Modal negative translations as a case study in The Big Programme

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- based mostly on a joint FSCD 2017 paper with **Miriam Polzer** and **Ulrich Rabenstein**
- my former students who wrote supporting Coq code

  `https://git8.cs.fau.de/software/dnegmod/`
- the proof assistant formalization played an important rôle in our proof theoretic turn

  presently though I am gradually convincing Miriam to join me again for a follow-up project completely subverting the message of this talk

short statement of the problem

- Consider the intuitionistic modal signature $\mathcal{L}_\Box$

  Connectives: $\wedge, \vee, \rightarrow, \bot, \Box$

  for starters, let's stick with the language with $\Box$ only, no $\Diamond$

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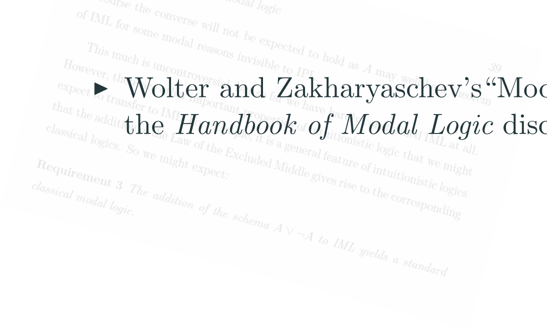
- Given any axiom $\alpha \in \mathcal{L}_\Box$ and any formula $\phi \in \mathcal{L}_\Box$, is it always true that $\phi \in \mathsf{c}_\Box\mathsf{K} + \alpha$ iff $\phi^t \in \mathsf{i}_\Box\mathsf{K} + \alpha$??

  Also, does it depend somehow on t?

what's the deal with "The Big Programme"?

- Wolter and Zakharyaschev's"Modal decision problems" in the *Handbook of Modal Logic* discusses the fate of



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- The bit relevant for us:

  for any meaningful property a logic might have, devise an algorithm taking $\alpha \in \mathcal{L}_\Box$ on input and deciding if $\mathsf{i}_\Box\mathsf{K} + \alpha$ (the normal logic axiomatized by α) has this property

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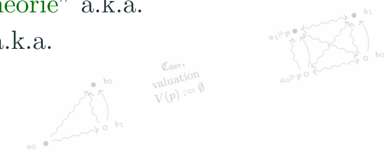

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- The Sixties full of optimism, the Seventies brought in brutal realism

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- But when investigating whole lattices of logics, semantic methods typically employed
- At the very least, purely algebraic ones
- More commonly, non-constructive dualities employed
- Our paper serves as a reminder it is not always necessary
- …even if proof calculi involved have few decent properties

- A normal modality $\Box$ is one that distributes over finite meets/products/conjunctions, i.e.,
  - ⁎ algebraically: $\Box(\phi \wedge \psi) = \Box\phi \wedge \Box\psi$ and $\Box\top = \top$
  - ⁎ categorically: monoidal wrt cartesian structure
  - ⁎ deductively: $\vdash \Box(\phi \to \psi) \to \Box\phi \to \Box\psi$ and $\dfrac{\vdash \psi}{\vdash \Box\psi}$

  The last rule known as Necessitation or Gödel rule

- The resulting extension of the intuitionistic propositional calculus (IPC) is denoted as $\mathsf{i}_{\Box}\mathsf{K}$

