negative translations and normal modality

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a standard notion at FSCD

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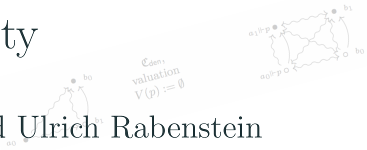

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increasingly so too, but . . .

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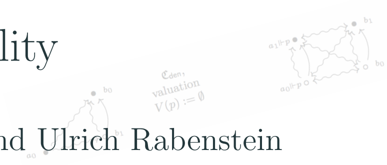


- A normal modality $\Box$ is one that distributes over finite meets/products/conjunctions, i.e.,
  - ⁕ algebraically: $\Box(\phi \wedge \psi) = \Box\phi \wedge \Box\psi$ and $\Box\top = \top$
  - ⁕ categorically: monoidal wrt cartesian structure
  - ⁕ deductively: $\vdash \Box(\phi \to \psi) \to \Box\phi \to \Box\psi$ and $\dfrac{\vdash \psi}{\vdash \Box\psi}$

    The last rule known as Necessitation or Gödel rule
- The resulting extension of the intuitionistic propositional calculus (IPC) is denoted as $\mathrm{i}_\Box\mathsf{K}$

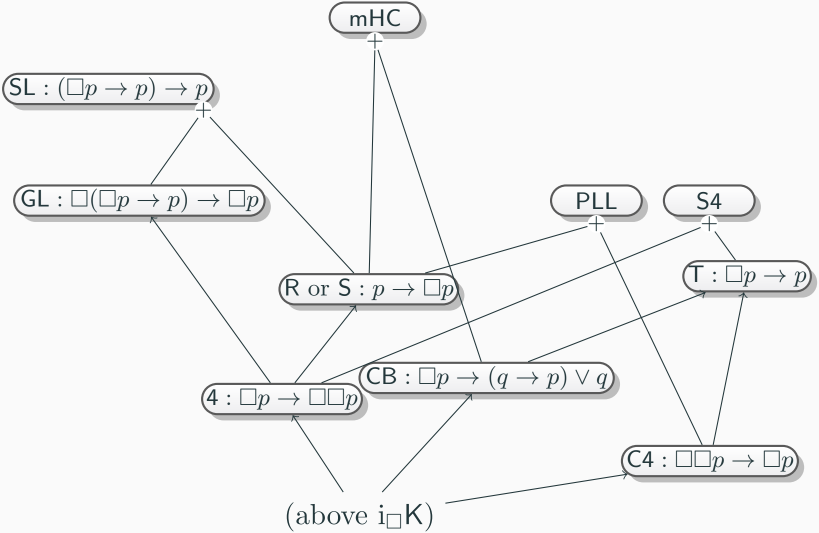
  Let us denote its language by $\mathcal{L}_\Box$
- Typically, one has more structure and more axioms

  We need a general notion
- Normal modal logic: a subset of $\mathcal{L}_\Box$ containing $\mathrm{i}_\Box\mathsf{K}$ and closed under Necessitation, MP and substitution

  Algebraically corresponds to a variety: an equational class of Heyting algebras with $\Box$

  Later, we'll see a Gentzen/ND-style formulation

some axioms to consider . . .



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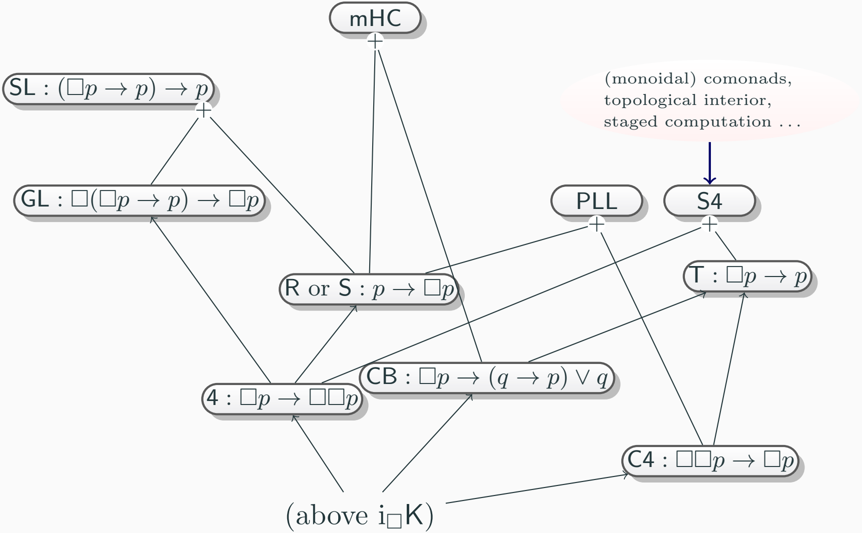

