negative translations and normal modality

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Informatik 8, FAU Erlangen-Nuremberg

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my students who provided a Coq formalization

those who heard ALCOP 2016 talk will see what changed

the more proof-theoretical helped us into FSCD 2017

- A normal modality $\Box$ is one that distributes over finite meets/products/conjunctions, i.e.,
  - ⁕ algebraically: $\Box(\phi \wedge \psi) = \Box\phi \wedge \Box\psi$ and $\Box\top = \top$
  - ⁕ categorically: monoidal wrt cartesian structure
  - ⁕ deductively: $\vdash \Box(\phi \to \psi) \to \Box\phi \to \Box\psi$ and $\dfrac{\vdash \psi}{\vdash \Box\psi}$

    The last rule known as Necessitation or Gödel rule
- The resulting extension of the intuitionistic propositional calculus (IPC) is denoted as $\mathrm{i}_{\Box}\mathsf{K}$

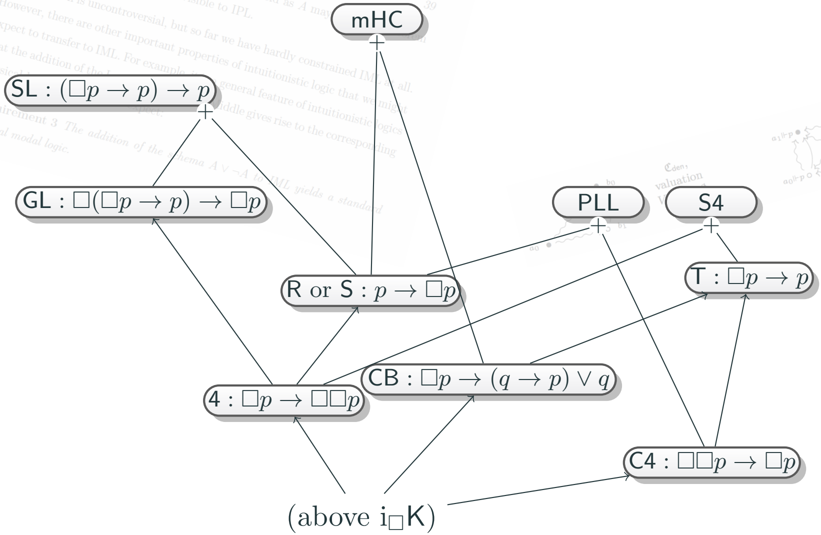
  Let us denote its language by $\mathcal{L}_{\Box}$
- Typically, one has more structure and more axioms

  We need a general notion
- Normal modal logic: a subset of $\mathcal{L}_{\Box}$ containing $\mathrm{i}_{\Box}\mathsf{K}$ and closed under Necessitation, MP and substitution

  Algebraically corresponds to a variety: an equational class of Heyting algebras with $\Box$

  Later, we'll see a Gentzen/ND-style formulation

some axioms to consider . . .



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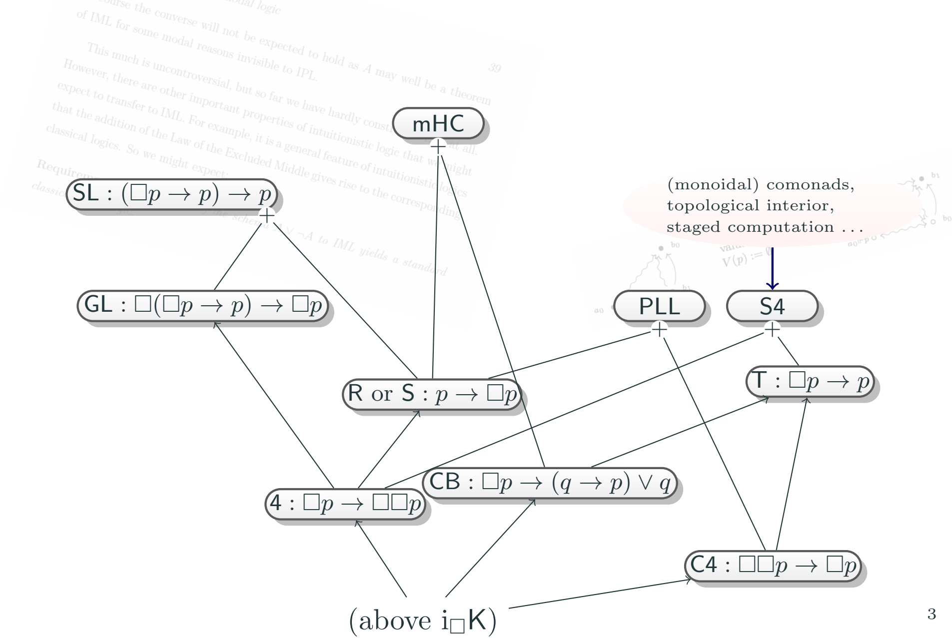

