# Quasipolynomial Computation of Nested Fixpoints

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Erlangen, Oberseminar – October 29 2019

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# Why Nested Fixpoints?

- Model checking for the $\mu$-calculus = solving parity games.
- Satisfiability checking for the $\mu$-calculus by solving parity games.
- Winning regions of parity games are nested fixpoints.
- Model checking and satisfiability checking for generalized $\mu$-calculi (graded, probabilistic, alternating-time) by nested fixpoints.
- Synthesis for linear-time logics (e.g. LTL).
- Computing generalized fair bisimulations.
- Type checking for inductive-coinductive types.

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**We show:**

- Nested fixpoints stabilize after quasipolynomially many iterations.
- The problem of computing nested fixpoints is in $\mathrm{NP} \cap \mathrm{co}\text{-}\mathrm{NP}$.
- Zielonka's algorithm can be adapted to compute nested fixpoints.

# Fixpoints of Set-Functions

Function $\alpha : \mathcal{P}(U)^{k+1} \to \mathcal{P}(U)$ is monotone if for all $U_i \subseteq V_i$, $0 \leq i \leq k$,

$$\alpha(U_0, \ldots, U_k) \subseteq \alpha(V_0, \ldots, V_k)$$

**Extremal fixpoints, systems of fixpoint equations**

Let $f : \mathcal{P}(U) \to \mathcal{P}(U)$, $f_i : \mathcal{P}(U)^{k+1} \to \mathcal{P}(U)$, $0 \leq i \leq k$ be monotone.

$$\mathsf{LFP}\, f = \bigcap \{Z \subseteq U \mid f(Z) \subseteq Z\}$$

$$\mathsf{GFP}\, f = \bigcup \{Z \subseteq U \mid Z \subseteq f(Z)\}$$

System $\overline{f}$ of fixpoint equations:

$$X_i =_{\eta_i} f_i(X_0, \ldots, X_k) \qquad 0 \leq i \leq k, \eta_i \in \{\mathsf{LFP}, \mathsf{GFP}\}$$

# Nested Fixpoints and Parity Games

Parity game $(V = V_\exists \mathbin{\dot\cup} V_\forall, E \subseteq V \times V, \Omega)$ with priorities 0 to $k$. Define:

$$\Omega_i = \{v \in V \mid \Omega(v) = i\}$$
$$\Diamond U = \{v \in V \mid E(v) \cap U \neq \emptyset\}$$
$$\Box U = \{v \in V \mid E(v) \subseteq U\}$$

$$\alpha_{\mathsf{PG}}(X_1, \dots, X_k) = (V_\exists \cap (\bigcup_{0 \le i \le k} \Omega_i \cap \Diamond X_i)) \cup (V_\forall \cap (\bigcup_{0 \le i \le k} \Omega_i \cap \Box X_i))$$

**Theorem (e.g. [Dawar,Grädel,2008],[Bruse,Falk,Lange,2014])**

$$\mathrm{win}_\exists = [\![X_k]\!]_{\alpha_{\mathsf{PG}}}$$

where

$$X_0 =_{\mathsf{GFP}} \alpha_{\mathsf{PG}}(X_0, \dots, X_k) \qquad X_i =_{\eta_i} X_{i-1}, i > 0$$

# A Tool: Fixpoint Parity Games (Venema, König et al.)

**Fixpoint Parity Game for $\overline{f}$**

Parity game $(V, E, \Omega)$, nodes: $V = (U \times [k]) \cup \mathcal{P}(U)^k$

| node | priority | owner | moves to |
|---:|---:|---:|---:|
| $(u, j) \in U$ | $j$ | $\exists$ | $\{\mathbf{U} \in \mathcal{P}(U)^k \mid u \in f_j(\mathbf{U})\}$ |
| $\mathbf{U}$ | $0$ | $\forall$ | $\{(v, i) \mid v \in U_i\}$ |

where $\mathbf{U} = (U_0, \ldots, U_k) \in \mathcal{P}(U)^k$.

**Theorem [König et al. 2019]**

Eloise wins node $(u, i)$ if and only if $u \in [\![X_i]\!]_{\overline{f}}$.

Problem: exponential size

- still useful for showing *history-freeness* for nested fixpoints.

