Towards (In)Equivalence Games via Categories of Relational Structures

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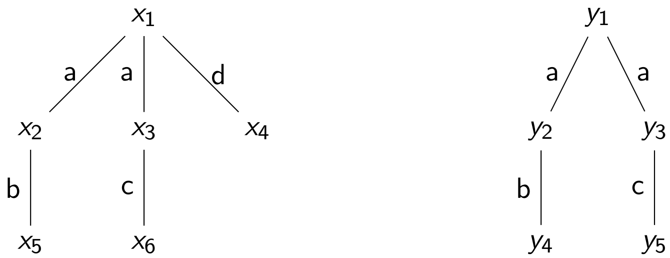
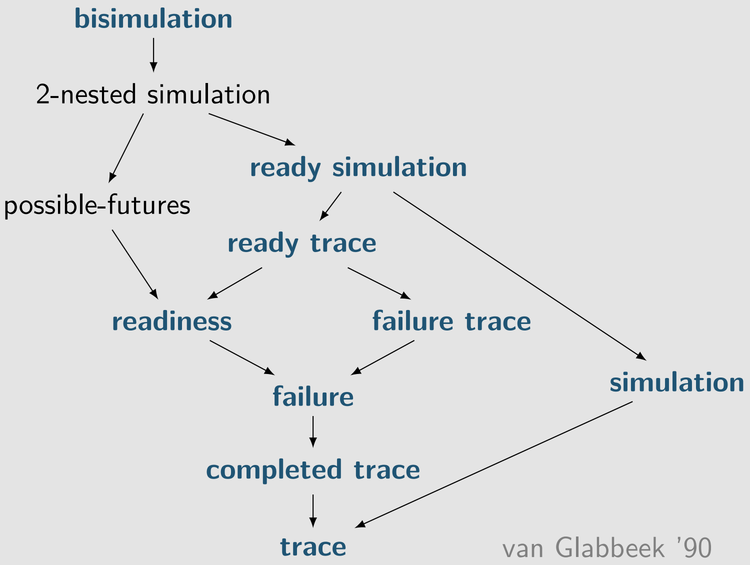
Motivation: Inequality

How do these systems relate?



The LT/BT Spectrum



Contributions

Existing Work

- Monads on Categories of Relational Structures Ford et al. '21
- Graded Behavioural Equivalence Games Ford et al. '22

In This Talk

Extension of behavioural equivalence games to semantics defined on relational structures.

Graded Monads

Graded Monads

A *graded monad* $\mathbb{M}$ consists of

- A family of functors $M_n\colon \mathbf{C} \to \mathbf{C}$ for $n \in \mathbb{N}$
- A family of natural transformations $\mu^{ij}\colon M_iM_j \Rightarrow M_{i+j}$
- A natural transformation $\eta\colon Id \Rightarrow M_0$

Subject to the usual monad laws (+ indices)

Graded Algebras

A *graded M_n-algebra A* consists of

- A family of $\mathbf{C}$-objects A_k for $k \leq n$
- A family of morphisms $a^{ij}\colon M_iA_j \Rightarrow A_{i+j}$ for $i + j \leq n$

Subject to the usual algebra laws (+ indices)

Graded Semantics

Graded Semantics

A *graded semantics* for G-coalgebras consists of a graded monad $\mathbb{M}$ and a natural transformation $\alpha\colon G \Rightarrow M_1$.

For $\gamma\colon X \to GX$ define inductively $\gamma^{(k)}\colon X \to M_k 1$:

$$\gamma^{(0)}\colon X \xrightarrow{\eta} M_0 X \xrightarrow{M_0 !} M_0 1$$

$$\gamma^{(k+1)}\colon X \xrightarrow{\alpha \cdot \gamma} M_1 X \xrightarrow{M_1 \gamma^{(k)}} M_1 M_k 1 \xrightarrow{\mu^{1k}} M_{k+1} 1$$

Depth-1 Algebras

Depth-1 Graded Monads

A graded monad is *depth-1* if the following diagram is a coequalizer:

$$M_1 M_0 M_0 \underset{\mu^{10} M_0}{\overset{M_1 \mu^{00}}{\rightrightarrows}} M_1 M_0 \xrightarrow{\mu^{10}} M_1$$

Canonical M_1 Algebras

Canonical Algebra

An M_1-algebra A is *canonical* if it is free over $(-)_0 \colon \mathbf{Alg}_1(\mathbb{M}) \to \mathbf{Alg}_0(\mathbb{M})$

$$A_0 \xrightarrow{f_0} B_0$$

$$\begin{tikzcd}
M_1 A_0 \arrow[r, "M_1 f_0"] \arrow[d, "a^{10}"] & M_1 B_0 \arrow[d, "b^{10}"] \\
A_1 \arrow[r, dashed, "f_1"] & B_1
\end{tikzcd}$$

(Pre)determinization

Lemma

If $\mathbb{M}$ is depth-1, then the M_1-algebra $(M_0X, M_1X, \mu^{0,0}, \mu^{0,1}, \mu^{1,0})$ is canonical.