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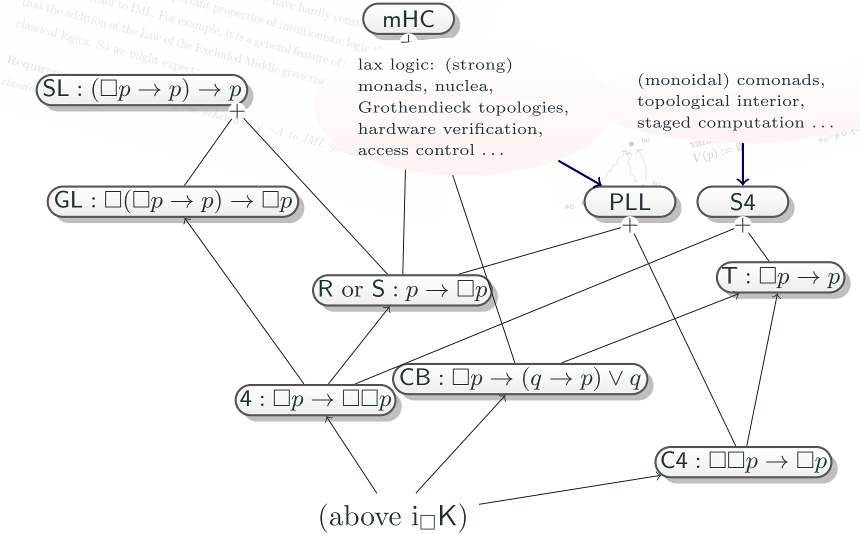

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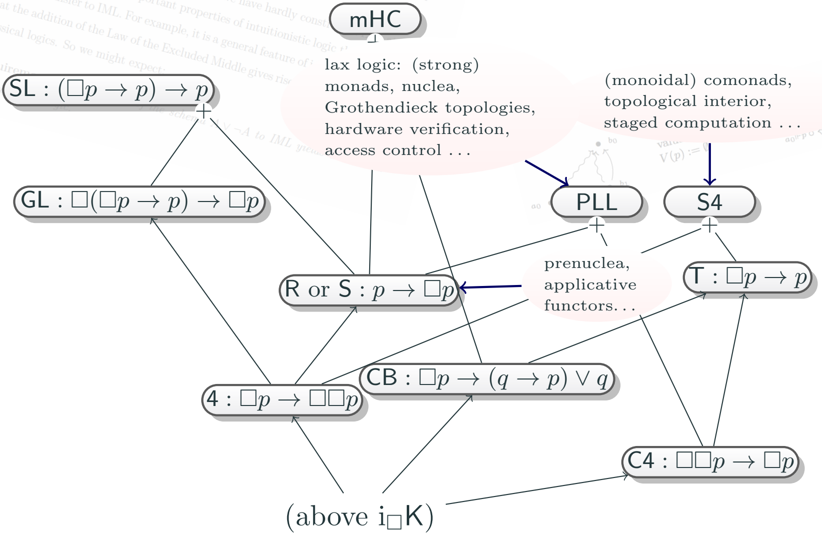

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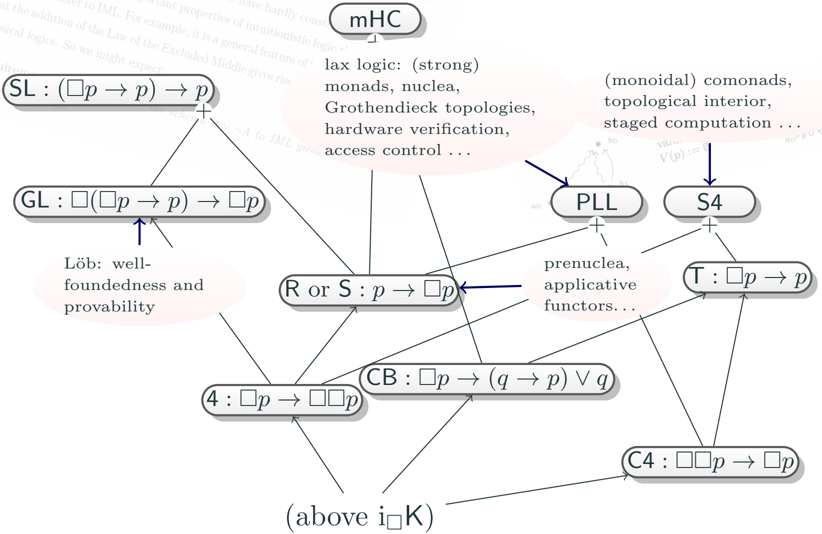

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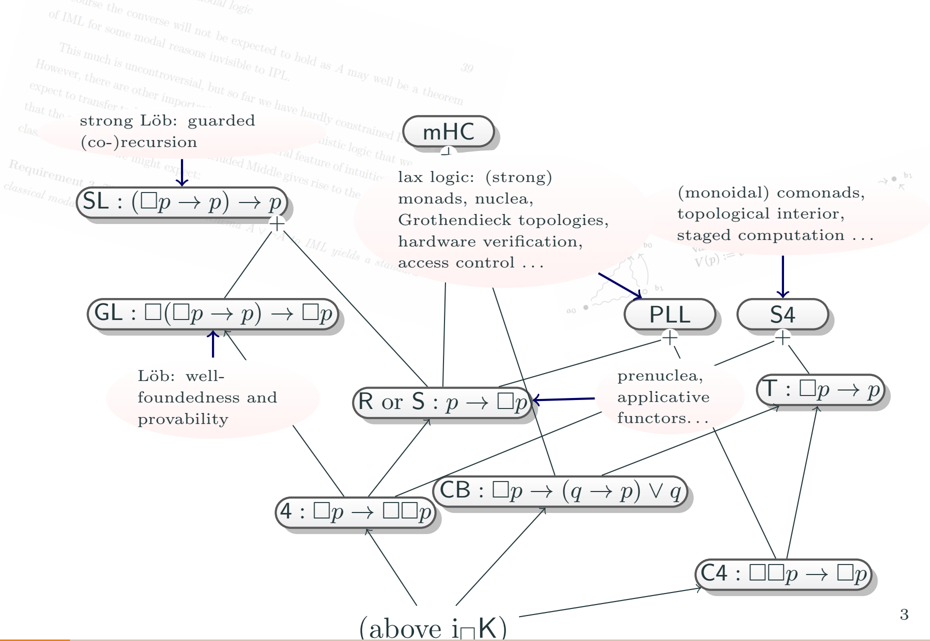