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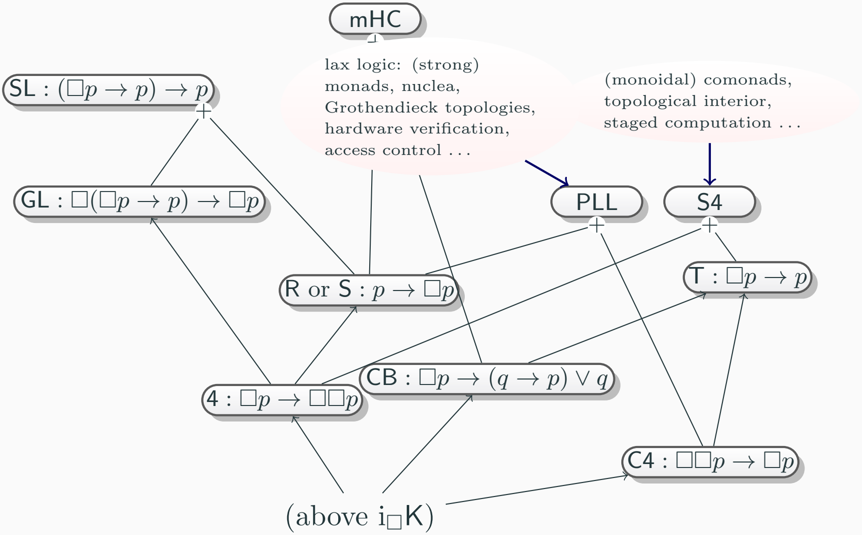

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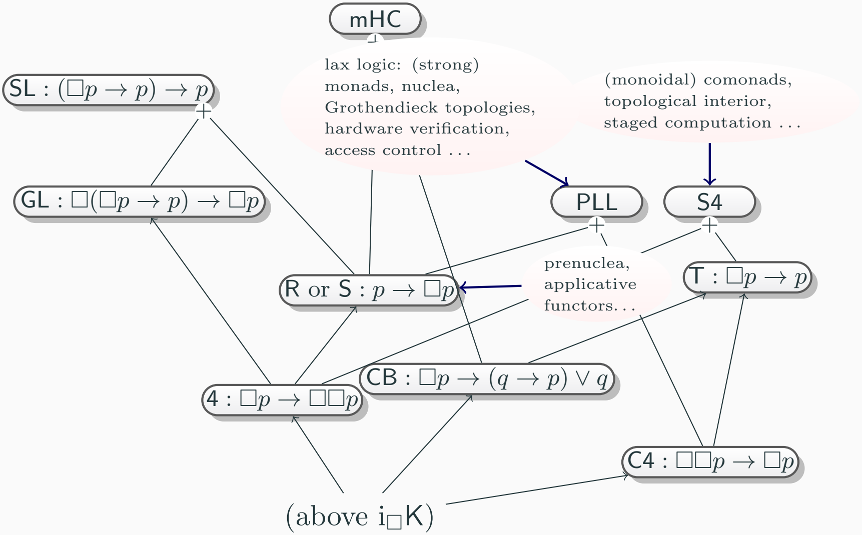

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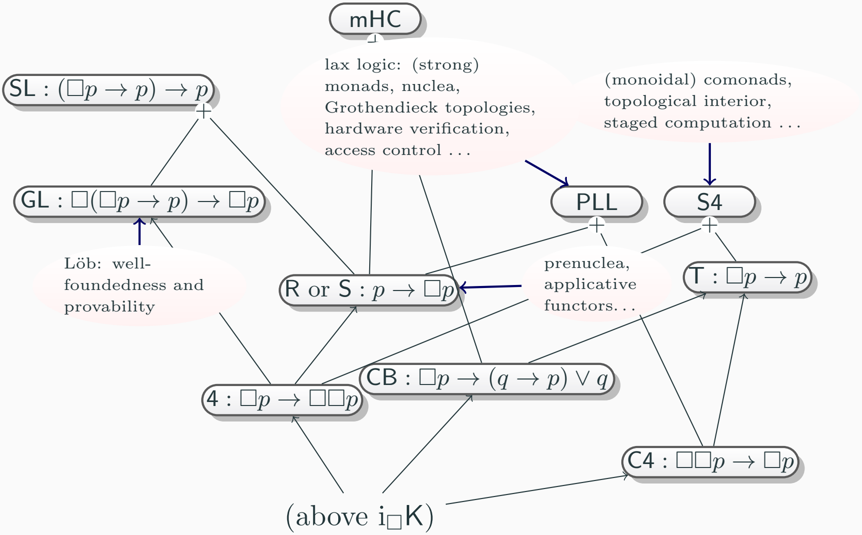

