it can be presented by $\widehat{\mathbf{T}}$-equations.*

Remark 4.21. We will see in the proof that the $\widehat{\mathbf{T}}$-equations presenting a given pseudovariety can be chosen to be of the form $e\colon (\widehat{T}\widehat{X}, \widehat{\mu}_{\widehat{X}}) \twoheadrightarrow (A, \alpha)$ where $e \in \widehat{\mathcal{E}}$, the object X is $\mathcal{E}$-projective in $\mathscr{D}$, and A is finite. Moreover, we can assume $X \in \mathsf{Var}$ for any class Var of objects as in Remark 3.9.

Proof of Theorem 4.20. Every class of finite $\mathbf{T}$-algebras presented by $\widehat{\mathbf{T}}$-equations is a pseudovariety – this is analogous to Proposition 3.8.

Conversely, let $\mathcal{V}$ be a pseudovariety in $\mathscr{D}_{\mathsf{f}}^{\mathbf{T}}$. For every finite $\mathbf{T}$-algebra (A, α) we have an $\mathcal{E}$-projective object X in $\mathscr{D}$ and a quotient $e\colon X \twoheadrightarrow A$ (see Assumption 3.1). Since $\widehat{e} \in \widehat{\mathcal{E}}$ by Remark 3.16(1), we have $\widehat{T}\widehat{e} \in \widehat{\mathcal{E}}$ by Lemma 4.12. Therefore the homomorphism $\overline{e}\colon (\widehat{T}\widehat{X}, \widehat{\mu}_X) \to (A, \alpha^+)$ extending $\widehat{e}$ lies in $\widehat{\mathcal{E}}$: we have $\overline{e} = \alpha^+ \cdot \widehat{T}\widehat{e}$, and α^+ is a split epimorphism by the unit law $\alpha^+ \cdot \widehat{\eta}_A = \mathsf{id}_A$. Since $(\widehat{\mathcal{E}}, \widehat{\mathcal{M}})$ is a proper factorization system and $\mathsf{Pro}\,\mathscr{D}_{\mathsf{f}}$ has finite coproducts, every split epimorphism lies in $\widehat{\mathcal{E}}$ [2, Thm. 14.11], whence $\alpha^+ \in \widehat{\mathcal{E}}$. Thus, we see that every finite $\mathbf{T}$-algebra is a quotient, in the category of $\widehat{\mathbf{T}}$-algebras, of $(\widehat{T}\widehat{X}, \widehat{\mu}_{\widehat{X}})$ for an $\mathcal{E}$-projective object X of $\mathscr{D}$. Each such quotient lies in $\mathsf{Pro}\,\mathscr{D}_{\mathsf{f}}^{\mathbf{T}}$. Indeed, the codomain, being a finite $\mathbf{T}$-algebra, does. To see that the domain also does, combine Remark 4.8(3) and Proposition 4.16.

In Remark 3.9 we can thus denote by Var the collection of all free algebras $(\widehat{T}\widehat{X}, \widehat{\mu}_{\widehat{X}})$ where X ranges over $\mathcal{E}$-projective objects of $\mathscr{D}$. Then Theorem 3.23 and Remark 3.9 yield our claim that every pseudovariety in $\mathscr{D}_{\mathsf{f}}^{\mathbf{T}}$ can be presented by $\widehat{\mathbf{T}}$-equations which are finite quotients of free algebras $(\widehat{T}\widehat{X}, \widehat{\mu}_{\widehat{X}})$ where X is $\mathcal{E}$-projective in $\mathscr{D}$. □

5 PROFINITE TERMS AND IMPLICIT OPERATIONS

In our presentation so far, we have worked with an abstract categorical notion of equations given by quotients of projective objects. In Reiterman's original paper [19] on pseudovarieties of Σ-algebras, a different concept is used: equations between *implicit operations*, or equivalently, equations between *profinite terms*. This raises a natural question: which categories $\mathscr{D}$ allow the simplification of equations in the sense of Definition 4.19 to equations between profinite terms? It turns out to be sufficient that $\mathscr{D}$ is cocomplete and has a finite dense set $\mathcal{S}$ of objects that are projective w.r.t. strong epimorphisms. Recall that density of $\mathcal{S}$ means that every object D of $\mathscr{D}$ is a canonical colimit of all morphisms from objects of $\mathcal{S}$ to D. More precisely, if we view $\mathcal{S}$ as a full subcategory of $\mathscr{D}$, then D is the colimit of the diagram

$$\mathcal{S}/D \to \mathscr{D} \quad \text{given by} \quad \left(s \xrightarrow{f} D\right) \mapsto s$$

with colimit cocone given by the morphisms f.

Assumption 5.1. Throughout this section $\mathscr{D}$ is a cocomplete category with a finite dense set $\mathcal{S}$ of objects projective w.r.t. strong epimorphisms. It follows (see Proposition 5.4 below) that $\mathscr{D}$ has $(\mathsf{StrongEpi}, \mathsf{Mono})$-factorizations, and we work with this factorization system. We denote by $\mathscr{D}_{\mathsf{f}}$ the collection of all objects D such that

$$\mathscr{D}(s, D) \text{ is finite for every object } s \in \mathcal{S}. \tag{5.1}$$

We will show in Proposition 5.4 below that every category $\mathscr{D}$ satisfying the above assumptions can be presented as a category of algebras over an $\mathcal{S}$-sorted signature. Throughout this section, let Σ be an $\mathcal{S}$-sorted algebraic signature, i.e. a signature without relation symbols. We denote by

$$\mathbf{Alg}\,\Sigma$$

the category of Σ-algebras and homomorphisms.

Example 5.2. (1) The category $\mathbf{Set}^{\mathcal{S}}$ satisfies Assumption 5.1. A finite dense set in $\mathbf{Set}^{\mathcal{S}}$ is given by the objects

$$\mathbf{1}_s \ (s \in \mathcal{S})$$

where $\mathbf{1}_s$ is the $\mathcal{S}$-sorted set that is empty in all sorts except s, and has a single element $*$ in sort s. Indeed, let A and B be $\mathcal{S}$-sorted sets and let a cocone of the canonical diagram for A be given:

$$\frac{\mathbf{1}_s \xrightarrow{f} A}{\mathbf{1}_s \xrightarrow{f^*} B}$$

By this we mean that we have morphisms $f^*\colon \mathbf{1}_s \to B$ for every $f\colon \mathbf{1}_s \to A$ (and observe that the cocone condition is void in this case because there are no connecting morphisms $\mathbf{1} \to \mathbf{1}_t$ for $s \neq t$). Then we are to prove that there exists a unique $\mathcal{S}$-sorted function $h\colon A \to B$ with $f^* = h \cdot f$ for all f. Uniqueness is clear: given $x \in A$ of sort s, let $f_x\colon \mathbf{1}_s \to A$ be the map with $f_x(*) = x$. Then $h \cdot f_x = f_x^*$ implies

$$h(x) = f_x^*(*).$$

Conversely, if h is defined by the above equation, then for every $s \in \mathcal{S}$ and $f\colon \mathbf{1}_s \to A$ we have $f^* = h \cdot f$ because $f = f_x$ for $x = f(*)$.

More generally, every set of objects $\mathbf{K}_s$ ($s \in \mathcal{S}$), where $\mathbf{K}_s$ is nonempty in sort s and empty in all other sorts, is dense in $\mathbf{Set}^{\mathcal{S}}$.