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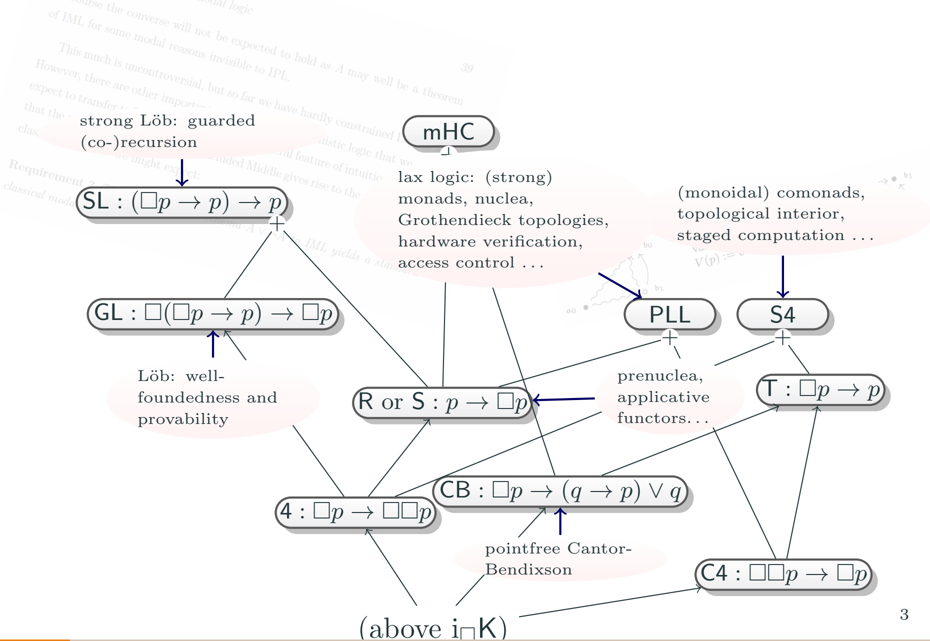

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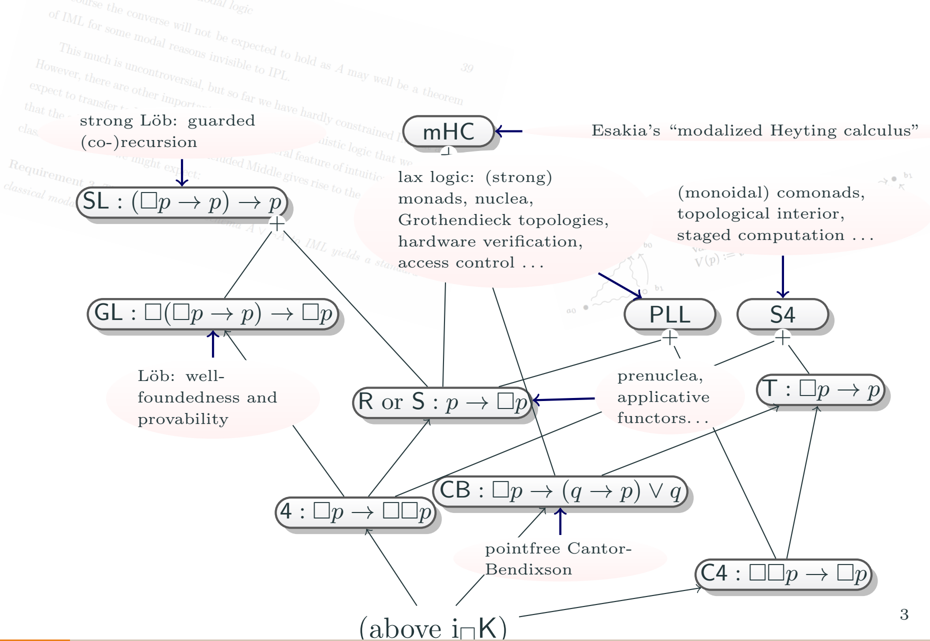

- One more example: for every x, $\forall x$ is such a $\Box$ over IQC (intuitionistic predicate calculus)
- $\exists x$, meanwhile, is obviously a $\Diamond$

For more on the resulting logic, see A. Prior, R. Bull, G. Bezhanishvili ...

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- In the intuitionistic setting, it matters whether or not $\Diamond$ is included in the syntax

  Not always natural to include in the examples from the previous slide

- Highlighted notationally: for any $\mathcal{Z} \subseteq \mathcal{L}_{\Box}$, $\mathrm{i}_{\Box}\mathcal{Z}$ is the logic axiomatized by $\mathcal{Z}$

  Other conceivable conventions: $\mathsf{iK}_{\Box} + \mathcal{Z}$ or $\mathsf{IntK}_{\Box} \oplus \mathcal{Z}$.