  Let us denote its language by $\mathcal{L}_{\Box}$

- Typically, one has more structure and more axioms

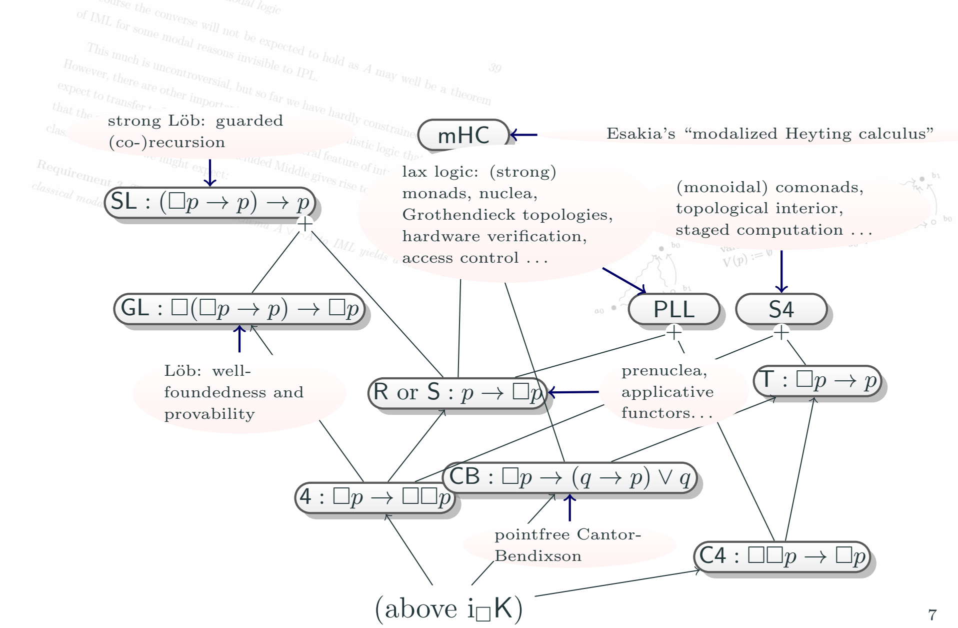
  We need a general notion

- Normal modal logic: a subset of $\mathcal{L}_{\Box}$ containing $\mathsf{i}_{\Box}\mathsf{K}$ and closed under Necessitation, MP and substitution

  Algebraically corresponds to a variety: an equational class of Heyting algebras with $\Box$

  Later, we'll see a Gentzen/ND-style formulation

some axioms to consider …



- One more example: for every x, $\forall x$ is such a $\Box$ over IQC (intuitionistic predicate calculus)
- $\exists x$, meanwhile, is obviously a $\Diamond$

For more on the resulting logic, see A. Prior, R. Bull, G. Bezhanishvili . . .

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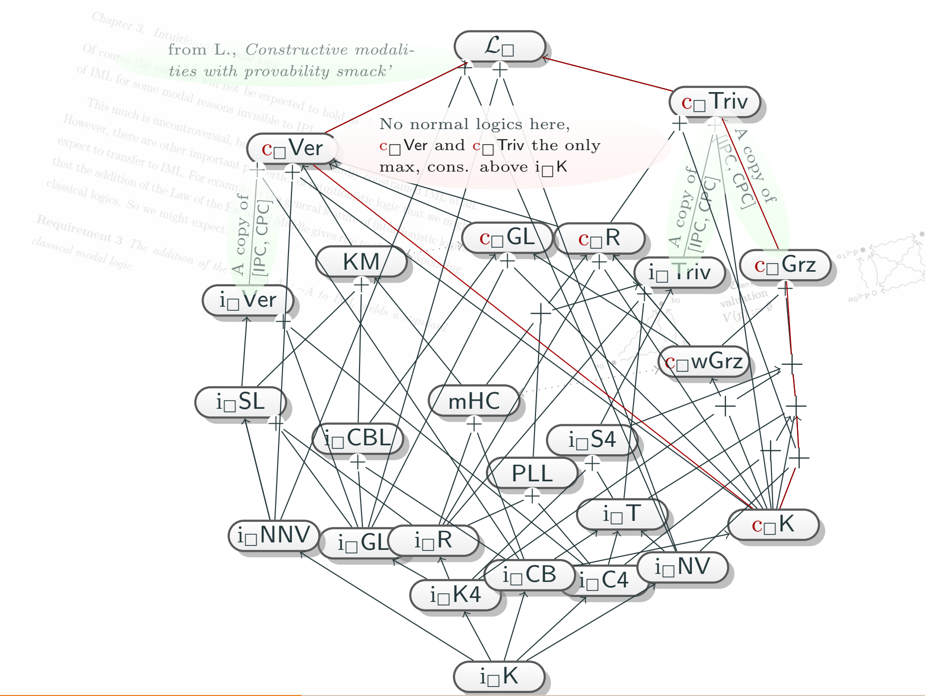
- Again, in the intuitionistic setting, it matters whether or not $\Diamond$ is included in the syntax
- Recall that here we work with $\Box$ only