(2) The category $\mathbf{Alg}\,\Sigma$ satisfies Assumption 5.1. Recall that strong epimorphisms are precisely the homomorphisms with surjective components, and monomorphisms are the homomorphisms with injective components. It follows easily that for the free-algebra functor $F_\Sigma\colon \mathbf{Set}^{\mathcal{S}} \to \mathbf{Alg}\,\Sigma$ all algebras $F_\Sigma X$ are projective w.r.t. strong epimorphisms. We present a finite dense set of free algebras.

Assume first that Σ is a unary signature, i.e. all operation symbols in Σ are of the form $\sigma\colon s \to t$. Then the free algebras

$$F_\Sigma \mathbf{1}_s \ (s \in \mathcal{S})$$

form a dense set in $\mathbf{Alg}\,\Sigma$. Indeed, let $U_\Sigma\colon \mathbf{Alg}\,\Sigma \to \mathbf{Set}^{\mathcal{S}}$ denote the forgetful functor and $\eta\colon \mathsf{Id} \to U_\Sigma F_\Sigma$ the unit of the adjunction $F_\Sigma \dashv U_\Sigma$. Given Σ-algebras A and B and a cocone of the canonical diagram as follows:

$$\frac{F_\Sigma \mathbf{1}_s \xrightarrow{f} A}{F_\Sigma \mathbf{1}_s \xrightarrow{f^*} B}$$

We are to prove that there exists a unique homomorphism $h\colon A \to B$ with $f^* = h \cdot f$ for every f. We obtain a corresponding cocone in $\mathbf{Set}^{\mathcal{S}}$ as follows:

$$\frac{\mathbf{1}_s \xrightarrow{\eta} U_\Sigma F_\Sigma \mathbf{1}_s \xrightarrow{U_\Sigma f} U_\Sigma A}{\mathbf{1}_s \xrightarrow{\eta} U_\Sigma F_\Sigma \mathbf{1}_s \xrightarrow{U_\Sigma f^*} U_\Sigma B}$$

Due to (1) there exists a unique function $k\colon U_\Sigma A \to U_\Sigma B$ with

$$U_\Sigma f^* \cdot \eta = (k \cdot U_\Sigma f) \cdot \eta \qquad \text{for all } f. \tag{5.2}$$

Here and in the following we drop the subscripts indicating components of η. It remains to prove that k is a homomorphism from A to B; then the universal property of η implies $f^* = k \cdot f$. Thus, given $\sigma\colon s \to t$ in Σ and $a \in A_s$ we need to prove $k(\sigma_A(a)) = \sigma_B(k(a))$. Consider the unique homomorphisms

$$\begin{aligned} f\colon F_\Sigma \mathbf{1}_t &\to A, & f(*) &= \sigma_A(a),\\ g\colon F_\Sigma \mathbf{1}_s &\to A, & g(*) &= a,\\ j\colon F_\Sigma \mathbf{1}_t &\to F_\Sigma \mathbf{1}_s, & j(*) &= \sigma(*). \end{aligned}$$

Then $f = g \cdot j$ and thus $f^* = g^* \cdot j$ because the morphisms $(-)^*$ form a cocone of the canonical diagram of A. It follows that

$$k(\sigma_A(a)) = k(f(*)) = f^*(*) = g^*(j(*)) = g^*(\sigma(*)) = \sigma_B(g^*(*)) = \sigma_B(k(g(*))) = \sigma_B(k(a)),$$

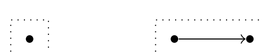
where the last but one equation holds by (5.2). Thus, k is a homomorphism as desired.

For a general signature Σ, let $k \in \mathbb{N} \cup \{\omega\}$ be an upper bound of the arities of operation symbols in Σ and let for every set $T \subseteq \mathcal{S}$ the following $\mathcal{S}$-sorted set X_T be given: X_T is empty for every sort outside of T, and for sorts $s \in T$ the elements are $(X_T)_s = \{\, i \mid i < k \,\}$. Then the set

$$F_\Sigma X_T \quad (T \subseteq \mathcal{S})$$

is dense in $\mathbf{Alg}\,\Sigma$. The proof is analogous to the unary case.

(3) The category of graphs, i.e. sets with a binary relation, and graph homomorphisms satisfies Assumption 5.1. Strong epimorphisms are precisely the surjective homomorphisms which are also surjective on all edges. Thus the two graphs shown below are clearly projective w.r.t. strong epimorphisms. Moreover, they form a dense set: every graph is a canonical colimit of all of its vertices and all of its edges.



(4) Every variety, and even every quasivariety of Σ-algebras (presented by implications) satisfies Assumption 5.1. This will follow from Proposition 5.4 below.

Definition 5.3. A full subcategory $\mathscr{D}$ of $\mathbf{Alg}\,\Sigma$ is said to be *closed under* $(\mathsf{StrongEpi}, \mathsf{Mono})$-*factorizations* if for every morphism $f\colon A \to B$ of $\mathscr{D}$ with factorization $f = A \xrightarrow{e} C \xrightarrow{m} B$, the object C lies in $\mathscr{D}$.

Proposition 5.4. *For every category $\mathscr{D}$ the following two statements are equivalent:*

(1) *$\mathscr{D}$ is cocomplete and has a finite dense set of objects which are projective w.r.t. strong epimorphisms.*

(2) *There exists a signature Σ such that $\mathscr{D}$ is equivalent to a full reflective subcategory of* $\mathbf{Alg}\,\Sigma$ *closed under* $(\mathsf{StrongEpi}, \mathsf{Mono})$*-factorizations.*

Moreover, Σ can always be chosen to be a unary signature.

Proof. (2) $\Rightarrow$ (1) Suppose that $\mathscr{D} \subseteq \mathbf{Alg}\,\Sigma$ is a full reflective subcategory closed under $(\mathsf{StrongEpi}, \mathsf{Mono})$-factorizations. Denote by $(-)^@\colon \mathbf{Alg}\,\Sigma \to \mathscr{D}$ the reflector (i.e. the left adjoint to the inclusion functor $\mathscr{D} \hookrightarrow \mathbf{Alg}\,\Sigma$) and by $\eta_X : X \to X^@$ the universal maps. From Example 5.2 we know $\mathbf{Alg}\,\Sigma$ has a finite dense set of projective objects A_i, $i \in I$. We prove that the objects $A_i^@$, $i \in I$, form a dense set in $\mathscr{D}$.

To verify the density, let $\mathscr{A}$ be the full subcategory of $\mathbf{Alg}\,\Sigma$ on $\{A_i\}_{i\in I}$. For every algebra $D \in \mathscr{D}$ the canonical diagram $\mathscr{A}/D \to \mathbf{Alg}\,\Sigma$ assigning A_i to each $f\colon A_i \to D$ has the canonical colimit D. Since the left adjoint $(-)^@$ preserves that colimit, we have that $D = D^@$ is a canonical colimit of all $f^@ : A_i^@ \to D$ for f ranging over $\mathscr{A}/D$, as required. (Indeed, observe that every morphism

$f\colon A_i^@ \to D$ in $\mathscr{D}$ has the form $f = f^@$ because the subcategory $\mathscr{D}$ is full and contains the domain and codomain of f.)