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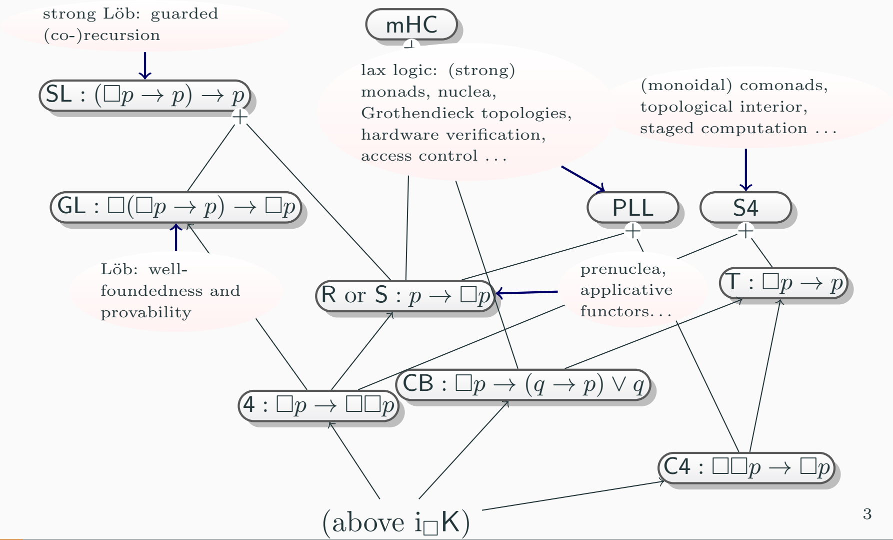

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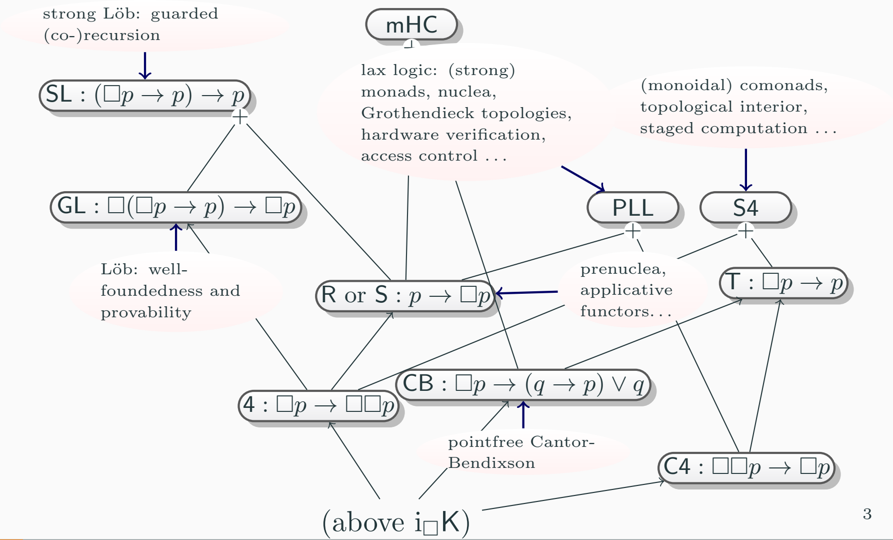

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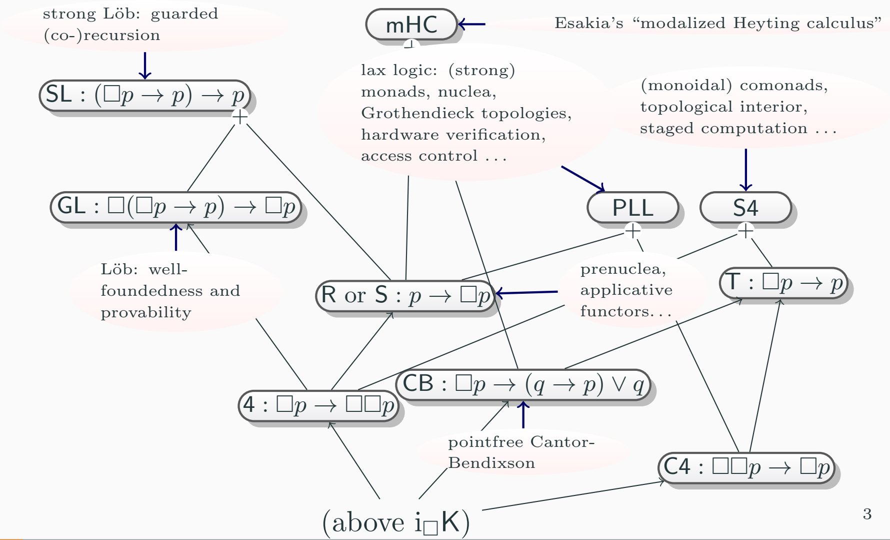

- One more example: for every x, $\forall x$ is such a $\Box$ over IQC

  (intuitionistic predicate calculus)
- $\exists x$, meanwhile, is obviously a $\Diamond$

  For more on the resulting logic, see A. Prior, R. Bull, G. Bezhanishvili ...

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- In the intuitionistic setting, it matters whether or not $\Diamond$ is included in the syntax

  Not always natural to include in the examples from the previous slide
- Highlighted notationally: for any $\mathcal{Z} \subseteq \mathcal{L}_\Box$, $\mathrm{i}_\Box\mathcal{Z}$ is the logic axiomatized by $\mathcal{Z}$

  Other conceivable conventions: $\mathsf{iK}_\Box + \mathcal{Z}$ or $\mathsf{IntK}_\Box \oplus \mathcal{Z}$.