from L., *Constructive modalities with provability smack'*
$\mathcal{L}_\Box$
$c_\Box$Triv
No normal logics here, $c_\Box$Ver and $c_\Box$Triv the only max, cons. above $i_\Box$K
$c_\Box$Ver
A copy of [IPC, CPC]
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$c_\Box$GL
$c_\Box$R
KM
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$c_\Box$Grz
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$i_\Box$SL
mHC
$i_\Box$CBL
$i_\Box$S4
PLL
$i_\Box$T
$c_\Box$K
$i_\Box$NNV
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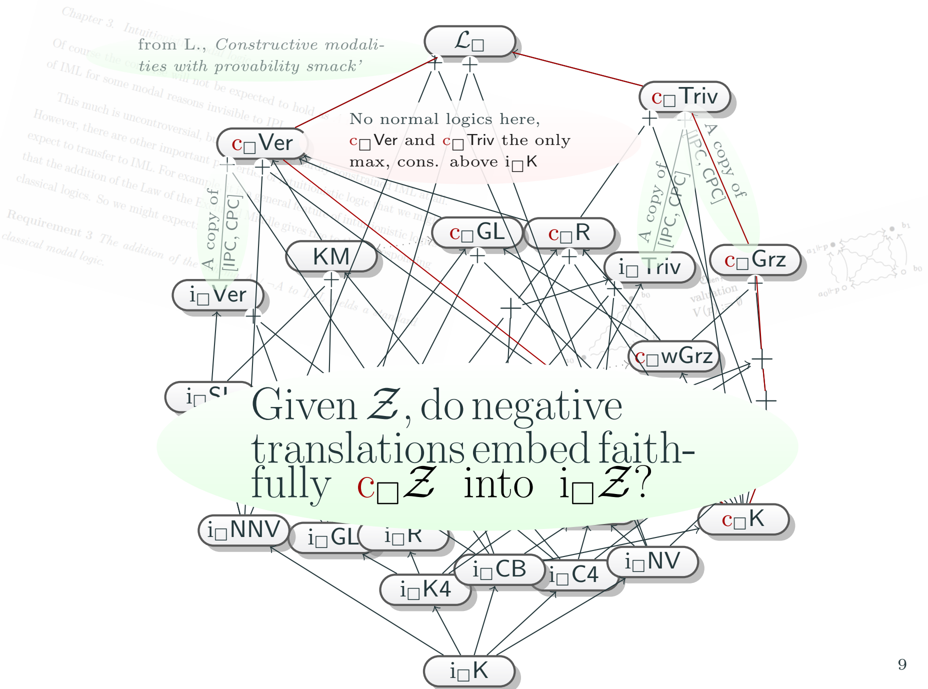
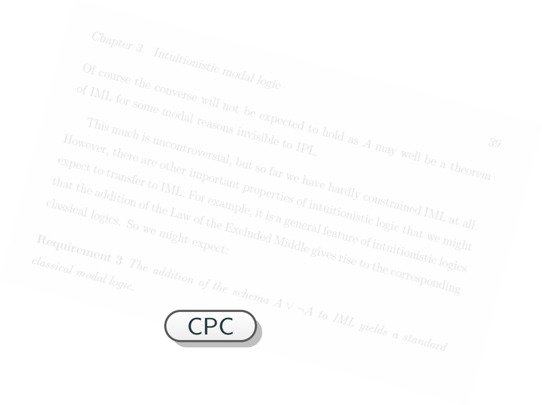


Given $\mathcal{Z}$, do negative translations embed faithfully $c_\Box\mathcal{Z}$ into $i_\Box\mathcal{Z}$?



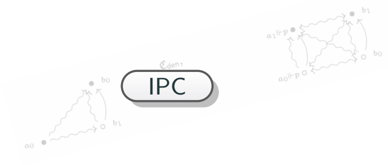
Chapter 3. Intuitionistic modal logic 39

Of course the converse will not be expected to hold as A may well be a theorem of IML for some modal reasons invisible to IPL.

This much is uncontroversial, but so far we have hardly constrained IML at all. However, there are other important properties of intuitionistic logic that we might expect to transfer to IML. For example, it is a general feature of intuitionistic logics that the addition of the Law of the Excluded Middle gives rise to the corresponding classical logics. So we might expect:

Requirement 3 *The addition of the schema $A \vee \neg A$ to IML yields a standard classical modal logic.*

CPC



IPC

CPC —— Glivenko, Gödel(-)Gentzen, Kolmogorov, Krivine(-Streicher-Reus) . . . ⟶ IPC

(See, e.g., Ferreira and Oliva or classical refs like Troelstra and van Dalen)

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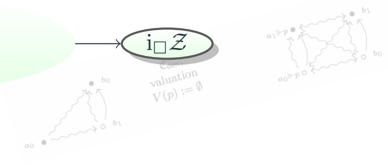
$c_\Box\mathcal{Z}$ — ??? → $i_\Box\mathcal{Z}$



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CQC — ~~Glivenko~~ Kuroda, Gödel(-)Gentzen, Kolmogorov, Krivine(-Streicher-Reus) ... → IQC

statement of our problem again

- Consider ζ, ϕ in the intuitionistic modal signature $\mathcal{L}_\Box$
- Pick a monotone modular or, better still, a regular negative translation t

  We'll make it precise in a second

  Essentially: adding no less $\neg\neg$ than either Kuroda or a suitable version of Gödel-Gentzen
- Is it always true that $\phi \in \mathrm{c}_\Box\mathsf{K} + \zeta$ iff $\phi^t \in \mathrm{i}_\Box\mathsf{K} + \zeta$??
- Quite well-understood for the base system $\mathrm{c}_\Box\mathsf{K}$ vs. $\mathrm{i}_\Box\mathsf{K}$, i.e., in the absence of ζ

  cf., e.g., Božić and Došen 1984
- ...in the general case, we found mostly a huge hole in the literature
- We had to fill it ourselves

but what can go wrong, actually?