Next, we observe that every strong epimorphism e of $\mathscr{D}$ is strongly epic in $\mathbf{Alg}\,\Sigma$. Indeed, take the (StrongEpi, Mono)-factorization $e = m \cdot e'$ of e in $\mathbf{Alg}\,\Sigma$. Since $\mathscr{D}$ is closed under factorizations, we have that $e', m \in \mathscr{D}$. Moreover, the morphism m is monic in $\mathscr{D}$ because it is monic in $\mathbf{Alg}\,\Sigma$. Since e is a strong (and thus extremal) epimorphism in $\mathscr{D}$, it follows that m is an isomorphism. Thus $e \cong e'$ is a strong epimorphism in $\mathbf{Alg}\,\Sigma$.

Since each A_i is projective w.r.t. strong epimorphisms in $\mathbf{Alg}\,\Sigma$, it thus follows that $A_i^@$ is projective w.r.t. strong epimorphisms $e\colon B \twoheadrightarrow C$ in $\mathscr{D}$. Indeed, given a morphism $h\colon A_i^@ \to C$, compose it with the universal arrow $\eta\colon A_i \to A_i^@$. Thus, $h \cdot \eta$ factorizes in $\mathbf{Alg}\,\Sigma$ through e:

$$\begin{array}{ccc} A_i & \xrightarrow{\ \eta\ } & A_i^@ \\ {\scriptstyle k}\downarrow & \swarrow{\scriptstyle \overline{k}} & \downarrow{\scriptstyle h} \\ B & \xrightarrow[e]{} & C \end{array}$$

The unique morphism $\overline{k}\colon A_i^@ \to B$ of $\mathscr{D}$ with $k = \overline{k} \cdot \eta$ then fulfils the desired equality $h = e \cdot \overline{k}$ since $h \cdot \eta = e \cdot \overline{k} \cdot \eta$.

(1)⇒(2) Let $\mathcal{S}$ be a finite dense set of objects projective w.r.t. strong epimorphisms, and consider $\mathcal{S}$ as a full subcategory of $\mathscr{D}$. Define an $\mathcal{S}$-sorted signature of unary symbols

$$\Sigma = \mathsf{Mor}(\mathcal{S}^{\mathrm{op}}) \setminus \{\, \mathrm{id}_s \mid s \in \mathcal{S} \,\}.$$

Every morphism $\sigma\colon s \to t$ of $\mathcal{S}^{\mathrm{op}}$ has arity as indicated: the corresponding unary operation has inputs of sort s and yields values of sort t. Define a functor

$$E\colon \mathscr{D} \to \mathbf{Alg}\,\Sigma$$

by assigning to every object D the $\mathcal{S}$-sorted set with sorts

$$(ED)^s = \mathscr{D}(s, D) \quad \text{for } s \in \mathcal{S}$$

endowed with the operations

$$\sigma_{ED}\colon \mathscr{D}(s, D) \to \mathscr{D}(s', D)$$

given by precomposing with $\sigma\colon s' \to s$ in $\mathcal{S} \subseteq \mathscr{D}$. To every morphism $f\colon D_1 \to D_2$ of $\mathscr{D}$ assign the Σ-homomorphism Ef with sorts

$$(Ef)^s\colon \mathscr{D}(s, D_1) \to \mathscr{D}(s, D_2)$$

given by postcomposing with f. To say that $\mathcal{S}$ is a dense set is equivalent to saying that E is full and faithful [3, Prop. 1.26]. Moreover, since $\mathscr{D}$ is cocomplete, E is a right adjoint [3, Prop. 1.27]. Thus, $\mathscr{D}$ is equivalent to a full reflective subcategory of $\mathbf{Alg}\,\Sigma$.

Next we show that $\mathscr{D}$ has the factorization system (StrongEpi, Mono). Indeed, being reflective in $\mathbf{Alg}\,\Sigma$, it is a complete category. Moreover, $\mathscr{D}$ is well-powered because the right adjoint $\mathscr{D} \hookrightarrow \mathbf{Alg}\,\Sigma$ preserves monomorphisms and $\mathbf{Alg}\,\Sigma$ is well-powered. Consequently, the factorization system exists [2, Cor. 14.21].

To prove closure under factorizations, observe first that a morphism $e\colon D_1 \to D_2$ is strongly epic in $\mathscr{D}$ iff Ee is strongly epic in $\mathbf{Alg}\,\Sigma$. Indeed, if e is strongly epic, then Ee has surjective sorts $(Ee)^s$ because s is projective w.r.t. e. Thus, Ee is a strong epimorphism in $\mathbf{Alg}\,\Sigma$. Conversely, if Ee is strongly epic in $\mathbf{Alg}\,\Sigma$, then for every commutative square $g \cdot e = m \cdot f$ in $\mathscr{D}$ with m monic, the morphism Em is monic in $\mathbf{Alg}\,\Sigma$ because E is a right adjoint, and thus a diagonal exists.

Now let $f: A \to B$ be a morphism in $\mathscr{D}$ and let $f = A \xrightarrow{e} C \xrightarrow{m} B$ be its (StrongEpi, Mono)-factorization in $\mathscr{D}$. Thus $C \in \mathscr{D}$ and since by the above argument Ee and Em are strong epimorphisms and monomorphisms in $\mathbf{Alg}\,\Sigma$, respectively, C is the image of f w.r.t. to the factorization system of $\mathbf{Alg}\,\Sigma$. □

Example 5.5. (1) If $\mathscr{D} = \mathbf{Set}$, we can take $S = \{\mathbf{1}\}$ where $\mathbf{1}$ is a singleton set. The one-sorted signature Σ in the above proof is empty, thus, $\mathbf{Alg}\,\Sigma = \mathbf{Set}$.

(2) In the category **Gra** of graphs we can take $S = \{G_1, G_2\}$, see Example 5.2(3). Here Σ is a 2-sorted signature with two operations $s, t: G_2 \to G_1$. A graph $G = (V, E)$ is represented as an algebra A with sorts $A_{G_1} = V$ and $A_{G_2} = E$ and s, t given by the source and target of edges, respectively. More precisely, Gra is equivalent to the full subcategory of all Σ-algebras (V, E) where for all $e, e' \in E$ with $s(e) = s(e')$ and $t(e) = t(e')$, one has $e = e'$.

Assumption 5.6. From now on we assume that

(1) The category $\mathscr{D}$ is a full reflective subcategory of Σ-algebras closed under (StrongEpi, Mono)-factorizations; the reflecting of a Σ-algebra A into $\mathscr{D}$ is denoted by $A^{@}$.

(2) The category $\mathscr{D}_\mathrm{f}$ consists of all Σ-algebras in $\mathscr{D}$ of finite cardinality in all sorts.

In the case where the arities of operations in Σ are bounded, our present choice of $\mathscr{D}_\mathrm{f}$ corresponds well with the previous one in Assumption 5.1: choosing the set $\mathcal{S}$ as in Example 5.2(2), a Σ-algebra D has finite cardinality iff the set of all morphisms from s to D (for $s \in S$) is finite.

Notation 5.7. For the profinite monad $\widehat{\mathbf{T}}$ of Definition 4.5 we denote by

$$U: (\mathrm{Pro}\,\mathscr{D}_\mathrm{f})^{\widehat{\mathbf{T}}} \to \mathbf{Set}^{\mathcal{S}}$$

the forgetful functor that assigns to a $\widehat{\mathbf{T}}$-algebra (A, α) the underlying $\mathcal{S}$-sorted set of A.