  We use the former for join of logics and drop set brackets, i.e.,

  write $\mathrm{i}_\Box\mathcal{X} + \zeta$ instead of $\mathrm{i}_\Box(\mathcal{X} \cup \{\zeta\})$

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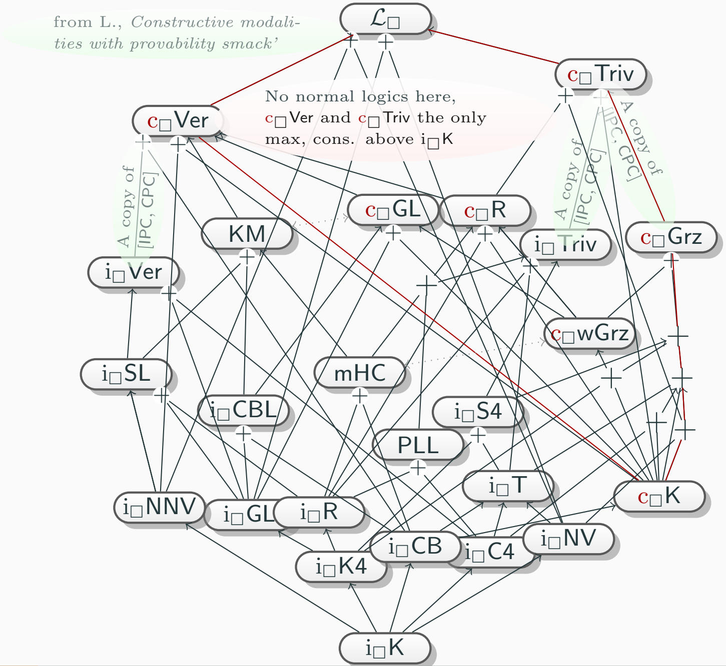
  write $\mathrm{i}_\Box\mathcal{X} + \zeta$ instead of $\mathrm{i}_\Box(\mathcal{X} \cup \{\zeta\})$
- Special case: $\mathrm{c}_\Box\mathcal{Z} := \mathrm{i}_\Box\mathcal{Z} + \neg\neg p \to p$
  i.e., logic axiomatized by $\mathcal{Z}$ over the classical propositional calculus (CPC)
- The lattice of all intuitionistic modal logics includes the lattice of all classical ones as a sublattice


from L., Constructive modalities with provability smack'
ℒ□
c□Triv
c□Ver
No normal logics here, c□Ver and c□Triv the only max, cons. above i□K
A copy of [IPC, CPC]
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A copy of [IPC, CPC]
c□GL
c□R
KM
i□Triv
c□Grz
i□Ver
c□wGrz
i□SL
mHC
i□CBL
i□S4
PLL
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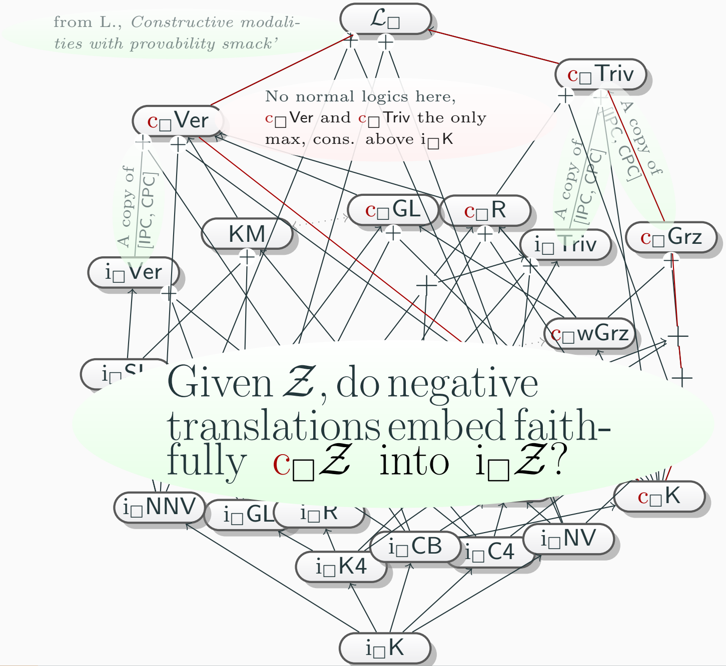
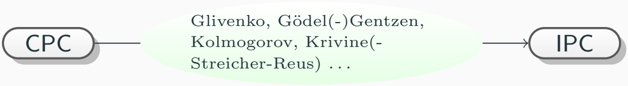


Given $\mathcal{Z}$, do negative translations embed faithfully $c_\Box\mathcal{Z}$ into $i_\Box\mathcal{Z}$?