39

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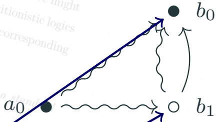
- No recursive translation from BBI to BI can be adequate
- We provided some simpler examples in $\Box$-only setting

$\mathfrak{C}_{\mathsf{den}}$: a frame ...

- validating $\mathsf{C4}: \Box\Box p \to \Box p$
- refuting $\neg\neg\Box\neg\neg\Box\neg\neg p \to \neg\neg\Box\neg\neg p$

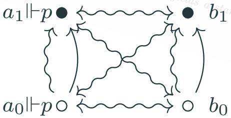


$\bullet$ $\rightsquigarrow$-irreflexive, $\circ$ $\rightsquigarrow$-reflexive

- just pick $V(p) := \emptyset$ and get
- $\overline{V}(\neg\neg p) = \emptyset$
- $\overline{V}(\Box\neg\neg p) = \{b_0\} = b_0\!\uparrow$
- $\overline{V}(\neg\neg\Box\neg\neg p) = \{b_0, b_1\} = b_1\!\uparrow$
- $\overline{V}(\Box\neg\neg\Box\neg\neg p)$ is the whole thing
- $a_0 \not\models \neg\neg\Box\neg\neg\Box\neg\neg p \to \neg\neg\Box\neg\neg p$

similarly, $\mathfrak{C}_{\mathsf{tr2}}$ is a frame . . .

- validating $\flat 4 := \Box\Box p \to \Box\Box\Box p$
- refuting $\neg\neg\Box\neg\neg\Box\neg\neg p \to \neg\neg\Box\neg\neg\Box\neg\neg\Box\neg\neg p$



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$$a_1{\uparrow} = \overline{W}(\Box\neg\neg\Box\neg\neg p), \qquad b_1{\uparrow} = \overline{W}(\Box\neg\neg p) = \overline{W}(\Box\neg\neg\Box\neg\neg\Box\neg\neg p),$$
$$a_0{\uparrow} = W(p) = \overline{W}(\neg\neg p) = \overline{W}(\neg\neg\Box\neg\neg\Box\neg\neg p), \qquad b_0{\uparrow} = \overline{W}(\neg\neg\Box\neg\neg p) = \overline{W}(\neg\neg\Box\neg\neg\Box\neg\neg\Box\neg\neg p).$$

additional motivation: Simpson's dissertation

Chapter 3. Intuitionistic modal logic 39

Of course the conv… well be a theorem of IML for s…

This … at all. Howev… might expect t… logics that the ad… esponding classical logics. S…

1. **too restrictive: important intuitionistic logics can be trivialized classically**
2. **too permissive: what if there are "standard" modal logics for which negative translations are not adequate?**

Requirement 3 *The addition of the schema $A \vee \neg A$ to IML yields a standard classical modal logic.*

our contributions I

- We isolate the well-behaved class of regular translations: $\mathrm{i}_{\Box}\mathsf{K}$-equivalent to Kolmogorov (at least for theoremhood)
- Even regular translations can go wrong though (*failure of* $\neg\neg$*-completeness*) with simple axioms like instances of *modal reduction principles*
  - ⁂ C4 : $\Box\Box p \to \Box p$
  - ⁂ ♭4 : $\Box\Box p \to \Box\Box\Box p$
- We provide a general criterion of $\neg\neg$-completeness Regular Adequacy:
  - ⁂ necessary and sufficient for logics axiomatized by a single ζ
  - ⁂ sufficient for logics axiomatized by a set of axioms
- Regular Adequacy: for a logic to be $\neg\neg$-complete, it is enough that whichever one of the good translations respects its axioms

Intuitionistic propositional rules:

$$\textsc{In}\ \frac{}{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \phi, \Gamma \Rightarrow \phi} \qquad \top_I\ \frac{}{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \top} \qquad \bot_E\ \frac{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \bot}{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \phi}$$

$$\rightarrow_I\ \frac{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma, \phi \Rightarrow \psi}{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \phi \rightarrow \psi} \qquad \rightarrow_E\ \frac{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \phi \rightarrow \psi \qquad \vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \phi}{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \psi}$$

$$\wedge_I\ \frac{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \phi \qquad \vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \psi}{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \phi \wedge \psi} \qquad \wedge_{E1}\ \frac{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \phi \wedge \psi}{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \phi} \qquad \wedge_{E2}\ \frac{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \phi \wedge \psi}{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \psi}$$

$$\vee_{I1}\ \frac{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \phi}{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \phi \vee \psi} \qquad \vee_{I2}\ \frac{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \psi}{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \phi \vee \psi}$$

$$\vee_E\ \frac{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \phi \vee \psi \qquad \vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \phi, \Gamma \Rightarrow \chi \qquad \vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \psi, \Gamma \Rightarrow \chi}{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \chi}$$