  We use the former for join of logics and drop set brackets, i.e.,

  write $\mathrm{i}_{\Box}\mathcal{X} + \zeta$ instead of $\mathrm{i}_{\Box}(\mathcal{X} \cup \{\zeta\})$

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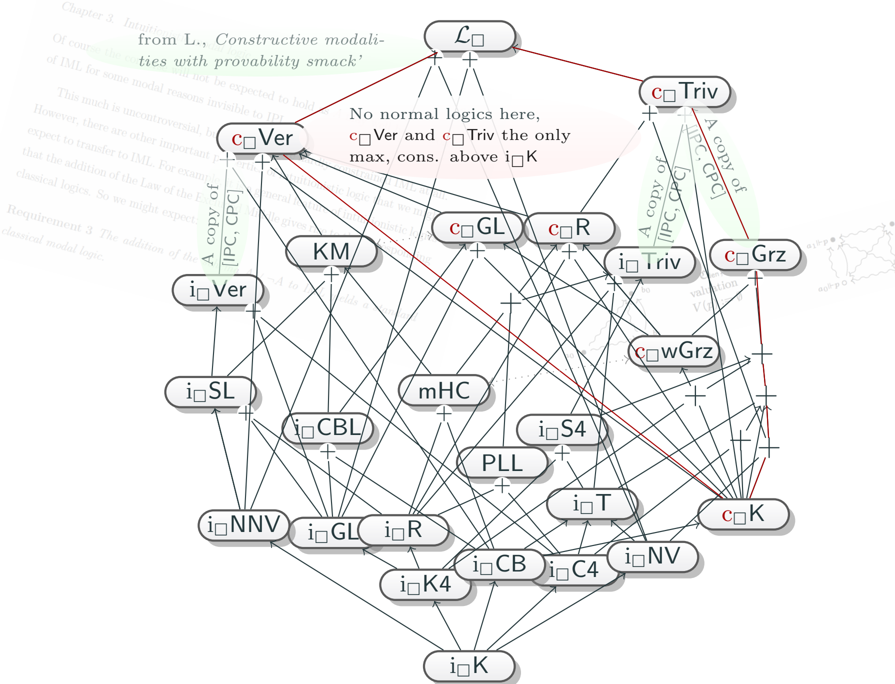
- Special case: $\mathrm{c}_\Box\mathcal{Z} := \mathrm{i}_\Box\mathcal{Z} + \neg\neg p \to p$

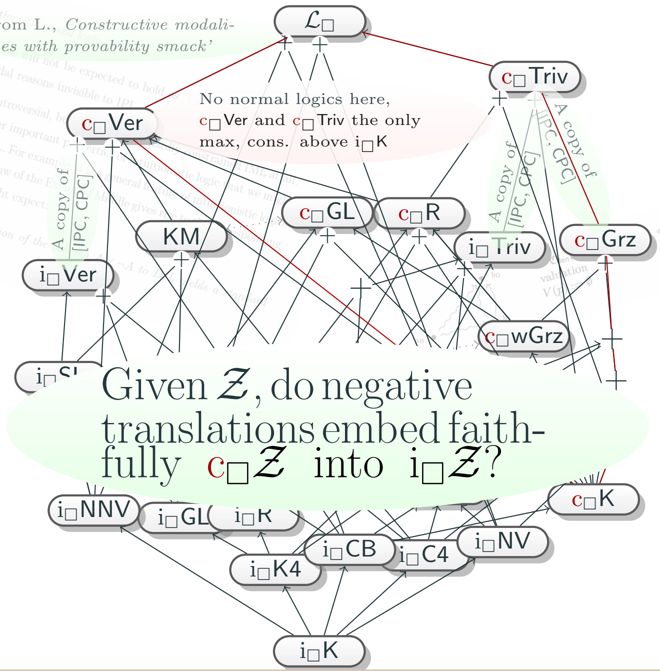
  i.e., logic axiomatized by $\mathcal{Z}$ over the classical propositional calculus (CPC)

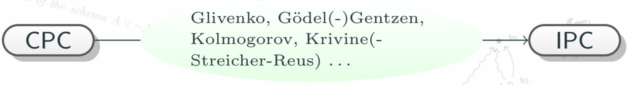
- The lattice of all intuitionistic modal logics includes the lattice of all classical ones as a sublattice



from L., *Constructive modalities with provability smack'*
$\mathcal{L}_\Box$
c□Triv
No normal logics here, c□Ver and c□Triv the only max, cons. above i□K
c□Ver
A copy of [IPC, CPC]
A copy of [IPC, CPC]
A copy of [IPC, CPC]
c□GL
c□R
KM
i□Triv
c□Grz
i□Ver
c□wGrz
i□SL
mHC
i□CBL
i□S4
PLL
i□T
c□K
i□NNV
i□GL
i□R
i□CB
i□C4
i□NV
i□K4
i□K

Given $\mathcal{Z}$, do negative translations embed faithfully $c_\Box\mathcal{Z}$ into $i_\Box\mathcal{Z}$?




CPC
Glivenko, Gödel(-)Gentzen,
Kolmogorov, Krivine(-
Streicher-Reus) ...
IPC

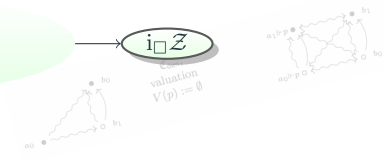
CPC — Glivenko, Gödel(-)Gentzen, Kolmogorov, Krivine(-Streicher-Reus) . . . → IPC

(See, e.g., Ferreira and Oliva or classical refs like Troelstra and van Dalen)

CQC — ~~Glivenko~~ Kuroda, Gödel(-)Gentzen, Kolmogorov, Krivine(-Streicher-Reus) . . . → IQC

CPC — Glivenko, Gödel(-)Gentzen, Kolmogorov, Krivine(-Streicher-Reus) . . . → IPC

$c_\Box\mathcal{Z}$ — ??? → $i_\Box\mathcal{Z}$



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- Consider ζ, ϕ in the intuitionistic modal signature $\mathcal{L}_{\Box}$

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- We had to fill it ourselves

what can go wrong . . . ?