Recall from Corollary 2.12 that $\mathrm{Pro}\,\mathscr{D}_\mathrm{f}$ is a full subcategory of $\mathbf{Stone}(\mathbf{Alg}\,\Sigma)$, the category of Stone Σ-algebras and continuous homomorphisms, closed under limits. From Example 3.20 and Proposition 3.18, we get the following

Lemma 5.8. *The factorization system* (StrongEpi, Mono) *on $\mathscr{D}$ is profinite and yields the factorization system on* $\mathrm{Pro}\,\mathscr{D}_\mathrm{f}$ *given by*

$$\widehat{\mathcal{E}} = \textit{continuous homomorphisms surjective in every sort, and}$$

$$\widehat{\mathcal{M}} = \textit{continuous homomorphisms injective in every sort.}$$

Notation 5.9. Let X be a finite $\mathcal{S}$-sorted set of variables.

(1) Denote by

$$F_\Sigma X$$

the free Σ-algebra of *terms*. It is carried by the smallest $\mathcal{S}$-sorted set containing X and such that for every operation symbol $\sigma: s_1, \dots, s_n \to s$ and every n-tuple of terms t_i of sorts s_i we have a term

$$\sigma(t_1, \dots, t_n) \quad \text{of sort } s.$$

(2) For the reflection $(F_\Sigma X)^{@}$, the free object of $\mathscr{D}$ on X, we put

$$X^{\oplus} = \widehat{(F_\Sigma X)^{@}}.$$

This is a free object of $\mathrm{Pro}\,\mathscr{D}_\mathrm{f}$ on X, see Lemma 3.13.

(3) Let (A, α) be a finite $\mathbf{T}$-algebra. An *interpretation* of the given variables in (A, α) is an S-sorted function f from X to the underlying sorted set $U(A, \alpha)$. We denote by

$$f^{@} \colon (F_\Sigma X)^{@} \to A$$

the corresponding morphism of $\mathscr{D}$. It extends to a unique homomorphism of $\widehat{\mathbf{T}}$-algebras (since (A, α^+) is a $\widehat{\mathbf{T}}$-algebra by Proposition 4.16) that we denote by

$$f^{\oplus} \colon \left(\widehat{T}X^{\oplus}, \mu_{X^{\oplus}}\right) \to (A, \alpha^+).$$

Definition 5.10. A *profinite term* over a finite S-sorted set X (of variables) is an element of $\widehat{T}X^{\oplus}$.

Example 5.11. Let $\mathscr{D} = \mathbf{Set}$ and $TX = X^*$ be the monoid monad. For every finite set $X = X^{@}$ we have that $\widehat{T}X^{\oplus}$ is the set of profinite words over X (see Example 4.11).

Definition 5.12. Let t_1, t_2 be profinite terms of the same sort in $\widehat{T}X^{\oplus}$. A finite $\mathbf{T}$-algebra is said to *satisfy the equation* $t_1 = t_2$ provided that for every interpretation f of X we have $f^{\oplus}(t_1) = f^{\oplus}(t_2)$.

Remark 5.13. In order to distinguish equations being pairs of profinite terms according to Definition 5.12 from equations being quotients according to Definition 4.19, we shall sometimes call the latter *equation morphisms.*

Theorem 5.14 (Generalized Reiterman Theorem for Monads on Σ-algebras)**.** *Let $\mathscr{D}$ be a full reflective subcategory of* $\mathbf{Alg}\,\Sigma$ *closed under* (StrongEpi, Mono)*-factorizations, and let* $\mathbf{T}$ *be a monad on $\mathscr{D}$ preserving strong epimorphisms. Then a collection of finite* $\mathbf{T}$*-algebras is a pseudovariety iff it can be presented by equations between profinite terms.*

Proof. (1) We first verify that all assumptions needed for applying Theorem 4.20 and Remark 4.21 are satisfied. Put

$$\mathsf{Var} := \{\, (F_\Sigma X)^{@} \mid X \text{ a finite } \mathcal{S}\text{-sorted set} \,\},$$

the set of all free objects of $\mathscr{D}$ on finitely many generators. We know from Lemma 5.8 that the factorization system (StrongEpi, Mono) is profinite.

(1a) Every object $(F_\Sigma X)^{@}$ of Var is projective w.r.t. strong epimorphisms. Indeed, given a strong epimorphism $e \colon A \twoheadrightarrow B$ in $\mathscr{D}$, it is a strong epimorphism in $\mathbf{Alg}\,\Sigma$, i.e. e has a splitting $i \colon B \rightarrowtail A$ in $\mathbf{Set}^{\mathcal{S}}$ with $e \cdot i = \mathsf{id}$. For every morphism $f \colon (F_\Sigma X)^{@} \to B$ of $\mathscr{D}$ we are to prove that f factorizes through e. The $\mathcal{S}$-sorted function $X \to A$ which is the domain-restriction of $i \cdot f \colon (F_\Sigma X)^{@} \to A$ has a unique extension to a morphism $g \colon (F_\Sigma X)^{@} \to A$ of $\mathscr{D}$. It is easy to see that $e \cdot i = \mathsf{id}$ implies $e \cdot g = f$, as required.

(1b) Every object $D \in \mathscr{D}_{\mathsf{f}}$ is a strong quotient $e \colon (F_\Sigma X)^{@} \twoheadrightarrow D$ of some $(F_\Sigma X)^{@}$ in Var. Indeed, let X be the underlying set of D. Then the underlying function of $\mathsf{id} \colon X \to D$ is a split epimorphism in $\mathbf{Set}^{\mathcal{S}}$, hence, $\mathsf{id}^{@} \colon (F_\Sigma X)^{@} \twoheadrightarrow D$ is a strong epimorphism.

(2) By applying Theorem 4.20 and Remark 4.21, all we need to prove is that the presentation of finite $\mathbf{T}$-algebras by equation morphisms

$$e \colon (\widehat{T}X^{\oplus}, \widehat{\mu}_{X^{\oplus}}) \twoheadrightarrow (A, \alpha), \quad X \text{ finite and } e \text{ strongly epic},$$

is equivalent to their presentation by equations between profinite terms.

(2a) Let $\mathcal{V}$ be a collection in $\mathscr{D}_{\mathsf{f}}^{\mathbf{T}}$ presented by equations $t_i = t_i'$ in $\widehat{T}X_i^{\oplus}$, $i \in I$. Using Theorem 4.20, we just need proving that $\mathcal{V}$ is a pseudovariety:

(i) Closure under finite products $\prod_{k\in K}(A_k,\alpha_k)$: Let f be an interpretation of X_i in the product. Then we have $f=\langle f_k\rangle_{k\in K}$ for interpretations f_k of X_i in (A_k,α_k). By assumption $f_k^{\oplus}(t_i)=f_k^{\oplus}(t_i')$ for every $k\in K$. Since the forgetful functor from $\widehat{\mathbf{T}}$-algebras to $\mathbf{Set}^{\mathcal{S}}$ preserves products, we have $f^{\oplus}=\langle f_k^{\oplus}\rangle_{k\in K}$, hence $f^{\oplus}(t_i)=f^{\oplus}(t_i')$.