The Bellin, de Paiva and Ritter rule for $\mathsf{i}_\Box\mathsf{K}$

$$\Box_{\mathsf{K}}\ \frac{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \Box\phi_1 \qquad \ldots \vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \Box\phi_n \qquad \vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \phi_1, \ldots, \phi_n \Rightarrow \psi}{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \Box\psi}$$

The Sobociński-style rule for additional axioms:

$$\textsc{AxSb}\ \frac{\zeta \in \mathcal{Z} \qquad s \text{ a substitution}}{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow s(\zeta)}$$

our contributions II

- We show that a large class of axioms called enveloped implications meets the regular adequacy criterion

  Again, syntactic proof in our calculus formalized in Coq

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  - $\neg\neg$-completeness holds
  - Glivenko translation is as good as regular ones

§ 3: regular negative translations

Kolmogorov's brutal saturation

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- that is:

$$\begin{array}{rclrclrcl}
\bot^{\mathsf{kol}} & := & \bot & p^{\mathsf{kol}} & := & \neg\neg p & (\phi \wedge \psi)^{\mathsf{kol}} & := & \neg\neg(\phi^{\mathsf{kol}} \wedge \psi^{\mathsf{kol}}) \\
(\phi \vee \psi)^{\mathsf{kol}} & := & \neg\neg(\phi^{\mathsf{kol}} \vee \psi^{\mathsf{kol}}) & (\phi \to \psi)^{\mathsf{kol}} & := & \neg\neg(\phi^{\mathsf{kol}} \to \psi^{\mathsf{kol}}) & (\Box\phi)^{\mathsf{kol}} & := & \neg\neg\Box\phi^{\mathsf{kol}}
\end{array}$$

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- more parsimoniously, the effect can be approximated "from the inside" and "from the outside" *drop it on the surface and let it sink*

$$\neg\neg(p \wedge (q \vee (r \to \Box s))) \dashv\vdash_{\mathsf{i}_\Box\mathsf{K}} \neg\neg(\neg\neg p \wedge \neg\neg(q \vee (r \to \Box s)))$$

outer route: Glivenko 1929

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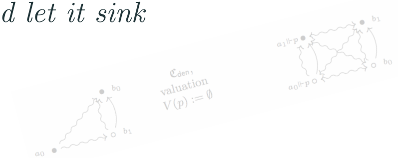

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$$\dashv\vdash_{\mathsf{i}_\Box\mathsf{K}} \neg\neg(\neg\neg p \wedge \neg\neg(\neg\neg q \vee \neg\neg(r \to \Box s)))$$

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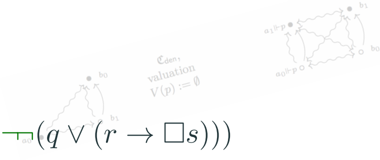

$$\neg\neg(p \wedge (q \vee (r \to \Box s))) \dashv\vdash_{\mathsf{i}_\Box\mathsf{K}} \neg\neg(\neg\neg p \wedge \neg\neg(q \vee (r \to \Box s)))$$

$$\dashv\vdash_{\mathsf{i}_\Box\mathsf{K}} \neg\neg(\neg\neg p \wedge \neg\neg(\neg\neg q \vee \neg\neg(r \to \Box s)))$$

exfalso here $\Rightarrow$ $\dashv\vdash_{\mathsf{i}_\Box\mathsf{K}} \neg\neg(\neg\neg p \wedge \neg\neg(\neg\neg q \vee \neg\neg(\neg\neg r \to \neg\neg\Box s)))$

outer route: Glivenko 1929

drop it on the surface and let it sink

$$\neg\neg(p \land (q \lor (r \to \Box s))) \dashv\vdash_{\mathsf{i}_\Box\mathsf{K}} \neg\neg(\neg\neg p \land \neg\neg(q \lor (r \to \Box s)))$$

$$\dashv\vdash_{\mathsf{i}_\Box\mathsf{K}} \neg\neg(\neg\neg p \land \neg\neg(\neg\neg q \lor \neg\neg(r \to \Box s)))$$

exfalso here $\Rightarrow$ $$\dashv\vdash_{\mathsf{i}_\Box\mathsf{K}} \neg\neg(\neg\neg p \land \neg\neg(\neg\neg q \lor \neg\neg(\neg\neg r \to \neg\neg\Box s)))$$

...but now what?

outer route: Glivenko 1929

drop it on the surface and let it sink

$$\neg\neg(p \wedge (q \vee (r \to \Box s))) \dashv\vdash_{\mathrm{i}_\Box\mathsf{K}} \neg\neg(\neg\neg p \wedge \neg\neg(q \vee (r \to \Box s)))$$

$$\dashv\vdash_{\mathrm{i}_\Box\mathsf{K}} \neg\neg(\neg\neg p \wedge \neg\neg(\neg\neg q \vee \neg\neg(r \to \Box s)))$$

exfalso here $\Rightarrow$ $$\dashv\vdash_{\mathrm{i}_\Box\mathsf{K}} \neg\neg(\neg\neg p \wedge \neg\neg(\neg\neg q \vee \neg\neg(\neg\neg r \to \neg\neg\Box s)))$$

…but now what?