$\bar{M}_1$

Let $E\colon \mathbf{Alg}_0(\mathbb{M}) \to \mathbf{Alg}_1(\mathbb{M})$ be the functor extending M_0-algebras to their canonical M_1-algebra.

$$\bar{M}_1\colon (\mathbf{Alg}_0(\mathbb{M}) \xrightarrow{E} \mathbf{Alg}_1(\mathbb{M}) \xrightarrow{(-)_1} \mathbf{Alg}_0(\mathbb{M}))$$

It is immediate that $M_1 = U\bar{M}_1F$

$$\frac{X \xrightarrow{\alpha\cdot\gamma} M_1X = U\bar{M}_1FXX}{FX \xrightarrow{\gamma^{\#}} \bar{M}_1FX}$$

Relevant Structures

Varieties of Algebras

$$\begin{tikzcd} \mathbf{Str}(\mathscr{H}) \arrow[r, bend left, "F_\Sigma"] \arrow[r, phantom, "\perp"] & \mathbf{Alg}(\Sigma) \arrow[l, bend left, "U"] \arrow[r, bend left, "\mathscr{R}_\mathbb{T}"] \arrow[r, phantom, "\perp"] & \mathbf{Alg}(\mathbb{T}) \arrow[l, bend left, "E_\Sigma"] \end{tikzcd}$$

Horn Models

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "F_\Pi"] \arrow[r, phantom, "\perp"] & \mathbf{Str}(\Pi) \arrow[l, bend left, "|\cdot|"] \arrow[r, bend left, "\mathscr{R}_\mathcal{A}"] \arrow[r, phantom, "\perp"] & \mathbf{Str}(\Pi, \mathcal{A}) \arrow[l, bend left, "E_\Pi"] \end{tikzcd}$$

Relational Structures

Relational Signature

Category $\mathbf{Str}(\Pi)$

Objects are tuples (X, E), where X is a set and E consists of pairs $\alpha(f)$ with $\alpha \in \Pi$ and $f\colon \mathrm{ar}(\alpha) \to X$. (*edges*)

Morphisms $g\colon (X, E) \to (Y, E')$ are maps $g\colon X \to Y$ such that $\alpha(f) \in E$ implies $\alpha(g \cdot f) \in E'$

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "F_\Pi"] \arrow[r, phantom, "\perp"] & \mathbf{Str}(\Pi) \arrow[l, bend left, "|\cdot|"] \arrow[r, bend left, "\mathscr{R}_\mathcal{A}"] \arrow[r, phantom, "\perp"] & \mathbf{Str}(\Pi, \mathcal{A}) \arrow[l, bend left, "E_\Pi"]
\end{tikzcd}$$

Horn Theories

Horn Axioms

Let $\mathcal{A}$ be a set of axioms of the form

$$\Phi \Rightarrow \psi$$

where ψ is a $\Pi \cup \{=\}$-edge in Var and Φ is a set of Π-edges in Var

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "F_\Pi"] \arrow[r, phantom, "\perp"] & \mathbf{Str}(\Pi) \arrow[l, bend left, "|\cdot|"] \arrow[r, bend left, "\mathscr{R}_\mathcal{A}"] \arrow[r, phantom, "\perp"] & \mathbf{Str}(\Pi, \mathcal{A}) \arrow[l, bend left, "E_\Pi"]
\end{tikzcd}$$

Examples of Horn Theories

Posets

Signature $\Pi = \{\leq\}$, Axioms

$$x \leq x \qquad \{x \leq y, y \leq z\} \Rightarrow x \leq z \qquad \{x \leq y, y \leq x\} \Rightarrow x = y$$

Metric Spaces

Signature $\Pi = \{=_\epsilon \mid \epsilon \in [0,1] \cap \mathbb{Q}\}$, Axioms

$$x =_0 x \qquad x =_0 x \Rightarrow x = x \qquad x =_\epsilon y \Rightarrow y =_\epsilon x$$

$$\{x =_\epsilon y, y =_{\epsilon'} z\} \Rightarrow x =_{\epsilon+\epsilon'} z$$

$$x =_\epsilon y \Rightarrow x =_{\epsilon+\epsilon'} y$$

$$\{x =_{\epsilon'} y \mid [0,1] \cap \mathbb{Q} \ni \epsilon' > \epsilon\} \Rightarrow x =_\epsilon y$$