(ii) Closure under subobjects $m\colon (A,\alpha)\rightarrowtail(B,\beta)$: Let f be an interpretation of X_i in (A,α). Then $g=(Um)\cdot f$ is an interpretation in (B,β), thus $g^{\oplus}(t_i)=g^{\oplus}(t_i')$. Since m is a homomorphism of $\widehat{\mathbf{T}}$-algebras, we have $g^{\oplus}=m\cdot f^{\oplus}$. Moreover, m is monic in every sort, whence $f^{\oplus}(t_i)=f^{\oplus}(t_i')$.

(iii) Closure under quotients $e\colon (B,\beta)\twoheadrightarrow(A,\alpha)$: Let f be an interpretation of X_i in A. Since Ue is a split epimorphism in $\mathbf{Set}^{\mathcal{S}}$, we can choose $m\colon UA\to UB$ with $(Ue)\cdot m=\mathrm{id}$. Then $g=m\cdot f$ is an interpretation of X_i in (B,β), thus, $g^{\oplus}(t_i)=g^{\oplus}(t_i')$. Since e is a homomorphism of $\widehat{\mathbf{T}}$-algebras, we have

$$e\cdot g^{\oplus}=(Ue\cdot g)^{\oplus}=(Ue\cdot m\cdot f)^{\oplus}=f^{\oplus}.$$

Using this, we obtain $f^{\oplus}(t_i)=f^{\oplus}(t_i')$.

(2b) For every equation morphism

$$e\colon (\widehat{T}X^{\oplus},\widehat{\mu}_{X^{\oplus}})\twoheadrightarrow(A,\alpha)$$

we consider the set of all profinite equations $t=t'$ where $t,t'\in\widehat{T}X^{\oplus}$ have the same sort and fulfil $e(t)=e(t')$. We prove that given a finite algebra (B,β), it satisfies e iff it satisfies all of those equations.

(i) Let (B,β) satisfy e and let f be an interpretation of X in it. Then the homomorphism $f^{\oplus}$ factorizes through e:

$$\begin{array}{ccc}
(\widehat{T}X^{\oplus},\widehat{\mu}_{X^{\oplus}}) & \xrightarrow{f^{\oplus}} & (B,\beta)\\
 & \searrow^{e} & \uparrow h\\
 & & (A,\alpha)
\end{array}$$

Thus, $f^{\oplus}(t)=f^{\oplus}(t')$ whenever $e(t)=e(t')$, as required.

(ii) Let (B,β) satisfy the given equations $t=t'$. We prove that every homomorphism $h\colon(\widehat{T}X^{\oplus},\widehat{\mu}_{X^{\oplus}})\to(B,\beta)$ factorizes through the given e, which lies in $(\mathrm{Pro}\,\mathscr{D}_{\mathrm{f}})^{\widehat{\mathbf{T}}}$. We clearly have

$$h=f^{\oplus}$$

for the interpretation $f\colon X\to U(B,\beta)$ obtained by the domain-restriction of Uh. Consequently, for all $t,t'\in\widehat{T}X^{\oplus}$ of the same sort, we know that

$$e(t)=e(t')\quad\text{implies}\quad h(t)=h(t').$$

This tells us precisely that Uh factorizes in $\mathbf{Set}^{\mathcal{S}}$ through Ue:

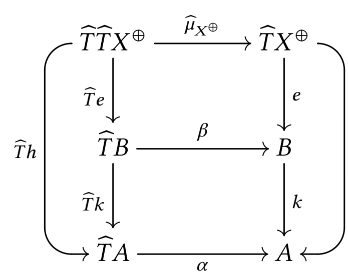
$$\begin{array}{ccc}
 & U(\widehat{T}X^{\oplus},\widehat{\mu}_{X^{\oplus}}) & \\
{}^{Ue}\swarrow & & \searrow^{Uh}\\
U(B,\beta) & \xrightarrow[k]{} & U(A,\alpha)
\end{array}$$

It remains to prove that k is a homomorphism of $\widehat{\mathbf{T}}$-algebras. Firstly, k preserves the operations of Σ and is thus a morphism $k\colon B\to A$ in $\mathscr{D}$. This follows from Ue being epic in $\mathbf{Set}^{\mathcal{S}}$: given

$\sigma\colon s_1, \ldots, s_n \to s$ in Σ and elements x_i of sort s_i in B, choose y_i of sort s_i in $U(\widehat{T}X^{\oplus}, \widehat{\mu}_{X^{\oplus}})$ with $Ue(y_i) = x_i$. Using that e and h are Σ-homomorphism we obtain the desired equation

$$k(\sigma(x_i)) = k(\sigma_B(Ue(y_i)) = k \cdot Ue(\sigma(y_i)) = Uh(\sigma(y_i)). = \sigma_A(h(y_i)) = \sigma_A(k(x_i)).$$

Moreover, $\widehat{T}e$ is epic by Lemma 5.8. In the following diagram



the outside and upper square commute because h and e are a homomorphism of $\widehat{\mathbf{T}}$-algebras, respectively, and the left hand and right hand parts commute because $k \cdot e = h$. Since $\widehat{T}e$ is epic, it follows that the lower square also commutes. □

Remark 5.15. We now show that profinite terms are just another view of the implicit operations that Reiterman used in his paper [19]. We start with a one-sorted signature Σ (for notational simplicity) and then return to the general case. We denote by

$$W\colon \mathscr{D}_{\mathsf{f}}^{\mathbf{T}} \to \mathbf{Set}$$

the forgetful functor assigning to every finite algebra (A, α) the underlying set A.

Definition 5.16. An n-ary *implicit operation* is a natural transformation $\varrho\colon W^n \to W$ for $n \in \mathbb{N}$. Thus if

$$U\colon \mathscr{D}_{\mathsf{f}} \to \mathbf{Set}$$

denotes the forgetful functor, then ϱ assigns to every finite $\mathbf{T}$-algebra (A, α) an n-ary operation on UA such that every homomorphism in $\mathscr{D}_{\mathsf{f}}^{\mathbf{T}}$ preserves that operation.

For the case of finitary Σ-algebras, i.e. finitary monads $\mathbf{T}$ on $\mathbf{Set}$, the above concept is due to Reiterman [19, Sec. 2].

Example 5.17. Let $\mathscr{D} = \mathbf{Set}$ and $TX = X^*$ be the monoid monad. Every element x of a finite monoid (A, α) has a unique idempotent power x^k for some $k > 0$, denoted by x^{ω}. Since monoid morphisms preserve idempotent powers, this yields a unary implicit operation ϱ with components $\varrho_{(A,a)}\colon x \mapsto x^{\omega}$.

Notation 5.18. Consider n as the set $\{0, \ldots, n-1\}$. Every profinite term $t \in \widehat{T}n^{\oplus}$ defines an n-ary implicit operation ϱ_t as follows: Given a finite $\mathbf{T}$-algebra (A, α) and an n-tuple $f\colon n \to UA$, we get the homomorphism $f^{\oplus}\colon (\widehat{T}n^{\oplus}, \widehat{\mu}_{n^{\oplus}}) \to (A, \alpha)$, and ϱ_t assigns to f the value

$$\varrho_t(f) = f^{\oplus}(t).$$

The naturality of ϱ_t is easy to verify.