The modal Double Negation Shift/Kuroda axiom

$$\mathsf{DNS}: \qquad \Box\neg\neg p \to \neg\neg\Box p.$$

not valid in general (but valid over $\mathrm{i}_\Box\mathsf{R}$!)

axiomatizes the same logic as $\neg\neg\Box p \leftrightarrow \neg\neg\Box\neg\neg p$

Kuroda's 1951: "saturated Glivenko"

$$\phi^{\mathsf{kur}} := \neg\neg\phi_{\mathsf{kur}},$$

with $(\cdot)_{\mathsf{kur}}$ defined as $(\Box\phi)_{\mathsf{kur}} := \Box\neg\neg\phi_{\mathsf{kur}}$ and identity in other inductive clauses, i.e.,

$$\begin{array}{lll} \bot_{\mathsf{kur}} := \bot & p_{\mathsf{kur}} := p & (\phi \wedge \psi)_{\mathsf{kur}} := \phi_{\mathsf{kur}} \wedge \psi_{\mathsf{kur}} \\ (\phi \vee \psi)_{\mathsf{kur}} := \phi_{\mathsf{kur}} \vee \psi_{\mathsf{kur}} & (\phi \to \psi)_{\mathsf{kur}} := \phi_{\mathsf{kur}} \to \psi_{\mathsf{kur}} & (\Box\phi)_{\mathsf{kur}} := \Box\neg\neg\phi_{\mathsf{kur}}. \end{array}$$

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This is exactly as with $\mathsf{CQC} \Rightarrow \mathsf{IQC}$.

How about the "inner route"?

inner route: Gödel and Gentzen 1933

plant $\neg\neg$ inside and let it grow

$$\Box(\neg\neg q \vee (\neg\neg r \to (\neg\neg p \wedge \neg\neg s))) \dashv\vdash_{\mathsf{i}_{\Box}\mathsf{K}} \Box(\neg\neg q \vee (\neg\neg r \to \neg\neg(\neg\neg p \wedge \neg\neg s)))$$

inner route: Gödel and Gentzen 1933

plant $\neg\neg$ inside and let it grow

$$\Box(\neg\neg q \vee (\neg\neg r \to (\neg\neg p \wedge \neg\neg s))) \dashv\vdash_{\mathsf{i}_\Box\mathsf{K}} \Box(\neg\neg q \vee (\neg\neg r \to \neg\neg(\neg\neg p \wedge \neg\neg s)))$$

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. . . now we need help

e.g., $(\phi \vee \psi)^{\mathsf{ggn}} := \neg\neg(\phi^{\mathsf{ggn}} \vee \psi^{\mathsf{ggn}})$

inner route: Gödel and Gentzen 1933

plant $\neg\neg$ inside and let it grow

$$\Box(\neg\neg q \vee (\neg\neg r \to (\neg\neg p \wedge \neg\neg s))) \dashv\vdash_{\mathsf{i}_\Box\mathsf{K}} \Box(\neg\neg q \vee (\neg\neg r \to \neg\neg(\neg\neg p \wedge \neg\neg s)))$$

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…now we need help

e.g., $(\phi \vee \psi)^{\mathsf{ggn}} := \neg\neg(\phi^{\mathsf{ggn}} \vee \psi^{\mathsf{ggn}})$

…but **unlike the predicate case**

still not enough!

$$\neg\neg\Box\neg\neg p \to \Box\neg\neg p \notin \mathsf{i}_\Box\mathsf{K}$$

Noted by Božić and Došen'84, more recently by Holliday in his work on possibility semantics

Thus, saturated Gödel-Gentzen:

$$
\begin{array}{llllll}
\bot^{\mathsf{ggs}} := \bot & & p^{\mathsf{ggs}} := \neg\neg p & & (\phi \wedge \psi)^{\mathsf{ggs}} := \phi^{\mathsf{ggs}} \wedge \psi^{\mathsf{ggs}} \\
(\phi \vee \psi)^{\mathsf{ggs}} := \neg\neg(\phi^{\mathsf{ggs}} \vee \psi^{\mathsf{ggs}}) & & (\phi \to \psi)^{\mathsf{ggs}} := \phi^{\mathsf{ggs}} \to \psi^{\mathsf{ggs}} & & (\Box\phi)^{\mathsf{ggs}} := \neg\neg\Box\phi^{\mathsf{ggs}}.
\end{array}
$$