CPC
Glivenko, Gödel(-)Gentzen,
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Streicher-Reus) . . .
IPC

CPC — Glivenko, Gödel(-)Gentzen, Kolmogorov, Krivine(-Streicher-Reus) . . . → IPC

(See, e.g., Ferreira and Oliva or classical refs like Troelstra and van Dalen)

CQC — ~~Glivenko~~ Kuroda, Gödel(-)Gentzen, Kolmogorov, Krivine(-Streicher-Reus) . . . → IQC

CPC → Glivenko, Gödel(-)Gentzen, Kolmogorov, Krivine(-Streicher-Reus) ... → IPC

$\mathrm{c}_{\Box}\mathcal{Z}$ → ??? → $\mathrm{i}_{\Box}\mathcal{Z}$

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- We had to fill it ourselves

what can go wrong . . . ?

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- No recursive translation from BBI to BI can be adequate

additional motivation: Simpson's dissertation

Of course the conv… well be a theorem of IML for s…

This … at all. Howev… might expect t… logics that the ad… esponding classical logics. S…

1. too restrictive: important intuitionistic logics can be trivialized classically
2. too permissive: what if there are "standard" modal logics for which negative translations are not adequate?

Requirement 3 *The addition of the schema $A \vee \neg A$ to IML yields a standard classical modal logic.*

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  - ⁕ natural to formalize in a proof assistant, see our Coq code

Intuitionistic propositional rules:

$$\textsc{In}\ \frac{}{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \phi, \Gamma \Rightarrow \phi} \qquad \top_I\ \frac{}{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \top} \qquad \bot_E\ \frac{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \bot}{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \phi}$$

$$\to_I\ \frac{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma, \phi \Rightarrow \psi}{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \phi \to \psi} \qquad \to_E\ \frac{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \phi \to \psi \qquad \vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \phi}{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \psi}$$

$$\wedge_I\ \frac{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \phi \qquad \vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \psi}{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \phi \wedge \psi} \qquad \wedge_{E1}\ \frac{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \phi \wedge \psi}{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \phi} \qquad \wedge_{E2}\ \frac{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \phi \wedge \psi}{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \psi}$$

$$\vee_{I1}\ \frac{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \phi}{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \phi \vee \psi} \qquad \vee_{I2}\ \frac{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \psi}{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \phi \vee \psi}$$

$$\vee_E\ \frac{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \phi \vee \psi \qquad \vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \phi, \Gamma \Rightarrow \chi \qquad \vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \psi, \Gamma \Rightarrow \chi}{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \chi}$$

The Bellin, de Paiva and Ritter rule for $\mathsf{i}_{\Box}\mathsf{K}$

$$\Box_{\mathsf{K}}\ \frac{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \Box\phi_1 \qquad \ldots \vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \Box\phi_n \qquad \vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \phi_1, \ldots, \phi_n \Rightarrow \psi}{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \Box\psi}$$

The Sobociński-style rule for additional axioms:

$$\textsc{AxSb}\ \frac{\zeta \in \mathcal{Z} \qquad s \text{ a substitution}}{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow s(\zeta)}$$

our contributions II

- We show that a large class of axioms called enveloped implications meets the regular adequacy criterion (Th. 18/Cor. 19)

  Again, syntactic proof in our calculus formalized in Coq

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  (*strong*, *pure* or *applicative* modality)
  - ⁂ $\neg\neg$-completeness holds
  - ⁂ Glivenko translation is as good as regular ones

§3: regular negative translations

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(\phi \vee \psi)^{\mathsf{kol}} & := & \neg\neg(\phi^{\mathsf{kol}} \vee \psi^{\mathsf{kol}}) & (\phi \to \psi)^{\mathsf{kol}} & := & \neg\neg(\phi^{\mathsf{kol}} \to \psi^{\mathsf{kol}}) & (\Box\phi)^{\mathsf{kol}} & := & \neg\neg\Box\phi^{\mathsf{kol}}
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- more parsimoniously, the effect can be approximated "from the inside" and "from the outside"

outer route: Glivenko 1929

drop it on the surface and let it sink

$$\neg\neg(p \land (q \lor (r \to \Box s))) \dashv\vdash_{\mathsf{i}_\Box\mathsf{K}} \neg\neg(\neg\neg p \land \neg\neg(q \lor (r \to \Box s)))$$

outer route: Glivenko 1929

drop it on the surface and let it sink

$$\neg\neg(p \wedge (q \vee (r \to \Box s))) \dashv\vdash_{\mathsf{i}_\Box\mathsf{K}} \neg\neg(\neg\neg p \wedge \neg\neg(q \vee (r \to \Box s)))$$

$$\dashv\vdash_{\mathsf{i}_\Box\mathsf{K}} \neg\neg(\neg\neg p \wedge \neg\neg(\neg\neg q \vee \neg\neg(r \to \Box s)))$$

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exfalso here $\Rightarrow$ $$\dashv\vdash_{\mathsf{i}_\Box\mathsf{K}} \neg\neg(\neg\neg p \wedge \neg\neg(\neg\neg q \vee \neg\neg(\neg\neg r \to \neg\neg\Box s)))$$

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drop it on the surface and let it sink

$$\neg\neg(p \wedge (q \vee (r \to \Box s))) \dashv\vdash_{\mathsf{i}_\Box\mathsf{K}} \neg\neg(\neg\neg p \wedge \neg\neg(q \vee (r \to \Box s)))$$

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...but now what?