# History-freeness for Nested Fixpoints

**History-free witnesses**

Even graph $S \subseteq (U \times [k]) \times [k] \times (U \times [k])$ s.t. for all $(u, j) \in \pi_1[S]$,

$$u \in f_j(S_0(u, j), \dots, S_k(u, j)),$$

where $S_i(u, j) = \{(w, i) \mid ((u, j), i, (w, i)) \in S\}$.

Note: $|S| \in \mathcal{O}(|U|^2)$

**Lemma**

There is history-free witness $S$ s.t. $(u, j) \in \pi_1[S]$ if and only if $u \in [\![X_j]\!]_{\overline{f}}$.

# Containment in $\mathbf{NP} \cap$ co-NP

**Theorem**

If all functions $f_i$ can be computed in polynomial time, the problem of solving $\overline{f}$ is in $\mathrm{NP} \cap \mathrm{co}\text{-}\mathrm{NP}$.

Proof: Each state $(u, i)$ is contained in $[\![X_i]\!]$ or in solution of dual nested fixpoint, hence containment in $\mathrm{NP}$ suffices. Guess *polynomial*-sized history-free witness containing $(u, i)$. Verify evenness and compatibility with functions $f_i$ in polynomial time.

# Parity Games in Quasipolynomial Time [Calude et al.,2017]

Idea: Annotate nodes with quasipolynomial histories ("statistics")

$$\overline{o} = (o_{\lceil \log n \rceil + 1}, \ldots, o_0) \qquad 1 \leq o_i \leq k$$

Define $\overline{o}@i = (o'_{\lceil \log n \rceil + 1}, \ldots, o'_0)$ as follows:

- $i$ even: pick greatest $j$ s.t. $i > o_j > 0$. If no such $j$ exists, then $j = *$.
- $i$ odd: pick greatest $j$ s.t.
  - **a)** $i > o_j > 0$ or
  - **b)** $o_j$ even for all $j' < j$, $o_{j'}$ odd (and if $o_j > 0$, $i < o_j$).
- If $j = *$, then $\overline{o}@i = \overline{o}$. Otherwise, $o'_{j'} = o_{j'}$ for $j' > j$, $o'_j = i$ and $o'_{j'} = 0$ for $j' < j$.

Move from $(v, \overline{o})$ to $(w, \overline{o}@\Omega(w))$ if move from $v$ to $w$ exists in original game. Solve safety game of quasipolynomial size $n \cdot k^{\lceil \log n \rceil + 2}$.

# Quasipolynomial Approximation

Use Calude et al.'s quasipolynomial histories to compute nested fixpoint:

Put $\mathrm{hi} = \{(o_{\lceil \log n \rceil + 1}, \dots, o_0) \mid 1 \leq o_i \leq k\}$ having $|hi| \leq k^{\lceil \log n \rceil + 2}$ and define $\gamma : \mathcal{P}(U \times [k] \times \mathrm{hi}) \to \mathcal{P}(U \times [k] \times \mathrm{hi})$ by

$$\gamma(Y) = \{(v, i, \overline{o}) \in U \times [k] \times \mathrm{hi} \mid v \in f(Y_0^{\overline{o}}, \dots, Y_k^{\overline{o}})\}$$

where

$$Y_j^{\overline{o}} = \begin{cases} \emptyset & \text{leftmost digit in } \overline{o}@j \text{ is not } 0 \\ \{u \in U \mid (u, j, \overline{o}@j) \in Y\} & \text{otherwise.} \end{cases}$$

**Theorem**

$[\![X_k]\!]_f = \pi_1[[\![Y_0]\!]_\gamma]$, where $Y_0 =_{\mathrm{GFP}} \gamma(Y_0)$.

# Universal Graphs

Labelled graph: $G = (W, \delta)$, $\delta \subseteq W \times [k] \times W$

**Definition - Universal Graphs**

Homomorphism from $G = (W, \delta)$ to $G' = (W', \delta')$: $\Phi : W \to W'$ s.t.

$$\text{for all } (v, p, w) \in \delta, \text{ we have } (\Phi(v), p, \Phi(w)) \in \delta'.$$

$(n, k)$-universal graph $S$: even labelled graph s.t. for all even labelled graphs $G$ with $|G| \leq n$, there is homomorphism from $G$ to $S$.