- Everything!

However, there are other important properties of intuitionistic logic that we might expect to transfer to IML. For example, it is a general feature of intuitionistic logics that the addition of the Law of the Excluded Middle gives rise to the corresponding classical logics. So we might expect:

Requirement 3 *The addition of the schema $A \vee \neg A$ to IML yields a standard classical modal logic.*

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BI vs. BBI

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- No recursive translation from BBI to BI can be adequate

additional motivation: Simpson's dissertation

Chapter 3. Intuitionistic modal logic *39*

Of course the conv… well be a theorem of IML for s…

This … at all. Howev… might expect t… logics that the ad… esponding classical logics. S…

1. **too restrictive: important intuitionistic logics can be trivialized classically**
2. **too permissive: what if there are "standard" modal logics for which negative translations are not adequate?**

Requirement 3 *The addition of the schema $A \vee \neg A$ to IML yields a standard classical modal logic.*

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- We isolate the well-behaved class of regular translations: $\mathsf{i}_{\Box}\mathsf{K}$-equivalent to Kolmogorov (at least for theoremhood)

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  - ⁎ a variant of Bellin et al. ND calculus plus combinators
  - ⁎ natural to formalize in a proof assistant, see our Coq code

Intuitionistic propositional rules:

$$\textsc{In}\ \frac{}{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \phi, \Gamma \Rightarrow \phi} \qquad \top_I\ \frac{}{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \top} \qquad \bot_E\ \frac{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \bot}{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \phi}$$

$$\to_I\ \frac{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma, \phi \Rightarrow \psi}{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \phi \to \psi} \qquad \to_E\ \frac{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \phi \to \psi \qquad \vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \phi}{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \psi}$$

$$\wedge_I\ \frac{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \phi \qquad \vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \psi}{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \phi \wedge \psi} \qquad \wedge_{E1}\ \frac{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \phi \wedge \psi}{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \phi} \qquad \wedge_{E2}\ \frac{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \phi \wedge \psi}{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \psi}$$

$$\vee_{I1}\ \frac{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \phi}{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \phi \vee \psi} \qquad \vee_{I2}\ \frac{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \psi}{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \phi \vee \psi}$$

$$\vee_E\ \frac{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \phi \vee \psi \qquad \vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \phi, \Gamma \Rightarrow \chi \qquad \vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \psi, \Gamma \Rightarrow \chi}{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \chi}$$

The Bellin, de Paiva and Ritter rule for $\mathsf{i}_\Box\mathsf{K}$

$$\Box_{\mathsf{K}}\ \frac{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \Box\phi_1 \qquad \ldots \vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \Box\phi_n \qquad \vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \phi_1, \ldots, \phi_n \Rightarrow \psi}{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow \Box\psi}$$

The Sobociński-style rule for additional axioms:

$$\textsc{AxSb}\ \frac{\zeta \in \mathcal{Z} \qquad s \text{ a substitution}}{\vdash_{\mathsf{Ni}_\Box \mathcal{Z}} \Gamma \Rightarrow s(\zeta)}$$

our contributions II

- We show that a large class of axioms called enveloped implications meets the regular adequacy criterion (Th. 18/Cor. 19)

  Again, syntactic proof in our calculus formalized in Coq

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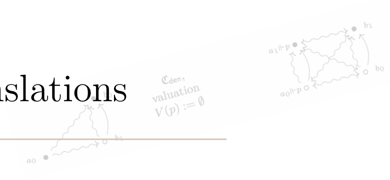
  ⁎ Glivenko translation is as good as regular ones

§ 3: regular negative translations



Kolmogorov's brutal saturation

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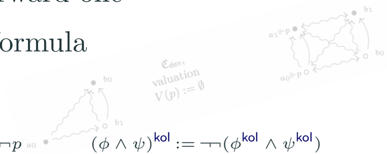

$$
\begin{array}{rclrclrcl}
\bot^{\mathsf{kol}} & := & \bot & p^{\mathsf{kol}} & := & \neg\neg p & (\phi \wedge \psi)^{\mathsf{kol}} & := & \neg\neg(\phi^{\mathsf{kol}} \wedge \psi^{\mathsf{kol}}) \\
(\phi \vee \psi)^{\mathsf{kol}} & := & \neg\neg(\phi^{\mathsf{kol}} \vee \psi^{\mathsf{kol}}) & (\phi \to \psi)^{\mathsf{kol}} & := & \neg\neg(\phi^{\mathsf{kol}} \to \psi^{\mathsf{kol}}) & (\Box\phi)^{\mathsf{kol}} & := & \neg\neg\Box\phi^{\mathsf{kol}}
\end{array}
$$

Kolmogorov's brutal saturation

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- that is:

$$\begin{array}{llllllll} \bot^{\mathsf{kol}} & := & \bot & p^{\mathsf{kol}} & := & \neg\neg p & (\phi \wedge \psi)^{\mathsf{kol}} := \neg\neg(\phi^{\mathsf{kol}} \wedge \psi^{\mathsf{kol}}) \\ (\phi \vee \psi)^{\mathsf{kol}} & := & \neg\neg(\phi^{\mathsf{kol}} \vee \psi^{\mathsf{kol}}) & (\phi \to \psi)^{\mathsf{kol}} & := & \neg\neg(\phi^{\mathsf{kol}} \to \psi^{\mathsf{kol}}) & (\Box\phi)^{\mathsf{kol}} := \neg\neg\Box\phi^{\mathsf{kol}} \end{array}$$

