the periodic sequence property/ies

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TACL, 27 June 2017

Informatik 8, FAU Erlangen-Nürnberg

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- the property deserves more attention though: a surprising generalization of local finiteness

what i did

- Formalized the proof in the Coq proof assistant $\Rightarrow$ obtaining a verified implementation

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- Carried out a comparative study with other natural classes of implicative logics ⇒
  obtaining a confirmation that Ruitenburg did something remarkable
- While Ruitenburg's paper discusses solely IPC, it is instructive to start with much bigger picture

a wild property in a zoo of logics

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- Define the obvious iterated substitution operation:

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- Question: modulo provable equivalence in CPC, when are you going to enter a cycle?
- Actually, how do you know you must enter a cycle at all?

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- Say L has "periodic sequences" property if we can always find b s.t. $\vdash_{\mathsf{L}} A^b \leftrightarrow A^{b+c}$ holds

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- One universal quantifier (over A) and two existential ones (over b and c), so several orderings possible:

	globally	locally
uniform	$\exists b, c.\forall A$	$\exists c.\forall A.\exists b$
parametric	$\exists b.\forall A.\exists c$	$\forall A.\exists b, c$

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  In fact, one can find a refuting valuation in the famous heap model of BBI
- The sequence does stabilize for the relevance logic RM ("R with Mingle") though ⇒ psp seems an open question

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- We can also break down most intutionistic modal logics using one of these two sequences: see the abstract.

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- Obviously, existence of a sequence not entering a cycle would directly contradict local finiteness

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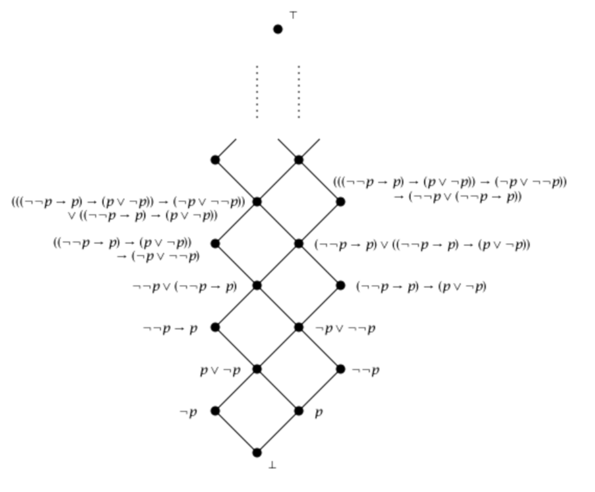
- In fact, I don't even know if globally parametric psp obtains in general for locally finite implicative ones, or even for finite implicative ones

the case of ipc

the rieger-nishimura lattice (wikipedia screenshot)



Not locally finite even in one variable

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  (although few references seem to state clearly the bounds involved; seem doubly exponential to me) Btw, we're going to learn tomorrow the same property holds for $\vee$-free PLL!
- But with $\vee$ in the full signature, can IPC have the psp?? So far, all positive examples we have seen were locally finite!

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Theorem

An extension of K4 *has the (local parametric) psp iff it is locally finite (iff it is of finite depth)*

Proof.

Use the sequence discussed above and proof of Theorem 12.21 in *Modal Logic* by Chagrov & Zakharyaschev □

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And yet . . .

THE JOURNAL OF SYMBOLIC LOGIC
Volume 49, Number 3, Sept. 1984

ON THE PERIOD OF SEQUENCES $(A^n(p))$ IN INTUITIONISTIC PROPOSITIONAL CALCULUS

WIM RUITENBURG

§0. Abstract. In classical propositional calculus for each proposition $A(p)$ the following holds: $\vdash A(p) \leftrightarrow A^3(p)$. In this paper we consider what remains of this in the intuitionistic case. It turns out that for each proposition $A(p)$ the following holds: there is an $n \in \mathbf{N}$ such that

$$\vdash A^n(p) \leftrightarrow A^{n+2}(p).$$

As a byproduct of the proof we give some theorems which may be useful elsewhere in propositional calculus.

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- Ruitenburg shows that quantifiers cannot be shifted

$$\forall b. \exists A. \quad \nvdash_{\mathsf{IPC}} A^b \quad \& \quad \vdash_{\mathsf{IPC}} A^{b+1} \quad \& \quad \vdash_{\mathsf{IPC}} A(\top)$$

  hence, $\forall b. \exists A$ s.t. $\nvdash_{\mathsf{IPC}} A^b \leftrightarrow A^{b+2}$

  his counterexample works even for the logic of linear orders LC

  a.k.a. the *Gödel-Dummet logic* or—in Johnstone's Elephant—the logic of *strong de Morgan law*—and this logic is locally finite

  (unlike its modal counterpart)

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- and then replace it with any sequence of formulas $C_1, \ldots, C_m$ s.t.

$$\forall i \leq n. \exists j \leq m \quad \text{s.t.} \quad \vdash_{\mathsf{IPC}} B_i(\top) \leftrightarrow C_j$$

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$$\forall i \leq n . \exists j \leq m \quad \text{s.t.} \quad \vdash_{\mathsf{IPC}} B_i(\top) \leftrightarrow C_j$$

- Now set $b := 2 * m + 2$

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- Some light on how it become the way it is and where it stands cast by email remarks by Albert Visser, from whom I learned about it:

> *Wim found the result as a PhD student. When he first presented it Carst Koymans and I did not believe it. We quickly found a mistake / gap in the proof. After a few days Wim came back with a repair. Again we shot at it and found a mistake / gap. This repeated itself a number of times until the proof seemed airtight. The whole thing is a singleton result. Nobody ever analysed the methods or connected it e.g. to Ghilardi's work or anything. I still think Wim did something remarkable —not only finding the proof but also asking the question— and should have gotten far more credit for it.*

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- The result remaining isolated, never analyzed from either semantic/dual or Curry-Howard point of view

formalization

- 3000∼4000 lines of code

  A better hacker would do better, I guess

- Formalizes the first part of Ruitenburg's paper (ca. 5 dense pages) which contains the actual syntactic proof of the main theorem

  The second part contains (counter-)examples, e.g, such as the one I showed you before, discussions of improvements possible in concrete cases and arguments why these cannot be generalized. It also involves Kripke frames, not just syntax

- Did not uncover any worthwhile errors
- Available at `https://git8.cs.fau.de/redmine/projects/ruitenburg1984`

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- Bound $b = 10$ would work. Now think what A^{10} is . . .

  To Coq's credit, after several minutes it actually computed the output as a mere 3.2 MB text file. Extracted Haskell hit stack overflow

- You may still be curious whether there were additional reasons to be interested in it
- …or, to put it differently, why Albert Visser told me about this particular result
- But at least at the moment of writing this slide, I am under the impression that at the corresponding moment in the talk, I may not have all that much time left
- So just an essence in six slides …

  and otherwise, see *Fixed point elimination in the Intuitionistic Propositional Calculus* by Ghilardi et al, Wed afternoon

- It is easy to define what it means for A to have a definable L-fixpoint B:

$$p \# B \text{ and } \vdash_{\mathsf{L}} A(B) \leftrightarrow B$$

- It is also easy to define L-least fixpoints: the satisfy in addition

$$\text{for some/any fresh } q,\ A(q) \to q \vdash_{\mathsf{L}} B \to q$$

- Dually, L-greatest fixpoints satisfy in addition

$$\text{for some/any fresh } q,\ q \to A(q) \vdash_{\mathsf{L}} q \to B$$

- If a fixpoint is both least and greatest, it is called unique

- It is easy to realize that the psp + presence of $\bot$ implies definable least L-fixpoints of monotone formulas
- It is easy to realize that the psp + presence of $\top$ implies definable greatest L-fixpoints of monotone formulas
- In other words, extending such logics with fixpoint operators does not improve expressivity!

  Well, one has to be a bit careful that B in question *preserves monotonicity/positivity*: Mardaev and other references discuss this
- But as it turns out, even when the logic itself does not have the psp, it may be useful to have a propositional reduct which has this property to establish fixpoint results

- Consider intuitionistic or classical logics with the Löb axiom $\Box(\Box p \to p) \to \Box p$

  in stronger, but classically useless variant $(\Box p \to p) \to p$

- In the classical setup, these logics arise as modal logics of well-founded transitive structures

  think of finite trees, for example

- In the intuitionistic setup, these logics arise as the Curry-Howard counterparts of calculi with guarded (co-)recursion

- We've seen that these logics do not have the psp
- But they do have unique fixpoints of guarded or modalized formulas

  i.e., those where every occurrence of p is within the scope of $\Box$

  curiously enough, in numerous calculi for guarded (co-)recursion—from Nakano LiCS 2000 to Clouston, Birkedal et al. FoSSaCS 2015—people found it useful to add explicit fixpoint operators for such formulas
- van Benthem around 2005: a semantic proof that this result can be used to prove definability of ordinary fixpoints in GL

  Seems that it was independently and even earlier found by Mardaev
- Visser around the same time: a syntactic proof relying on the psp of CPC...
- In fact, we can even define not-quite-least fixpoints of mixed formulas which contain both positive and guarded occurrences of p ...

and finally, one more puzzling connection

- There is a well-known trick of defining fixpoints (of monotone formulas) using propositional quantifiers in, say, system F:

$$\mu p.A = \forall p.(A \to p) \to p.$$

  See, e.g., Wadler's *Recursive types for free!* manuscript for a discussion of this in connection with parametricity

- We know the name for definability of (a certain kind of) propositional quantifiers in L ...
- ...it's uniform interpolation, of course!
- How does it relate to definability of fixpoints? And, for that matter, to the psp?

- For an arbitrary L, uniform interpolation does not imply definability of fixpoints of monotone formulas
- Consider K and $\mu p.\Box p \dots$
- What one needs in addition is a form of global deduction theorem ...
- ... which in the modal context, boils down to definability of master modality

  ... and for AAL, it boils down to the EDPC—equationally definable principal congruences
- A trick used by d'Agostino and Lenzi in TCS 2005 when showing that PDL with bisimulation quantifiers is equivalent to μ-calculus

- IPC, though, has trivially such a global deduction theorem, as long as we do not add modalities etc. to it
- So here lies another route for fixpoints
- You are going to hear more tomorrow, I guess
- Funnily enough, even though Ruitenburg's paper was written many years before Pitts, it does hint at a connection with uniform interpolation at the very end!

future work

- Attack the Lax logic PLL, perhaps also RN or R

  Cf. *The Variety of Nuclear Implicative Semilattices is Locally Finite*

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- One of referees: is psp related to "good" properties of 1-generated free algebras?