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$$\neg\neg(p \wedge (q \vee (r \to \Box s))) \dashv\vdash_{\mathsf{i}_\Box\mathsf{K}} \neg\neg(\neg\neg p \wedge \neg\neg(q \vee (r \to \Box s)))$$

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. . . but now what?

The modal Double Negation Shift/Kuroda axiom

$$\mathsf{DNS}: \qquad \Box\neg\neg p \to \neg\neg\Box p.$$

not valid in general (but valid over $\mathsf{i}_\Box\mathsf{R}$!)

axiomatizes the same logic as $\neg\neg\Box p \leftrightarrow \neg\neg\Box\neg\neg p$

Kuroda's 1951: "saturated Glivenko"

$$\phi^{\mathsf{kur}} := \neg\neg\phi_{\mathsf{kur}},$$

with $(\cdot)_{\mathsf{kur}}$ defined as $(\Box\phi)_{\mathsf{kur}} := \Box\neg\neg\phi_{\mathsf{kur}}$ and identity in other inductive clauses, i.e.,

$$\begin{array}{rclrclrcl} \bot_{\mathsf{kur}} & := & \bot & p_{\mathsf{kur}} & := & p & (\phi \wedge \psi)_{\mathsf{kur}} & := & \phi_{\mathsf{kur}} \wedge \psi_{\mathsf{kur}} \\ (\phi \vee \psi)_{\mathsf{kur}} & := & \phi_{\mathsf{kur}} \vee \psi_{\mathsf{kur}} & (\phi \to \psi)_{\mathsf{kur}} & := & \phi_{\mathsf{kur}} \to \psi_{\mathsf{kur}} & (\Box\phi)_{\mathsf{kur}} & := & \Box\neg\neg\phi_{\mathsf{kur}}. \end{array}$$

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This is exactly as with $\mathsf{CQC} \Rightarrow \mathsf{IQC}$.

How about the "inner route"?

inner route: Gödel and Gentzen 1933

plant $\neg\neg$ *inside and let it grow*

$$\Box(\neg\neg q \vee (\neg\neg r \rightarrow (\neg\neg p \wedge \neg\neg s))) \dashv\vdash_{\mathsf{i}_\Box\mathsf{K}} \Box(\neg\neg q \vee (\neg\neg r \rightarrow \neg\neg(\neg\neg p \wedge \neg\neg s)))$$

inner route: Gödel and Gentzen 1933

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. . . now we need help

e.g., $(\phi \vee \psi)^{\mathsf{ggn}} := \neg\neg(\phi^{\mathsf{ggn}} \vee \psi^{\mathsf{ggn}})$

inner route: Gödel and Gentzen 1933

plant $\neg\neg$ *inside and let it grow*

$$\Box(\neg\neg q \vee (\neg\neg r \to (\neg\neg p \wedge \neg\neg s))) \dashv\vdash_{\mathrm{i}_\Box \mathsf{K}} \Box(\neg\neg q \vee (\neg\neg r \to \neg\neg(\neg\neg p \wedge \neg\neg s)))$$

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... now we need help

e.g., $(\phi \vee \psi)^{\mathsf{ggn}} := \neg\neg(\phi^{\mathsf{ggn}} \vee \psi^{\mathsf{ggn}})$

... but **unlike the predicate case**
still not enough!

$$\neg\neg\Box\neg\neg p \to \Box\neg\neg p \notin \mathrm{i}_\Box\mathsf{K}$$

Noted by Božić and Došen'84, more recently by Holliday in his work on possibility semantics

Thus, saturated Gödel-Gentzen:

$$\begin{array}{llllll} \bot^{\mathsf{ggs}} := \bot & p^{\mathsf{ggs}} := \neg\neg p & (\phi \wedge \psi)^{\mathsf{ggs}} := \phi^{\mathsf{ggs}} \wedge \psi^{\mathsf{ggs}} \\ (\phi \vee \psi)^{\mathsf{ggs}} := \neg\neg(\phi^{\mathsf{ggs}} \vee \psi^{\mathsf{ggs}}) & (\phi \to \psi)^{\mathsf{ggs}} := \phi^{\mathsf{ggs}} \to \psi^{\mathsf{ggs}} & (\Box\phi)^{\mathsf{ggs}} := \neg\neg\Box\phi^{\mathsf{ggs}}. \end{array}$$

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A regular negative translation is a modular translation (Ferreira, Oliva) that uses at least as many $\neg\neg$ as either

- the Kuroda translation or
- the saturated Gödel-Gentzen translation,

but apart from possibly adding new occurrences of $\neg\neg$, leaves everything else unchanged

e.g., no flipping polarities like Krivine(-Streicher-Reus)