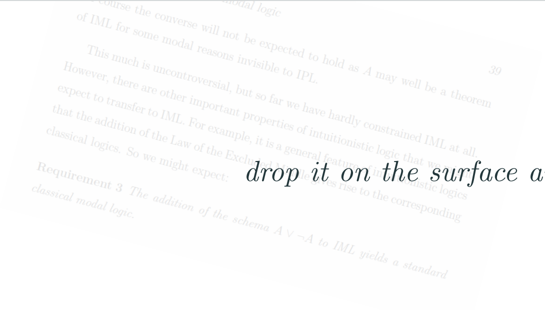
- more parsimoniously, the effect can be approximated "from the inside" and "from the outside"

outer route: Glivenko 1929

drop it on the surface and let it sink

$$\neg\neg(p \wedge (q \vee (r \to \Box s))) \dashv\vdash_{\mathsf{i}_\Box \mathsf{K}} \neg\neg(\neg\neg p \wedge \neg\neg(q \vee (r \to \Box s)))$$

outer route: Glivenko 1929

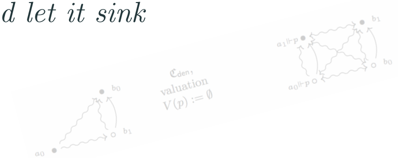


...odal logic

...course the converse will not be expected to hold as A may well be a theorem of IML for some modal reasons invisible to IPL. 39

This much is uncontroversial, but so far we have hardly constrained IML at all. However, there are other important properties of intuitionistic logic that we might expect to transfer to IML. For example, it is a general feature of intuitionistic logics that the addition of the Law of the Excluded Middle gives rise to the corresponding classical logics. So we might expect:

Requirement 3 *The addition of the schema $A \vee \neg A$ to IML yields a standard classical modal logic.*



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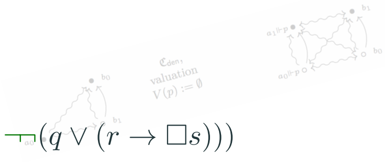

$$\neg\neg(p \wedge (q \vee (r \to \Box s))) \dashv\vdash_{\mathsf{i}_\Box\mathsf{K}} \neg\neg(\neg\neg p \wedge \neg\neg(q \vee (r \to \Box s)))$$

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exfalso here $\Rightarrow$

$$\dashv\vdash_{\mathsf{i}_\Box\mathsf{K}} \neg\neg(\neg\neg p \wedge \neg\neg(\neg\neg q \vee \neg\neg(\neg\neg r \to \neg\neg\Box s)))$$

outer route: Glivenko 1929

drop it on the surface and let it sink

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outer route: Glivenko 1929

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. . . but now what?

The modal Double Negation Shift/Kuroda axiom

DNS: $\Box\neg\neg p \to \neg\neg\Box p.$

not valid in general (but valid over $\mathsf{i}_\Box\mathsf{R}$!)

axiomatizes the same logic as $\neg\neg\Box p \leftrightarrow \neg\neg\Box\neg\neg p$

Kuroda's 1951: "saturated Glivenko"

$$\phi^{\mathsf{kur}} := \neg\neg\phi_{\mathsf{kur}},$$

with $(\cdot)_{\mathsf{kur}}$ defined as $(\Box\phi)_{\mathsf{kur}} := \Box\neg\neg\phi_{\mathsf{kur}}$ and identity in other inductive clauses, i.e.,

$$\begin{array}{rclcrclcrcl}
\bot_{\mathsf{kur}} & := & \bot & & p_{\mathsf{kur}} & := & p & & (\phi \wedge \psi)_{\mathsf{kur}} & := & \phi_{\mathsf{kur}} \wedge \psi_{\mathsf{kur}} \\
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\end{array}$$

This is exactly as with $\mathsf{CQC} \Rightarrow \mathsf{IQC}$.

How about the "inner route"?

inner route: Gödel and Gentzen 1933

plant $\neg\neg$ *inside and let it grow*

$$\Box(\neg\neg q \vee (\neg\neg r \to (\neg\neg p \wedge \neg\neg s))) \dashv\vdash_{\mathsf{i}_\Box \mathsf{K}} \Box(\neg\neg q \vee (\neg\neg r \to \neg\neg(\neg\neg p \wedge \neg\neg s)))$$

inner route: Gödel and Gentzen 1933

plant $\neg\neg$ inside and let it grow

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…now we need help

e.g., $(\phi \vee \psi)^{\mathsf{ggn}} := \neg\neg(\phi^{\mathsf{ggn}} \vee \psi^{\mathsf{ggn}})$

inner route: Gödel and Gentzen 1933

plant $\neg\neg$ *inside and let it grow*

$$\Box(\neg\neg q \vee (\neg\neg r \to (\neg\neg p \wedge \neg\neg s))) \dashv\vdash_{\mathsf{i}_\Box\mathsf{K}} \Box(\neg\neg q \vee (\neg\neg r \to \neg\neg(\neg\neg p \wedge \neg\neg s)))$$