**Theorem [Czerwiński,Daviaud,Fijalkow,Jurdziński,Lazić,Parys,19]**

There is a deterministic $(n, k)$-universal graph of size $n^{\log k + \mathcal{O}(1)}$.
Every $(n, k)$-universal graph has size at least $n^{\log \frac{k}{\log n} - 1}$.

# Solving Equation Systems using Universal Graphs

Fix deterministic $((n(k+1), k+1)$-universal graph $S = (W, \delta)$.

**Definition - Product fixpoint**

Define $g : \mathcal{P}(U \times [k] \times W) \to \mathcal{P}(U \times [k] \times W)$ by

$$g(X) = \{(v, p, q) \in U \times [k] \times W \mid v \in f_p(X_0^q, \dots, X_k^q)\}$$

where

$$X_i^q = \{u \in U \mid (u, i, \delta((q, p), i)) \in X\}.$$

$Y_0 =_{\mathsf{GFP}} g(Y_0)$ is product fixpoint of $f$ and $S$.

**Theorem**

For $0 \leq i \leq k$, we have $u \in [\![X_i]\!]_{\overline{f}}$ if and only if $(u, i) \in \pi_1[[\![Y_0]\!]_g]$.

# Zielonka's Algorithm for Solving Parity Games

Define

$$\mathrm{Attr}_{\exists}^{\mathsf{PG}}(G,F) = \mu X. G \cap (F \cup \alpha_{PG}(X,\dots,X))$$
$$\mathrm{Attr}_{\forall}^{\mathsf{PG}}(G,F) = \mu X. G \cap (F \cup \overline{\alpha_{PG}}(X,\dots,X))$$

**Algorithm: Solve parity game $(G,E,\Omega)$ [Zielonka]**

**procedure** $\mathrm{SOLVE}_{\exists}(G,i)$ ▷ $i$ even
  $N_i := \{v \in G \mid \Omega(v) = i\}$; ▷ maximal priority nodes
  $H := G \setminus \mathrm{Attr}_{\exists}^{\mathsf{PG}}(G,N_i)$; ▷ exclude Eloise-attractor of $N_i$
  $W_{\forall} := \mathrm{SOLVE}_{\forall}(H, i-1)$; ▷ solve smaller game
  $G := G \setminus \mathrm{Attr}_{\forall}^{\mathsf{PG}}(G,W_{\forall})$; ▷ remove Abelard-attractor of $W_{\forall}$
  **if** $W_{\forall} \neq \emptyset$ **then** $\mathrm{GOTO}\ 2$:
  **else** $\mathrm{RETURN\ G}$.

# Zielonka's Algorithm for Computing Nested Fixpoints

Define

$$\mathsf{Attr}_{\exists}(G,F) = \mu X.\, G \cap (F \cup f(X,\ldots,X))$$
$$\mathsf{Attr}_{\forall}(G,F) = \mu X.\, G \cap (F \cup \overline{f}(X,\ldots,X))$$

**Algorithm: Compute nested fixpoint**

1: **procedure** $\mathrm{SOLVE}_{\exists}(G,i)$ ▷ $i$ even
2: $N_i := \{v \in G \mid \Omega(v) = i\}$; ▷ maximal priority nodes
3: $H := G \setminus \mathsf{Attr}_{\exists}(G,N_i)$; ▷ exclude Eloise-attractor of $N_i$
4: $W_{\forall} := \mathrm{SOLVE}_{\forall}(H, i-1)$; ▷ compute smaller fixpoint
5: $G := G \setminus \mathsf{Attr}_{\forall}(G,W_{\forall})$; ▷ remove Abelard-attractor of $W_{\forall}$
6: **if** $W_{\forall} \neq \emptyset$ **then** $\mathrm{GOTO\ 2:}$
7: **else** $\mathrm{RETURN\ G.}$

# The Fixpoint Law behind Zielonka's Algorithm

A system of equations:

$$X_i =_{\mathsf{LFP}} X_{i-1} \qquad i > 1, i \text{ odd}$$
$$X_i =_{\mathsf{GFP}} X_{i-1} \qquad i \text{ even}$$
$$X_1 =_{\mathsf{GFP}} f(X_1, \ldots, X_k)$$

A second system of equations:

$$Y_i =_{\mathsf{LFP}} (\Omega_{>}(i) \cup f(Y_i, \ldots, Y_i) \cup Y_{i-1}) \cap (\Omega_{\leq}(i) \cup Y_{i+1}) \qquad i \text{ odd}$$
$$Y_i =_{\mathsf{GFP}} (\Omega_{\leq}(i) \cap f(Y_i, \ldots, Y_i) \cap Y_{i-1}) \cup (\Omega_{>}(i) \cap Y_{i+1}) \qquad i \text{ even}$$

**Theorem:**

$[\![X_k]\!] = [\![Y_k]\!]$.