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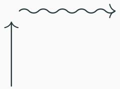
- The paper has a more formal definition in §3.1
- Regular translations are equivalent to Kolmogorov

§4: failure of $\neg\neg$-completeness

brief recap of intuitionistic frames

- intuitionistic order $\uparrow$ and modal relation $\rightsquigarrow$
- valuations $\uparrow$-upward closed
- clause for Heyting implication: scan all $\uparrow$-successors
- clause for $\Box$: scan all $\rightsquigarrow$-successors
- interaction condition:
  - ⁎ Minimal (Božić and Došen'84): $\uparrow ; \rightsquigarrow \quad \subseteq \quad \rightsquigarrow ; \uparrow$



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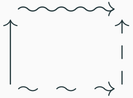
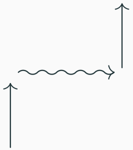


  - Canonical (most references) : $\uparrow;\rightsquigarrow;\uparrow \quad \subseteq \quad \rightsquigarrow$



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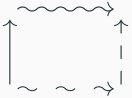

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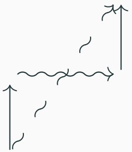

$\mathfrak{C}_{\mathsf{den}}$: a frame ...

- validating $\mathsf{C4} : \Box\Box p \to \Box p$

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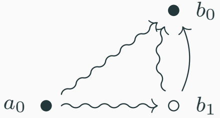

• $\rightsquigarrow$-irreflexive, ∘ $\rightsquigarrow$-reflexive

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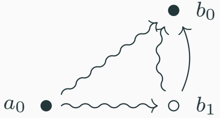

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- just pick $V(p) := \emptyset$ and get

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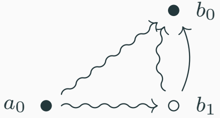

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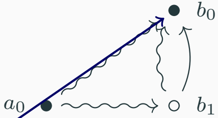

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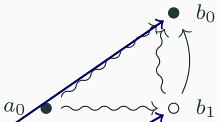

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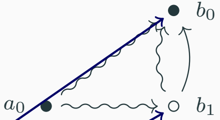

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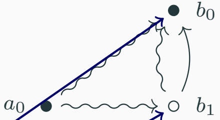
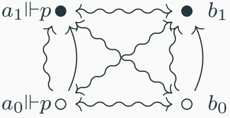


$\bullet$ $\rightsquigarrow$-irreflexive, $\circ$ $\rightsquigarrow$-reflexive

- just pick $V(p) := \emptyset$ and get
- $\overline{V}(\neg\neg p) = \emptyset$
- $\overline{V}(\Box\neg\neg p) = \{b_0\} = b_0\!\uparrow$
- $\overline{V}(\neg\neg\Box\neg\neg p) = \{b_0, b_1\} = b_1\!\uparrow$
- $\overline{V}(\Box\neg\neg\Box\neg\neg p)$ is the whole thing
- $a_0 \not\models \neg\neg\Box\neg\neg\Box\neg\neg p \to \neg\neg\Box\neg\neg p$

similarly, $\mathfrak{C}_{\mathsf{tr2}}$ is a frame ...

- validating $\flat 4 := \Box\Box p \to \Box\Box\Box p$
- refuting $\neg\neg\Box\neg\neg\Box\neg\neg p \to \neg\neg\Box\neg\neg\Box\neg\neg\Box\neg\neg p$



-

$$a_1{\uparrow} = \overline{W}(\Box\neg\neg\Box\neg\neg p), \qquad b_1{\uparrow} = \overline{W}(\Box\neg\neg p) = \overline{W}(\Box\neg\neg\Box\neg\neg\Box\neg\neg p),$$
$$a_0{\uparrow} = W(p) = \overline{W}(\neg\neg p) = \overline{W}(\neg\neg\Box\neg\neg\Box\neg\neg p), \qquad b_0{\uparrow} = \overline{W}(\neg\neg\Box\neg\neg p) = \overline{W}(\neg\neg\Box\neg\neg\Box\neg\neg\Box\neg\neg p).$$

- See Example 13 in the paper

- These counterexamples were well-behaved in at least one sense

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- Recall that our problem is:

$$\text{given } \zeta \in \mathcal{L}_\Box\text{, is there any } \phi \text{ s.t.}$$
$$\phi \in \mathrm{c}_\Box\mathsf{K} + \zeta \text{ but } \phi^t \notin \mathrm{i}_\Box\mathsf{K} + \zeta$$

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$$\phi \in \mathrm{c}_\Box\mathsf{K} + \zeta \text{ but } \phi^t \notin \mathrm{i}_\Box\mathsf{K} + \zeta$$

- In both cases, we just took $\phi = \zeta$
- Can it happen that ζ wouldn't yield a counterexample, but some other ϕ would?
- No—and this is what the first of our main results (Regular Adequacy) is saying!