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. . . now we need help

e.g., $(\phi \vee \psi)^{\mathsf{ggn}} := \neg\neg(\phi^{\mathsf{ggn}} \vee \psi^{\mathsf{ggn}})$

. . . but **unlike the predicate case**

still not enough!

$$\neg\neg\Box\neg\neg p \to \Box\neg\neg p \notin \mathrm{i}_\Box\mathsf{K}$$

Noted by Božić and Došen'84, more recently by Holliday in his work on possibility semantics

Thus, saturated Gödel-Gentzen:

$$
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A regular negative translation is a modular translation (Ferreira, Oliva) that uses at least as many $\neg\neg$ as either

- the Kuroda translation or
- the saturated Gödel-Gentzen translation,

but apart from possibly adding new occurrences of $\neg\neg$, leaves everything else unchanged

e.g., no flipping polarities like Krivine(-Streicher-Reus)

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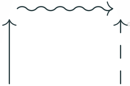
- The paper has a more formal definition in § 3.1
- Regular translations are equivalent to Kolmogorov

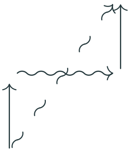
§ 4: failure of ¬¬-completeness

brief recap of intuitionistic frames

- intuitionistic order $\uparrow$ and modal relation $\rightsquigarrow$
- valuations $\uparrow$-upward closed
- clause for Heyting implication: scan all $\uparrow$-successors
- clause for $\Box$: scan all $\rightsquigarrow$-successors
- interaction condition:
  - ⁎ Minimal (Božić and Došen'84): $\uparrow;\rightsquigarrow \quad \subseteq \quad \rightsquigarrow;\uparrow$



  - ⁎ Canonical (most references) : $\uparrow;\rightsquigarrow;\uparrow \quad \subseteq \quad \rightsquigarrow$



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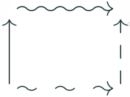

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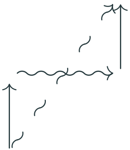
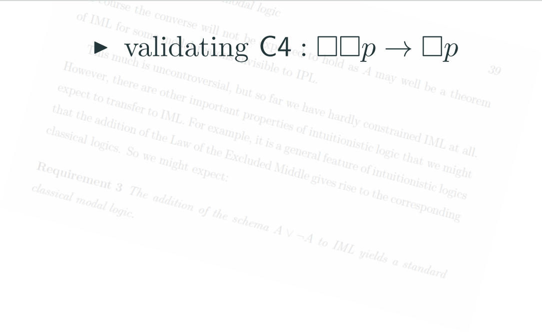


$\mathfrak{C}_{\mathsf{den}}$: a frame ...



course the converse ... modal logic
... may well be a theorem
... visible to IPL. 39

This much is uncontroversial, but so far we have hardly constrained IML at all. However, there are other important properties of intuitionistic logic that we might expect to transfer to IML. For example, it is a general feature of intuitionistic logics that the addition of the Law of the Excluded Middle gives rise to the corresponding classical logics. So we might expect:

Requirement 3 *The addition of the schema $A \vee \neg A$ to IML yields a standard classical modal logic.*

- validating $\mathsf{C4}$: $\Box\Box p \to \Box p$

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- validating $\mathsf{C4} : \Box\Box p \to \Box p$
- refuting $\neg\neg\Box\neg\neg\Box\neg\neg p \to \neg\neg\Box\neg\neg p$

$\bullet$ $\rightsquigarrow$-irreflexive, $\circ$ $\rightsquigarrow$-reflexive

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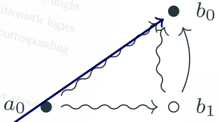

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- $\overline{V}(\Box\neg\neg p) = \{b_0\} = b_0\!\uparrow$

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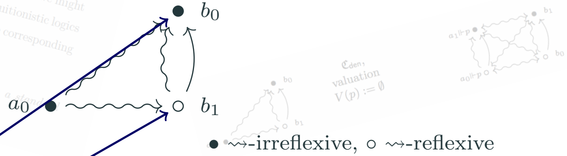

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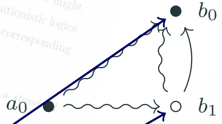

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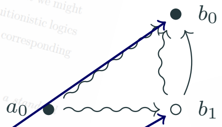
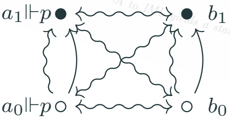


$\bullet$ $\rightsquigarrow$-irreflexive, $\circ$ $\rightsquigarrow$-reflexive

- just pick $V(p) := \emptyset$ and get
- $\overline{V}(\neg\neg p) = \emptyset$
- $\overline{V}(\Box\neg\neg p) = \{b_0\} = b_0\uparrow$
- $\overline{V}(\neg\neg\Box\neg\neg p) = \{b_0, b_1\} = b_1\uparrow$
- $\overline{V}(\Box\neg\neg\Box\neg\neg p)$ is the whole thing
- $a_0 \not\models \neg\neg\Box\neg\neg\Box\neg\neg p \to \neg\neg\Box\neg\neg p$

similarly, $\mathfrak{C}_{\mathsf{tr2}}$ is a frame …

- validating $\flat 4 := \Box\Box p \to \Box\Box\Box p$
- refuting $\neg\neg\Box\neg\neg\Box\neg\neg p \to \neg\neg\Box\neg\neg\Box\neg\neg\Box\neg\neg p$