Lemma 5.19. *Implicit n-ary operations correspond bijectively to profinite terms in $\widehat{T}n^{\oplus}$ via $t \mapsto \varrho_t$.*

Proof. Recall from Corollary 2.12 that Pro $\mathscr{D}_\mathrm{f}$ is a full subcategory of $\mathbf{Stone}(\mathbf{Alg}\,\Sigma)$ closed under limits. The forgetful functor of the latter preserves limits, hence, so does the forgetful functor $\overline{U}\colon$ Pro $\mathscr{D}_\mathrm{f} \longrightarrow \mathbf{Set}$. Recall further from Construction 4.6 that

$$\widehat{T}n^{\oplus} = \lim Q_{n^{\oplus}}$$

where $Q_{n^{\oplus}}\colon n^{\oplus}/K \longrightarrow$ Pro $\mathscr{D}_\mathrm{f}$ is the diagram of all morphisms

$$a\colon n^{\oplus} \longrightarrow K(A,\alpha) = A \quad \text{of Pro } \mathscr{D}_\mathrm{f}.$$

Thus, profinite terms $t \in \widehat{T}n^{\oplus}$ are elements of the limit of

$$\overline{U} \cdot Q_{n^{\oplus}}\colon n^{\oplus}/K \longrightarrow \mathbf{Set}$$

By the well-known description of limits in **Set**, to give t means to give a compatible collection of elements of UA, i.e. for every $n^{\oplus} \xrightarrow{a} K(A,\alpha)$ one gives $t_a \in UA$ such that for every morphism of $n^{\oplus}/K$:

$$\begin{array}{ccc} & n^{\oplus} & \\ {\scriptstyle a}\swarrow & & \searrow{\scriptstyle b} \\ K(A,\alpha) & \xrightarrow[Kh]{} & K(B,\beta) \end{array}$$

we have $Uh(t_a) = t_b$.

Now observe that an object of $n^{\oplus}/K$ is precisely a finite $\mathbf{T}$-algebra (A,α) together with an n-tuple a_0 of elements of UA. Thus, the given collection $a \mapsto t_a$ is precisely an n-ary operation on UA for every finite algebra (A,α). Moreover, the compatibility means precisely that every homomorphism $h\colon (A,\alpha) \longrightarrow (B,\beta)$ of finite $\mathbf{T}$-algebras preserves that operation. Thus, $\widehat{T}n^{\oplus}$ consists of precisely the n-ary implicit operations. Finally, it is easy to see that the resulting operation is ϱ_t of Notation 5.18 for every $t \in \widehat{T}n^{\oplus}$. □

Remark 5.20. (1) For $\mathcal{S}$-sorted signatures this is completely analogous. Let $W^s\colon \mathscr{D}_\mathrm{f}^{\mathbf{T}} \longrightarrow \mathbf{Set}$ assign to every finite $\mathbf{T}$-algebra (A,α) the component of sort s of the underlying $\mathcal{S}$-sorted set UA. An *implicit operation* of arity

$$\varrho\colon s_1,\dots,s_n \longrightarrow s$$

is a natural transformation

$$\varrho\colon W^{s_1} \times \cdots \times W^{s_n} \longrightarrow W^s$$

Thus ϱ assigns to every finite $\mathbf{T}$-algebra (A,α) an operation

$$\varrho_{(A,a)}\colon UA^{s_1} \times \dots UA^{s_n} \longrightarrow UA^s$$

that all homomorphisms in $\mathscr{D}_\mathrm{f}^{\mathbf{T}}$ preserve.

(2) Recall that we identify every natural number n with the set $\{0,\dots,n-1\}$. For every arity $s_1,\dots,s_n \longrightarrow s$ we choose a finite $\mathcal{S}$-sorted set X such that for every sort t we have

$$X^t = \{\, i \in \{1,\dots,n\} \mid t = s_i \,\}.$$

Then for every finite $\mathbf{T}$-algebra (A,α), to give an n-tuple $a_i \in A_{s_i}$ is the same as to give $\mathcal{S}$-sorted function $f\colon X \longrightarrow UA$.

(3) Notation 5.18 has the following generalization: given a profinite term $t \in \widehat{T}X^{\oplus}$ over X of sort s, we define an implicit operation $\varrho_t\colon s_1,\dots,s_n \longrightarrow s$ by its components at all finite $\mathbf{T}$-algebras (A,α) as follows:

$$\varrho_t(f) = f^{\oplus}(t) \quad \text{for all } f\colon X \longrightarrow UA.$$

This yields a bijection between $\widehat{T}X^{\oplus}$ and implicit operations of arity $s_1, \ldots, s_n \to s$ for X in (2). The proof is completely analogous to that of Lemma 5.19.

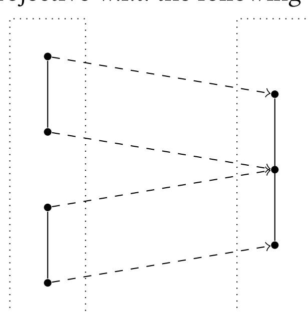
Definition 5.21. Let ϱ and ϱ' be implicit operations of the same arity. A finite algebra (A, α) *satisfies the equation* $\varrho = \varrho'$ if their components $\varrho_{(A,\alpha)}$ and $\varrho'_{(A,\alpha)}$ coincide.

The above formula $\varrho_t(f) = f^{\oplus}(t)$ shows that given profinite terms $t, t' \in \widehat{T}X^{\oplus}$ of the same sort, a finite algebra satisfies the profinite equation $t = t'$ if and only if it satisfies the implicit equation $\varrho_t = \varrho_{t'}$. Consequently:

Corollary 5.22. *Under the hypotheses of Theorem 5.14, a collection of finite* $\mathbf{T}$*-algebras is a pseudovariety iff it can be presented by equations between implicit operations.*

6 PROFINITE INEQUATIONS

Whereas for varieties $\mathscr{D}$ of algebras the equation morphisms in the Reiterman Theorem 4.20 can be substituted by equations $t = t'$ between profinite terms, this does not hold for varieties $\mathscr{D}$ of ordered algebras (i.e. classes of ordered Σ-algebras specified by inequations $t \leq t'$ between terms). The problem is that **Pos** does not have a set dense of objects projective w.r.t. strong epimorphisms. Indeed, only discrete posets are projective w.r.t. the following regular epimorphism:



We are going to show that for $\mathscr{D} = \mathbf{Pos}$ (and more generally varieties $\mathscr{D}$ of ordered algebras) a change of the factorization system from (StrongEpi, Mono) to (surjective, order-reflecting) enables us to apply the results of Section 4 to the proof that pseudovarieties of finite ordered $\mathbf{T}$-algebras are presentable by inequations between profinite terms. This generalizes results of Pin and Weil [17] who proved a version of Reiterman's theorem (without monads) for ordered algebras, in fact, for general first-order structures. We begin with monads on **Pos**, and then show how this yields results for monads on varieties $\mathscr{D}$ of ordered algebras.

Notation 6.1. Given an $\mathcal{S}$-sorted signature Σ of operation symbols, let $\Sigma_{\leq}$ denote the $\mathcal{S}$-sorted first-order signature with operation symbols Σ and a binary relation symbol $\leq_s$ for every $s \in \mathcal{S}$. Moreover, let

$$\mathbf{Alg}\,\Sigma_{\leq}$$

be the full subcategory of $\Sigma_{\leq}$-**Str** for which $\leq_s$ is interpreted as a partial order on the sort s for every $s \in \mathcal{S}$, and moreover every Σ-operation is monotone w.r.t. these orders. Thus, objects are ordered Σ-algebras, morphisms are monotone Σ-homomorphisms. Recall from Remark 2.15 our factorization system with

$$\mathcal{E} = \text{morphisms surjective in all sorts, and}$$
$$\mathcal{M} = \text{morphisms order-reflecting in all sorts.}$$

Thus a Σ-homomorphisms m lies in $\mathcal{M}$ iff for all x, y in the same sort of its domain we have $x \le y$ iff $m(x) \le m(y)$. The notion of a subcategory $\mathscr{D}$ of $\mathbf{Alg}\,\Sigma_{\le}$ being closed under factorizations is analogous to Definition 5.3.