# The Coalgebraic $\mu$-Calculus [Cîrstea et al., 2011]

Set **V** of fixpoint variables, set $\Lambda$ of modalities, closed under duals.

**Syntax:**

$\phi, \psi := \top \mid \bot \mid \phi \wedge \psi \mid \phi \vee \psi \mid X \mid \heartsuit\psi \mid \mu X.\psi \mid \nu X.\psi \qquad \heartsuit \in \Lambda, X \in \mathbf{V}$

**Set**-endofunctor $T$, *predicate lifting*[1] for $\heartsuit \in \Lambda$: natural transformation

$$[\![\heartsuit]\!] : \mathcal{Q} \to \mathcal{Q} \circ T^{op}$$

E.g. for $T = \mathcal{P}$,

$$[\![\Diamond]\!](A) = \{B \in \mathcal{P}(C) \mid B \cap A \neq \emptyset\}$$
$$[\![\Box]\!](A) = \{B \in \mathcal{P}(C) \mid B \subseteq A\}$$

[1] [Pattinson, 2001]

# The Coalgebraic $\mu$-Calculus [Cîrstea et al., 2011]

Assume monotonicity of predicate liftings ($A \subseteq B \Rightarrow [\![\heartsuit]\!]A \subseteq [\![\heartsuit]\!]B$)

**Semantics:**

Models: *T-coalgebras* $(C, \xi : C \to TC)$, extension of formulas:

$$[\![X]\!]_\sigma = \sigma(X) \qquad [\![\heartsuit\psi]\!]_\sigma = \xi^{-1}[[\![\heartsuit]\!][\![\psi]\!]_\sigma]$$

$$[\![\mu X.\, \psi]\!]_\sigma = \mathsf{LFP}([\![\psi]\!]^X_\sigma) \qquad [\![\nu X.\, \psi]\!]_\sigma = \mathsf{GFP}([\![\psi]\!]^X_\sigma)$$

where $\sigma : \mathbf{V} \to \mathcal{P}(C)$, where $[\![\psi]\!]^X_\sigma(A) = [\![\psi]\!]_{\sigma[X \mapsto A]}$ for $A \subseteq C$ and where $(\sigma[X \mapsto A])(X) = A$, $(\sigma[X \mapsto A])(Y) = \sigma(Y)$ for $X \neq Y$.

# Instances of the Coalgebraic $\mu$-Calculus

- $T = \mathcal{P}$: transition systems $(C, \xi : C \to \mathcal{P}(C))$
  - modalities: $\Diamond$,$\Box$
  - standard $\mu$-calculus, e.g. $\mu X.\, \psi \vee \Diamond X$
- $T = \mathcal{B}$ (bag functor): graded transition systems $(C, \xi : C \to \mathcal{B}(C))$
  - modalities: $\langle g \rangle$, $[g]$, $g \in \mathbb{N}$
  - graded $\mu$-calculus[2], e.g. $\mu X.\, \psi \vee \langle 1 \rangle X$
- $T = \mathcal{G}$: concurrent game frames
  - Set $N$ of agents, modalities $[D]$,$\langle D \rangle$, $D \subseteq N$
  - alternating-time $\mu$-calculus[3], e.g. $\nu X.\, \psi \wedge [D]X$
- $T = \mathcal{D}$: Markov chains
  - modalities $\langle p \rangle$,$[p]$, $p \in \mathbb{Q} \cap [0, 1]$
  - (two-valued) probabilistic $\mu$-calculus, e.g. $\nu X.\, \psi \wedge \langle 0.5 \rangle X$

[2] [Kupferman et al.,2002]
[3] [Alur et al., 2002]

# Recent Results on the Coalgebraic $\mu$-Calculus

- Reduce model checking [H,Schröder,CONCUR 2019] and satisfiability checking [H,Schröder,FoSSaCS 2019] for the coalgebraic $\mu$-calculus to computing nested fixpoints.