- $$a_1{\uparrow} = \overline{W}(\Box\neg\neg\Box\neg\neg p),$$
  $$a_0{\uparrow} = W(p) = \overline{W}(\neg\neg p) = \overline{W}(\neg\neg\Box\neg\neg\Box\neg\neg p),$$
  $$b_1{\uparrow} = \overline{W}(\Box\neg\neg p) = \overline{W}(\Box\neg\neg\Box\neg\neg\Box\neg\neg p),$$
  $$b_0{\uparrow} = \overline{W}(\neg\neg\Box\neg\neg p) = \overline{W}(\neg\neg\Box\neg\neg\Box\neg\neg\Box\neg\neg p).$$
- See Example 13 in the paper

- These counterexamples were well-behaved in at least one sense

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- Recall that our problem is:

$$\text{given } \zeta \in \mathcal{L}_{\Box}, \text{ is there any } \phi \text{ s.t.}$$

$$\phi \in \mathrm{c}_{\Box}\mathsf{K} + \zeta \text{ but } \phi^{t} \notin \mathrm{i}_{\Box}\mathsf{K} + \zeta$$

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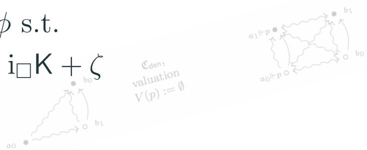

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- In both cases, we just took $\phi = \zeta$
- Can it happen that ζ wouldn't yield a counterexample, but some other ϕ would?
- No—and this is what the first of our main results (Regular Adequacy) is saying!

§ 5: what axioms are $\neg\neg$-complete?

enveloped implications

$s_{\neg\neg}(\beta)$: a substitution replacing all p occurring in β by their double negations

enveloped implications

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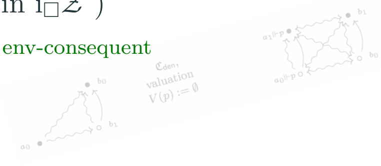
$$\neg\neg s_{\neg\neg}(\beta) \vdash_{\mathrm{i}_\Box\mathcal{Z}} \beta^{\mathsf{kol}}$$

a ($\neg\neg$-) pre-envelope

(in $\mathrm{i}_\Box\mathcal{Z}$)

or env-consequent



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$$\beta^{\mathsf{kol}} \vdash_{\mathrm{i}_\Box \mathcal{Z}} \neg\neg s_{\neg\neg}(\beta)$$

a ($\neg\neg$-) post-envelope (in $\mathrm{i}_\Box \mathcal{Z}$)

or env-antecedent

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an $\neg\neg$-envelope (in $\mathrm{i}_\Box\mathcal{Z}$)

$\beta \to \gamma$, where:
β a post-envelope
γ a pre-envelope

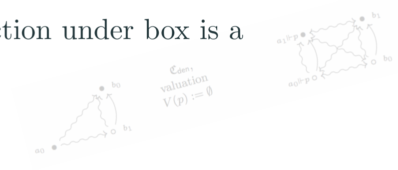
an enveloped implication

lemma 17 & theorem 18

- A box-free formula is a ¬¬-envelope

lemma 17 & theorem 18

- A box-free formula is a $\neg\neg$-envelope
- A shallow formula with no disjunction under box is a $\neg\neg$-envelope



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- **Any logic axiomatized by enveloped implications is $\neg\neg$-complete**

applications (corollary 19)

$\neg\neg$-completeness holds for logics axiomatized over $\mathrm{i}_\Box\mathsf{K}$ by combinations of the following axioms

(cf. Sotirov'84, Th. 10 and Litak'14, Tab. 2)

R	$p \to \Box p,$	CB	$\Box p \to (q \to p) \vee q,$	♭em	$\Box p \vee \Box\neg p,$
4	$\Box p \to \Box\Box p,$	NV	$\neg\Box\bot,$	em♭	$\Box p \vee \neg\Box p,$
T	$\Box p \to p,$	NNV	$\neg\neg\Box\bot,$	$\mathsf{T}^{\neg\neg}$	$\Box p \to \neg\neg p,$
coK	$(\Box p \to \Box q) \to \Box(p \to q),$	♭R	$p \to \Box\Box p,$	$\mathsf{T}^{\neg}$	$\Box\neg p \to \neg p,$
♭Lin	$\Box(p \to q) \vee \Box(q \to p),$	Lin♭	$(\Box p \to q) \vee (\Box q \to p),$	wem♭$^{\neg}$	$\neg\Box p \vee \neg\Box\neg p,$

or any superintuitionistic axiom, i.e., a formula in modality-free $\mathcal{L}$

An example of a standard logic obtained by a such combination:

$$\mathrm{i}_\Box\mathsf{S4} = \mathrm{i}_\Box\mathsf{4} + \mathsf{T}$$

§ 6: Glivenko for strength and purity

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  in any extension of $\mathrm{i}_\Box\mathsf{DNS}$, the Glivenko translation becomes equivalent to the regular ones/Kolmogorov, i.e., for every ϕ, we have that: $\vdash_{\mathrm{i}_\Box\mathsf{DNS}} \phi^{\mathsf{kol}} \leftrightarrow \phi^{\mathsf{glv}}$

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- DNS is a theorem of $\mathrm{i}_{\Box}\mathsf{R}$ (Th. 23)

The derivation has a rather CPS-like flavour ...

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- Extend, reuse the Coq framework and possibly merge with other developments

  `git clone git://git8.cs.fau.de/dnegmod`

  `https://cal8.cs.fau.de/redmine/projects/dnegmod`

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- . . . the notion of $C(W)$-variety: $C(W) \subseteq W$