Assumption 6.2. Throughout this section, $\mathscr{D}$ denotes a full reflective subcategory of $\mathbf{Alg}\,\Sigma_{\le}$ closed under factorizations. Moreover, $\mathscr{D}_{\mathrm{f}}$ is the full subcategory of $\mathscr{D}$ given by all algebras which are finite in every sort.

Thus, every variety of ordered algebras (presented by inequations $t \le t'$ betweens terms) can serve as $\mathscr{D}$, as well as every quasivariety (presented by implications between inequations).

Remark 6.3. (1) Recall from Corollary 2.12 that Pro $\mathscr{D}_{\mathrm{f}}$ is a full subcategory of $\mathbf{Stone}(\mathbf{Alg}\,\Sigma_{\le})$, the category of ordered Stone Σ-algebras.

(2) The factorization system on $\mathscr{D}$ inherited from $\mathbf{Alg}\,\Sigma_{\le}$ is profinite, see Example 3.20. Moreover, the induced factorization system $\widehat{\mathcal{E}}$ and $\widehat{\mathcal{M}}$ of Pro $\mathscr{D}_{\mathrm{f}}$ is given by the surjective and order-reflecting morphisms of Pro $\mathscr{D}_{\mathrm{f}}$, respectively (see Proposition 3.18).

Notation 6.4. (1) We again denote by $(-)^{@}\colon\ \mathbf{Alg}\,\Sigma_{\le} \to \mathscr{D}$ the reflector.

(2) For every finite $\mathcal{S}$-sorted set X we have the free algebra $F_\Sigma X$ (discretely ordered).

(3) The free object of Pro $\mathscr{D}_{\mathrm{f}}$ on a sorted set X is again denoted by $X^{\oplus}$ (in lieu of $\widehat{(F_\Sigma X)^{@}}$). For every finite $\mathbf{T}$-algebra (A, α), given an interpretation f of X in (A, α), we obtain a homomorphism

$$f^{\oplus}\colon (\widehat{T}X^{\oplus}, \widehat{\mu}_{X^{\oplus}}) \to (A, \alpha)$$

Definition 6.5. By a *profinite term* on a finite $\mathcal{S}$-sorted set X of variables is meant an element of $\widehat{T}X^{\oplus}$.

Given profinite terms t_1, t_2 of the same sort s, a finite $\mathbf{T}$-algebra (A, α) is said to *satisfy the inequation*

$$t_1 \le t_2$$

provided that for every interpretation f of X we have $f^{\oplus}(t_1) \le f^{\oplus}(t_2)$.

Theorem 6.6. *Let $\mathscr{D}$ be a full reflective subcategory of* $\mathbf{Alg}\,\Sigma_{\le}$ *closed under factorizations, and let* $\mathbf{T}$ *be a monad on $\mathscr{D}$ preserving sortwise surjective morphisms. Then a collection of finite* $\mathbf{T}$*-algebras is a pseudovariety iff it can be presented by inequations between profinite terms.*

Proof. In complete analogy to the proof of Theorem 5.14, we put

$$\mathsf{Var} = \{\, (F_\Sigma X)^{@} \mid X \text{ a finite } \mathcal{S}\text{-sorted set}\,\}.$$

and observe that Theorem 4.20 and Remark 4.21 can be applied.

(1) If $\mathcal{V}$ is a collection of finite $\mathbf{T}$-algebras presented by inequations $t_i \le t_i'$, we need to verify that $\mathcal{V}$ is a pseudovariety. This is analogous to the proof of Theorem 5.14; in part (2) we use that m reflects the relation symbols $\le_s$, hence from $m \cdot f^{\oplus}(t_i) \le_s m \cdot f^{\oplus}(t_i')$ we derive $f^{\oplus}(t_i) \le_s f^{\oplus}(t_i')$.

(2) Given an equation morphism $e\colon (\widehat{T}X^{\oplus}, \widehat{\mu}_{X^{\oplus}}) \twoheadrightarrow (A, \alpha)$, consider all inequations $t \le_s t'$ where t and t' are profinite terms of sort s with $Ue(t) \le Ue(t')$ in A. We verify that a finite $\widehat{\mathbf{T}}$-algebra (B, β) satisfies those inequations iff it satisfies e. This is again completely analogous to the corresponding argument in the proof of Theorem 5.14; just at the end we need to verify, additionally, that

$$x \le_s x' \text{ in } B \qquad \text{implies} \qquad h(x) \le_s h(x') \text{ in } A.$$

Denote by $U\colon (\mathrm{Pro}\,\mathscr{D}_{\mathrm{f}})^{\widehat{\mathbf{T}}} \to \mathbf{Pos}^{\mathcal{S}}$ the forgetful functor. Since Ue has surjective components, we have terms t, t' in $\widehat{T}X^{\oplus}$ of sort s with $x = Ue(t)$ and $x' = Ue(t')$, thus $t \le t'$ is one of the above inequations. The algebra (B, β) satisfies $t \leq t'$ and (like in Theorem 5.14) we get $h = f^{\oplus}$, hence $Uh(t) \leq Uh(t')$. From $Uh = k \cdot Ue$, this yields $k(x) \leq_s k(x')$. □

Remark 6.7. In particular, if $\mathscr{D}$ is a variety of ordered one-sorted Σ-algebras and $\mathbf{T}$ a monad preserving surjective morphisms, pseudovarieties of $\mathbf{T}$-algebras can be described by inequations between profinite terms. This generalizes the result of Pin and Weil [17].

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A IND- AND PRO-COMPLETIONS

The aim of this appendix is to characterize, for an arbitrary small category $\mathscr{C}$, the free completion $\mathsf{Pro}\,\mathscr{C}$ under cofiltered limits and its dual concept, the free completion $\mathsf{Ind}\,\mathscr{C}$ under filtered colimits (see Notation 2.2). Let us first recall the construction of the latter:

Remark A.1. For any small category $\mathscr{C}$, the ind-completion is given up to equivalence by the full subcategory $\mathscr{L}$ of the presheaf category $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ on filtered colimits of representables, and the Yoneda embedding

$$E\colon \mathscr{C} \rightarrowtail \mathscr{L}, \quad C \mapsto \mathscr{C}(-, C).$$

We usually leave the embedding E implicit and view $\mathscr{C}$ as a full subcategory of $\mathscr{L}$.

Dually to Remark 2.1, an object A of a category $\mathscr{C}$ is called *finitely presentable* if the functor $\mathscr{A}(A, -)\colon \mathscr{C} \to \mathbf{Set}$ is finitary, i.e. preserves filtered colimits.

