Permutation Games for the Weakly Aconjunctive \( \mu \)-Calculus

Daniel Hausmann  
Lutz Schröder  
Hans-Peter Deifel  

daniel.hausmann@fau.de  
lutz.schroeder@fau.de  
hans-peter.deifel@fau.de

Chair for Theoretical Computer Science (http://www8.cs.fau.de)  
Friedrich-Alexander-Universität Erlangen-Nürnberg, Germany

From Limit-deterministic Parity Automata to Deterministic Parity Automata

A parity automaton (PA) or Büchi automaton (BA) is limit-deterministic (LD) if every accepting run in it is deterministic from some point on.

**Limit-deterministic PA to Limit-deterministic BA**

Eventually guess even highest priority. E.g.: 

\[ A: \]

\[ \text{Büchi}(A): \]

Deterministic since \( A \) is LD

Transforms LDP\( A \) of size \( n \) with \( k \) priorities to LDB\( 1 \) Büchi\( (A) \) of size \( O(nk) \)

**Theorem:** If \( A \) is a limit-deterministic PA of size \( n \) with \( k \) priorities, then \( L(A) = L(\text{Det}(\text{Büchi}(A))) \); the latter is of size \( O(\lfloor nk \rfloor) \) and has \( O(\lfloor nk \rfloor) \) priorities.

**Limit-deterministic BA to DPA**

Safraless determination, using permutations of states. E.g.: 

\[ B: \]

\[ \text{Det}(B): \]

Determines LDB\( 1 \) Büchi\( (B) \) of size \( n \) to DPA Det\( (B) \) of size \( O(n) \) with \( O(n) \) priorities.

The Aconjunctive \( \mu \)-Calculus, Limit-deterministic Tracking Automata, Permutation Games

The Aconjunctive \( \mu \)-Calculus [Kozen, 1983]

Modal formulas with fixpoint variables and fixpoint operators; e.g. \( \nu X. \mu Y. ((p \land \exists X) \lor (\neg p \land \exists Y)) \) ("on all paths, infinitely often \( p \)"), interpreted over standard Kripke structures. Formula \( \phi \) is aconjunctive if for all conjunctions occurring as subformula in \( \phi \), at most one conjunct contains an active \( \mu \)-variable. ("All fixpoints have at most one trace per path")

Construct tracking automaton \( A_{\phi} \), i.e. a parity automaton that tracks subformulas of \( \phi \) through rule applications in pre-tableaux.

**Example (aconjunctive formula \( \phi \), tracking automaton \( A_{\phi} \) and pre-tableau for \( \phi \))**

\[ \phi = \mu X. (p \land \nu Y. (\ell(Y \land p) \lor \ell X)) \]

\[ \begin{align*}
\psi &= \mu X. \psi \\
\theta &= \nu Y. \chi \\
\nu &= \neg \pi \\
\sigma &= [X \Rightarrow \phi] \\
\end{align*} \]

**Theorem:** If \( \phi \) is aconjunctive, then the tracking automaton \( A_{\phi} \) is a limit-deterministic PA of size \( O(|\phi|) \) and with \( O(\text{ad}(\phi)) \) priorities.

Then determine \( A_{\phi} \) through limit-deterministic BA with permutation method; complement resulting DPA; obtain DPA \( D_{\phi} \) that accepts exactly the good branches (in which no least fixpoint is unfolded infinitely often) in pre-tableaux for \( \phi \). Build permutation game \( G_{\phi} \) over carrier of \( D_{\phi} \).

**Theorem:** For aconjunctive \( \phi \), Eloise wins \( G_{\phi} \) if and only if \( \phi \) is satisfiable in a model of size \( O((nk)! \), where \( n = |\phi| \) and \( k = \text{ad}(\phi) \).

Implementation as part of COOL (Coalgebraic Ontology Logic Reasoner)

Solves permutation satisfiability games on-the-fly and in coalgebraic generality: https://www8.cs.fau.de/research/software:cool

---