Thus, saturated Gödel-Gentzen:

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\begin{array}{llllll}
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\end{array}
$$

A regular negative translation is a modular translation (Ferreira, Oliva) that uses at least as many $\neg\neg$ as either

- the Kuroda translation or
- the saturated Gödel-Gentzen translation,

but apart from possibly adding new occurrences of $\neg\neg$, leaves everything else unchanged

e.g., no flipping polarities like Krivine(-Streicher-Reus)

Thus, saturated Gödel-Gentzen:

$$\begin{array}{llllll} \bot^{\mathsf{ggs}} := \bot & p^{\mathsf{ggs}} := \neg\neg p & (\phi \wedge \psi)^{\mathsf{ggs}} := \phi^{\mathsf{ggs}} \wedge \psi^{\mathsf{ggs}} \\ (\phi \vee \psi)^{\mathsf{ggs}} := \neg\neg(\phi^{\mathsf{ggs}} \vee \psi^{\mathsf{ggs}}) & (\phi \to \psi)^{\mathsf{ggs}} := \phi^{\mathsf{ggs}} \to \psi^{\mathsf{ggs}} & (\Box\phi)^{\mathsf{ggs}} := \neg\neg\Box\phi^{\mathsf{ggs}}. \end{array}$$

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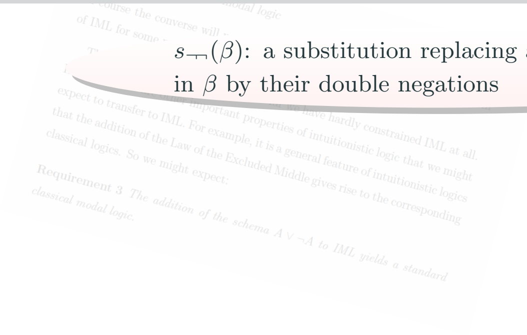
- The paper has a more formal definition in § 3.1

Regular translations are equivalent to Kolmogorov

§ 5: what axioms are $\neg\neg$-complete?

enveloped implications

$s_{\neg\neg}(\beta)$: a substitution replacing all p occurring in β by their double negations


expect to transfer to IML. For example, it is a general feature of intuitionistic logics that the addition of the Law of the Excluded Middle gives rise to the corresponding classical logics. So we might expect:

Requirement 3 *The addition of the schema $A \vee \neg A$ to IML yields a standard classical modal logic.*

enveloped implications

$s_{\neg\neg}(\beta)$: a substitution replacing all p occurring in β by their double negations

$$\neg\neg s_{\neg\neg}(\beta) \vdash_{\mathrm{i}_\Box \mathcal{Z}} \beta^{\mathsf{kol}}$$

a $(\neg\neg\text{-})$ pre-envelope

(in $\mathrm{i}_\Box \mathcal{Z}$)

or env-consequent

enveloped implications

$s_{\neg\neg}(\beta)$: a substitution replacing all p occurring in β by their double negations

$\neg\neg s_{\neg\neg}(\beta) \vdash_{\mathrm{i}_\Box\mathcal{Z}} \beta^{\mathsf{kol}}$

a ($\neg\neg$-) pre-envelope (in $\mathrm{i}_\Box\mathcal{Z}$)

or env-consequent

$\beta^{\mathsf{kol}} \vdash_{\mathrm{i}_\Box\mathcal{Z}} \neg\neg s_{\neg\neg}(\beta)$

a ($\neg\neg$-) post-envelope (in $\mathrm{i}_\Box\mathcal{Z}$)

or env-antecedent

enveloped implications

$s_{\neg\neg}(\beta)$: a substitution replacing all p occurring in β by their double negations

$\neg\neg s_{\neg\neg}(\beta) \vdash_{\mathrm{i}_\Box\mathcal{Z}} \beta^{\mathsf{kol}}$ — a $(\neg\neg\text{-})$ pre-envelope (in $\mathrm{i}_\Box\mathcal{Z}$) or env-consequent

$\beta^{\mathsf{kol}} \vdash_{\mathrm{i}_\Box\mathcal{Z}} \neg\neg s_{\neg\neg}(\beta)$ — a $(\neg\neg\text{-})$ post-envelope (in $\mathrm{i}_\Box\mathcal{Z}$) or env-antecedent

$\neg\neg s_{\neg\neg}(\beta) \dashv\vdash_{\mathrm{i}_\Box\mathcal{Z}} \beta^{\mathsf{kol}}$ — an $\neg\neg$-envelope (in $\mathrm{i}_\Box\mathcal{Z}$)

enveloped implications

$s_{\neg\neg}(\beta)$: a substitution replacing all p occurring in β by their double negations