**Corollary**

Model checking for coalgebraic $\mu$-calculi is in $\mathrm{QP}$ and in $\mathrm{NP} \cap \mathrm{Co\text{-}NP}$.

**Corollary**

Satisfiability checking for coalgebraic $\mu$-calculi can be done in time $\mathcal{O}(2^{nk \log n})$ (down from $\mathcal{O}(2^{n^2 k^2 \log n})$).

# Introducing: Coalgebraic Parity Games

**Definition - Coalgebraic parity game:**

$T$-coalgebra $(C, \xi : C \to TC)$ with mappings $\Omega : C \to \mathbb{N}$, $m : C \to \Lambda$.

Eloise wins node $c \in C$ if there is even graph $(D, R)$ on $C$ s.t.

$$\text{for all } d \in D,\ \xi(d) \in [\![m(d)]\!]R(d).$$

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e.g.

- $T = \mathcal{P}$: parity game for $T$ is graph $(C, \xi : C \to \mathcal{P}(C))$ with priority map $\Omega$ and node ownership map $m : C \to \{\Diamond, \Box\}$.

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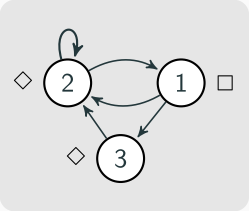
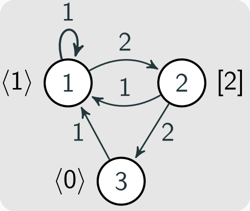
- $T = \mathcal{D}$: parity game for $T$ is Markov chain $(C, \xi : C \to \mathcal{D}(C))$ with priority map $\Omega$ and map $m : C \to \{\langle p \rangle, [p] \mid p \in \mathbb{Q} \cap [0, 1]\}$ .

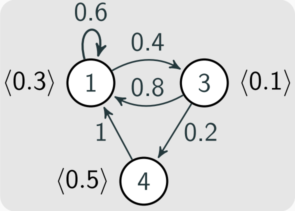
# Coalgebraic Parity Games, examples




$T = \mathcal{P}$: standard

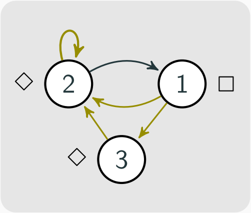



$T = \mathcal{B}$: graded




$T = \mathcal{D}$: probabilistic

# Coalgebraic Parity Games, examples, strategies

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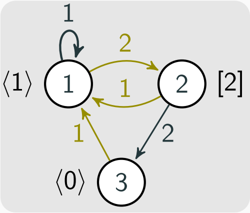


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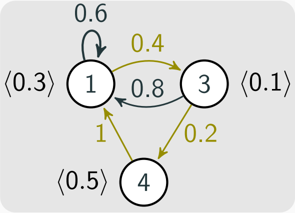


$T = \mathcal{D}$: probabilistic

# Solving Coalgebraic Parity Games

Winning regions in coalgebraic parity games are nested fixpoints:

Given game $(C, \xi, m, \Omega)$, define $f : \mathcal{P}(C)^k \to \mathcal{P}(C)$ by

$$f(X_0, \ldots, X_k) = \{v \mid \exists i, \heartsuit \in \Lambda.\, m(v) = \heartsuit, \Omega(v) = i \text{ and } \xi(v) \in [\![\heartsuit]\!] X_i\}$$

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**Theorem [H,Schröder,CONCUR 2019]:**

Player Eloise wins $u$ in coalgebraic parity game if and only if $u \in [\![X_k]\!]_f$.

Coalgebraic $\mu$-calculus model checking = solving coalgebraic parity games.
Enables on-the-fly model checking: Start with initial node, expand nodes step by step, compute $[\![X_k]\!]_f$ at any point (solving a partial game).

# Conclusion

**Results:**

- Computing nested fixpoints by
  - (fixpoint iteration),
  - Calude et al.'s quasipolynomial algorithm
  - universal graphs
  - Zielonka's algorithm
- Computing nested fixpoints also is in $\mathrm{NP} \cap \mathrm{Co\text{-}NP}$.
- Reduction of satisfiability checking and model checking for the *coalgebraic* $\mu$-calculus to computing nested fixpoints.

**Future work:**

- Computing fair bisimulations as nested fixpoints.
- Type checking for inductive-coinductive types by computing nested fixpoints.

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