§5: what axioms are $\neg\neg$-complete?

enveloped implications

$s_{\neg\neg}(\beta)$: a substitution replacing all p occurring in β by their double negations

enveloped implications

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$$\neg\neg s_{\neg\neg}(\beta) \vdash_{\mathrm{i}_\Box\mathcal{Z}} \beta^{\mathsf{kol}}$$

a ($\neg\neg$-) pre-envelope

(in $\mathrm{i}_\Box\mathcal{Z}$)

or env-consequent

enveloped implications

$s_{\neg\neg}(\beta)$: a substitution replacing all p occurring in β by their double negations

$\neg\neg s_{\neg\neg}(\beta) \vdash_{\mathrm{i}_\Box \mathcal{Z}} \beta^{\mathsf{kol}}$

a ($\neg\neg$-) pre-envelope (in $\mathrm{i}_\Box \mathcal{Z}$)

or env-consequent

$\beta^{\mathsf{kol}} \vdash_{\mathrm{i}_\Box \mathcal{Z}} \neg\neg s_{\neg\neg}(\beta)$

a ($\neg\neg$-) post-envelope (in $\mathrm{i}_\Box \mathcal{Z}$)

or env-antecedent

enveloped implications

$s_{\neg\neg}(\beta)$: a substitution replacing all p occurring in β by their double negations

$\neg\neg s_{\neg\neg}(\beta) \vdash_{\mathrm{i}_{\Box}\mathcal{Z}} \beta^{\mathsf{kol}}$ — a ($\neg\neg$-) pre-envelope (in $\mathrm{i}_{\Box}\mathcal{Z}$) or env-consequent

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$\neg\neg s_{\neg\neg}(\beta) \dashv\vdash_{\mathrm{i}_{\Box}\mathcal{Z}} \beta^{\mathsf{kol}}$ — an $\neg\neg$-envelope (in $\mathrm{i}_{\Box}\mathcal{Z}$)

enveloped implications

$s_{\neg\neg}(\beta)$: a substitution replacing all p occurring in β by their double negations

$\neg\neg s_{\neg\neg}(\beta) \vdash_{\mathrm{i}_\Box \mathcal{Z}} \beta^{\mathsf{kol}}$

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a ($\neg\neg$-) post-envelope (in $\mathrm{i}_\Box \mathcal{Z}$) or env-antecedent

$\neg\neg s_{\neg\neg}(\beta) \dashv\vdash_{\mathrm{i}_\Box \mathcal{Z}} \beta^{\mathsf{kol}}$

an $\neg\neg$-envelope (in $\mathrm{i}_\Box \mathcal{Z}$)

$\beta \to \gamma$, where:

β a post-envelope

γ a pre-envelope

an enveloped implication

lemma 17 & theorem 18

- A box-free formula is a $\neg\neg$-envelope

lemma 17 & theorem 18

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lemma 17 & theorem 18

- A box-free formula is a $\neg\neg$-envelope
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- An implication-free formula is a pre-envelope (env-consequent)
- A negation of a pre-envelope is a post-envelope
- **Any logic axiomatized by enveloped implications is $\neg\neg$-complete**

applications (corollary 19)

$\neg\neg$-completeness holds for logics axiomatized over $\mathrm{i}_\Box\mathsf{K}$ by combinations of the following axioms

(cf. Sotirov'84, Th. 10 and Litak'14, Tab. 2)

R	$p \to \Box p$,	CB	$\Box p \to (q \to p) \vee q$,	$\flat$em	$\Box p \vee \Box\neg p$,
4	$\Box p \to \Box\Box p$,	NV	$\neg\Box\bot$,	em$\flat$	$\Box p \vee \neg\Box p$,
T	$\Box p \to p$,	NNV	$\neg\neg\Box\bot$,	$\mathsf{T}^{\neg\neg}$	$\Box p \to \neg\neg p$,
coK	$(\Box p \to \Box q) \to \Box(p \to q)$,	$\flat$R	$p \to \Box\Box p$,	$\mathsf{T}^{\neg}$	$\Box\neg p \to \neg p$,
$\flat$Lin	$\Box(p \to q) \vee \Box(q \to p)$,	Lin$\flat$	$(\Box p \to q) \vee (\Box q \to p)$,	wem$\flat^{\neg}$	$\neg\Box p \vee \neg\Box\neg p$,

or any superintuitionistic axiom, i.e., a formula in modality-free $\mathcal{L}$

An example of a standard logic obtained by a such combination:

$$\mathrm{i}_\Box\mathsf{S4} = \mathrm{i}_\Box\mathsf{4} + \mathsf{T}$$

§6: Glivenko for strength and purity

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  in any extension of $\mathrm{i}_\Box\mathsf{DNS}$, the Glivenko translation becomes equivalent to the regular ones/Kolmogorov, i.e.,
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- DNS is a theorem of $\mathrm{i}_\Box\mathsf{R}$ (Th. 23)

  The derivation has a rather CPS-like flavour ...

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- Extend, reuse the Coq framework and possibly merge with other developments

  `git clone git://git8.cs.fau.de/dnegmod`

  `https://cal8.cs.fau.de/redmine/projects/dnegmod`