$\neg\neg s_{\neg\neg}(\beta) \vdash_{\mathrm{i}_{\square}\mathcal{Z}} \beta^{\mathsf{kol}}$

a ($\neg\neg$-) pre-envelope (in $\mathrm{i}_{\square}\mathcal{Z}$)

or env-consequent

$\beta^{\mathsf{kol}} \vdash_{\mathrm{i}_{\square}\mathcal{Z}} \neg\neg s_{\neg\neg}(\beta)$

a ($\neg\neg$-) post-envelope (in $\mathrm{i}_{\square}\mathcal{Z}$)

or env-antecedent

$\neg\neg s_{\neg\neg}(\beta) \dashv\vdash_{\mathrm{i}_{\square}\mathcal{Z}} \beta^{\mathsf{kol}}$

an $\neg\neg$-envelope (in $\mathrm{i}_{\square}\mathcal{Z}$)

$\beta \to \gamma$, where:

β a post-envelope

γ a pre-envelope

an enveloped implication

main positive results

- A box-free formula is a $\neg\neg$-envelope

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- A box-free formula is a ¬¬-envelope
- A shallow formula with no disjunction under box is a ¬¬-envelope

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- A shallow formula with no disjunction under box is a ¬¬-envelope
- An implication-free formula is a pre-envelope (env-consequent)
- A negation of a pre-envelope is a post-envelope
- **Any logic axiomatized by enveloped implications is ¬¬-complete**

applications

$\neg\neg$-completeness holds for logics axiomatized over $\mathrm{i}_\Box\mathsf{K}$ by combinations of the following axioms

(cf. Sotirov'84, Th. 10 and L.'14, Tab. 2)

R	$p \to \Box p,$	CB	$\Box p \to (q \to p) \vee q,$	♭em	$\Box p \vee \Box\neg p,$
4	$\Box p \to \Box\Box p,$	NV	$\neg\Box\bot,$	em♭	$\Box p \vee \neg\Box p,$
T	$\Box p \to p,$	NNV	$\neg\neg\Box\bot,$	$\mathsf{T}^{\neg\neg}$	$\Box p \to \neg\neg p,$
coK	$(\Box p \to \Box q) \to \Box(p \to q),$	♭R	$p \to \Box\Box p,$	$\mathsf{T}^{\neg}$	$\Box\neg p \to \neg p,$
♭Lin	$\Box(p \to q) \vee \Box(q \to p),$	Lin♭	$(\Box p \to q) \vee (\Box q \to p),$	wem♭$^{\neg}$	$\neg\Box p \vee \neg\Box\neg p,$

or any superintuitionistic axiom, i.e., a formula in modality-free $\mathcal{L}$

An example of a standard logic obtained by a such combination:

$$\mathrm{i}_\Box\mathsf{S4} = \mathrm{i}_\Box\mathsf{4} + \mathsf{T}$$

Glivenko for strength and purity

- The modal Double Negation Shift/Kuroda axiom

$$\mathsf{DNS}: \qquad \Box\neg\neg p \to \neg\neg\Box p.$$

axiomatizes the same logic as $\neg\neg\Box p \leftrightarrow \neg\neg\Box\neg\neg p$

- An analogue of an observation in Troelstra & van Dalen:

  in any extension of $\mathsf{i}_\Box\mathsf{DNS}$, the Glivenko translation becomes equivalent to the regular ones/Kolmogorov, i.e., for every ϕ, we have that: $\vdash_{\mathsf{i}_\Box\mathsf{DNS}} \phi^{\mathsf{kol}} \leftrightarrow \phi^{\mathsf{glv}}$

- Any t saturating glv is adequate for any extension of $\mathsf{i}_\Box\mathsf{DNS}$

- DNS is a theorem of $\mathsf{i}_\Box\mathsf{R}$

  Recall that $\mathsf{i}_\Box\mathsf{R}$ is $p \to \Box p$

  The derivation has a rather CPS-like flavour ...

Future work and challenges

- Semantic characterizations in terms of stability under ↑-cofinal subframes and stability under nuclea

  cf. algebraic approach to subframe logics by G. Bezhanishvili and S. Ghilardi

- Relate to similar work syntactically investigating negative translations (mostly Glivenko) for classes of substructural logics (Ono or Ono & Farahani)
- Extend to $\Diamond$, to Krivine(-Streicher-Reus) ...

  Note that G. Bezhanishvili showed that Glivenko works for MIPC

  Note also that this is where the semantic approach could shine

- Replace $\neg\neg$ by other nuclea/monads

  Connection to Aczel's *The Russell-Prawitz modality* and Escardó & Oliva's *The Peirce Translation and the Double Negation Shift*

- Extend, reuse the Coq framework and possibly merge with other developments