Definition A.2. Let L be an object of a category $\mathscr{L}$. Its *canonical diagram* w.r.t. a full subcategory $\mathscr{C}$ of $\mathscr{L}$ is the diagram D^L of all morphisms from objects of $\mathscr{C}$ to L:

$$D^L\colon \mathscr{C}/L \to \mathscr{L}, \quad (C \xrightarrow{c} L) \mapsto L.$$

Lemma A.3. *Let $\mathscr{C}$ be a full subcategory of $\mathscr{L}$ such that each object $C \in \mathscr{C}$ is finitely presentable in $\mathscr{L}$. An object L of $\mathscr{L}$ is a colimit of some filtered diagram in $\mathscr{C}$ if and only if its canonical diagram is filtered and the canonical cocone $(C \xrightarrow{c} L)_{c \in \mathscr{C}/L}$ is a colimit.*

Proof sketch. The *if* part is trivial. Conversely, if L is a colimit of some filtered diagram, then we can view it as a final subdiagram of its canonical diagram. Therefore, their colimits coincide. □

Theorem A.4. *Let $\mathscr{C}$ be a small category. A category $\mathscr{L}$ containing $\mathscr{C}$ as a full subcategory is an ind-completion of $\mathscr{C}$ if and only if the following conditions hold:*

(1) *$\mathscr{L}$ has filtered colimits,*

(2) *every object of $\mathscr{L}$ is the colimit of a filtered diagram in $\mathscr{C}$, and*

(3) *every object of $\mathscr{C}$ is finitely presentable in $\mathscr{L}$.*

Proof. (1) The *only if* part follows immediately from the construction of $\mathsf{Ind}\,\mathscr{C}$ in Remark A.1: (1) is obvious, (3) follows from the Yoneda Lemma, and (2) follows from Lemma A.3 and the fact that $\mathscr{C}$ is dense in $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$.

(2) We now prove the *if* part. Suppose that (1)–(3) hold. Let $F\colon \mathscr{C} \to \mathscr{K}$ be any functor to a category $\mathscr{K}$ with filtered colimits.

(2a) First, define the extension $\overline{F}\colon \mathscr{L} \to \mathscr{K}$ of F as follows. For any object $L \in \mathscr{L}$ expressed as the canonical colimit $(C \xrightarrow{c} L)_{c \in \mathscr{C}/L}$, the colimit of FD^L exists since the canonical diagram is filtered by condition (2) and $\mathscr{K}$ has filtered colimits. Thus $\overline{F}$ on objects can be given by a choice of a colimit:

$$\overline{F}L \coloneqq \mathsf{colim}\left(\mathscr{C}/L \xrightarrow{D^L} \mathscr{C} \xrightarrow{F} \mathscr{K}\right)$$

We choose the colimits such that $\overline{F}L = L$ if L is in $\mathscr{C}$. For any morphism $f\colon L \to L'$, each colimit injection $\tau_c\colon FC \to \overline{F}L$, for $C \xrightarrow{c} L$, associates with another colimit injection $\tau'_{f\cdot c}\colon FC \to \overline{F}L'$. Hence, there is a unique morphism $\overline{F}f\colon \overline{F}L \to \overline{F}L'$ such that $\tau'_{f\cdot c} = \overline{F}f \cdot \tau_c$. By the uniqueness of mediating morphisms, $\overline{F}$ preserves identities and composition. Therefore, $\overline{F}$ extends F.

(2b) Second, we show that $\overline{F}$ is finitary. Observe that $\overline{F}$ is in fact a pointwise left Kan extension of F along the embedding $E\colon \mathscr{C} \rightarrowtail \mathscr{L}$. By [14, Cor. X.5.4] we have, equivalently, that for every $L \in \mathscr{C}$ and $K \in \mathscr{K}$ the following map from $\mathscr{K}(\overline{F}-, K)$ to the set of natural transformations from $\mathscr{L}(E-, L)$ to $\mathscr{K}(F-, K)$ is a bijection: it assigns to a morphism $f\colon \overline{F}L \to K$ the natural transformation whose components are

$$\left(EC \xrightarrow{c} L\right) \mapsto \left(FC = \overline{F}EC \xrightarrow{\overline{F}c} \overline{F}L \xrightarrow{f} K\right).$$

Hence, given any colimit cocone $(C_i \to L)_{i \in I}$ of a filtered diagram, we have the following chain of isomorphisms, natural in K:

$$\begin{aligned}
\mathscr{K}(\overline{F}L, K) &\cong [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}](\mathscr{L}(E-, L), \mathscr{K}(F-, K)) && \text{see above}\\
&\cong [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}](\operatorname*{colim}_i \mathscr{L}(E-, C_i), \mathscr{K}(F-, K)) && \text{by condition (3)}\\
&\cong \lim_i [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}](\mathscr{L}(E-, C_i), \mathscr{K}(F-, K)) &&\\
&\cong \lim_i \mathscr{K}(\overline{F}C_i, K) && \text{see above}\\
&\cong \mathscr{K}(\operatorname{colim} \overline{F}C_i, K) &&
\end{aligned}$$

Thus, by Yoneda Lemma, $\operatorname{colim} \overline{F}C_i = \overline{F}L$, i.e. $\overline{F}$ is finitary.

(2c) The essential uniqueness of $\overline{F}$ is clear, since this functor is given by a colimit construction. □

By dualizing Theorem A.4, we obtain an analogous characterization of pro-completions:

Corollary A.5. *Let $\mathscr{C}$ be a small category. The pro-completion of $\mathscr{C}$ is characterized, up to equivalence of categories, as a category $\mathscr{L}$ containing $\mathscr{C}$ as a full subcategory such that*

(1) *$\mathscr{L}$ has cofiltered limits,*

(2) *every object of $\mathscr{L}$ is a cofiltered limit of a diagram in $\mathscr{C}$, and*

(3) *every object of $\mathscr{C}$ is finitely copresentable in $\mathscr{L}$.*

Remark A.6. Let $\mathscr{C}$ be a small category.

(1) $\operatorname{Pro}\mathscr{C}$ is unique up to equivalence.

(2) $\operatorname{Pro}\mathscr{C}$ can be constructed as the full subcategory of $[\mathscr{C}, \mathbf{Set}]^{\mathrm{op}}$ given by all cofiltered limits of representable functors. The category $\mathscr{C}$ has a full embedding into $\operatorname{Pro}\mathscr{C}$ via the Yoneda embedding $E\colon C \mapsto \mathscr{C}(C, -)$. This follows from the description of Ind-completions in Remark A.1 and the fact that

$$\operatorname{Pro}\mathscr{C} = (\operatorname{Ind}\mathscr{C}^{\mathrm{op}})^{\mathrm{op}}.$$

(3) If the category $\mathscr{C}$ is finitely complete, then $\operatorname{Pro}\mathscr{C}$ can also be described as the dual of the category of all functors in $[\mathscr{C}, \mathbf{Set}]$ preserving finite limits. Again, E is given by the Yoneda embedding. This is dual to [3, Thm. 1.46]. Moreover, it follows that $\operatorname{Pro}\mathscr{C}$ is complete and cocomplete.

(4) Given a small category $\mathscr{K}$ with cofiltered limits, denote by $[\operatorname{Pro}\mathscr{C}, \mathscr{K}]_{\mathrm{cfin}}$ the full subcategory of $[\operatorname{Pro}\mathscr{C}, \mathscr{K}]$ given by cofinitary functors. Then the pre-composition by E defines an equivalence of categories

$$(-) \cdot E\colon [\operatorname{Pro}\mathscr{C}, \mathscr{K}]_{\mathrm{cfin}} \xrightarrow{\simeq} [\mathscr{C}, \mathscr{K}],$$

where the inverse is given by right Kan extension along E.