Internal Hom on $\mathbf{Str}(\Pi, \mathcal{A})$

Pointwise Structure on Morphisms

The set of morphisms $\mathbf{Str}(\Pi, \mathcal{A})(X, Y)$ itself carries a $\mathbf{Str}(\Pi, \mathcal{A})$-structure, where

$$\mathsf{E}(X, Y) := \{e \mid \forall x \in X.\pi_x \cdot e \in \mathsf{E}(Y)\}$$

This defines the internal hom

$$[-, -] \colon \mathbf{Str}(\Pi, \mathcal{A}) \times \mathbf{Str}(\Pi, \mathcal{A})^{op} \to \mathbf{Str}(\Pi, \mathcal{A})$$

Algebras of Relational Structures

Σ-Algebras

Set Σ of symbols σ, each with arity given by an object $\mathbf{Str}(\Pi, \mathcal{A})$ object $\mathrm{ar}(\sigma)$ and depth $d(\sigma) \in \mathbb{N}$.

A Σ-Algebra A is a family of $\mathbf{Str}(\Pi, \mathcal{A})$-objects $(A_i)_{i \in \mathbb{N}}$ and a family of morphisms

$$\sigma^A_m \colon [\mathrm{ar}(\sigma), A_m] \to A_{m+d(\sigma)}$$

$$\begin{tikzcd} \mathbf{Str}(\mathscr{H}) \arrow[r, bend left, "F_\Sigma"] \arrow[r, phantom, "\perp"] & \mathbf{Alg}(\Sigma) \arrow[l, bend left, "U"] \arrow[r, bend left, "\mathscr{R}_\mathbb{T}"] \arrow[r, phantom, "\perp"] & \mathbf{Alg}(\mathbb{T}) \arrow[l, bend left, "E_\Sigma"] \end{tikzcd}$$

Varieties of Σ-Algebras

Relations in Context

Relational theories $(\Sigma, \mathcal{E})$ are parametric over a set of Axioms $\mathcal{E}$ of the form

$$X \vdash_k R(t)$$

where an algebra A satisfies $\mathcal{E}$ if all defined substitution instances of axioms hold in A.

$$\begin{tikzcd} \mathbf{Str}(\mathscr{H}) \arrow[r, bend left, "F_\Sigma"] \arrow[r, phantom, "\perp"] & \mathbf{Alg}(\Sigma) \arrow[l, bend left, "U"] \arrow[r, bend left, "\mathscr{R}_\mathbb{T}"] \arrow[r, phantom, "\perp"] & \mathbf{Alg}(\mathbb{T}) \arrow[l, bend left, "E_\Sigma"] \end{tikzcd}$$

Varieties of Σ-Algebras

Sequent Calculus

Judgments of the form $X \vdash_k R(e)$ and $X \vdash\downarrow t$ where

- $X \in \mathbf{Str}(\mathscr{H})$
- $R(e)$ a Π-edge in $T_{\Sigma,k}(X)$
- $t \in T_{\Sigma,k}(X)$

Rule(s) (Incomplete selection)

$$(\mathrm{Ax})\,\frac{\{X \vdash_k R(\tau \cdot e) \mid R(e) \in Y\} \cup \{X \vdash_k\downarrow \tau(y) \mid y \in Y\}}{X \vdash_{m+k} Q(\bar{\tau}_m \cdot t)}$$

Where $Y \vdash_m Q(t) \in \mathcal{E}$ and $\tau\colon Y \to T_{\Sigma,k}(X)$

Relational Behaviours

Behavioural

Let $(\alpha, \mathbb{M})$ be a relational semantics for G-coalgebra and fix a G-coalgebra (X, γ).

We define sets of Π-edges $E^{\alpha,n}(X)$ in X, where

$$e(f) \in E^{\alpha,n}(X) \qquad \text{iff} \qquad e(\gamma^{(n)} \cdot f) \in E(M_n 1)$$

$E^{\alpha}(X)$ is defined as $\bigcap_{n \in \mathbb{N}} E^{\alpha,n}(X)$, closed under the axioms in $\mathcal{A}$.

Goal of the Game

Bisimilarity

Let $(X, \to)$ be a labelled transition system over A.
(That is $\to \subseteq X \times A \times X$)

A *bisimulation* is a relation $R \subseteq X \times X$ such that for all xRy

- $x \xrightarrow{a} x'$ implies that there is y' with $y \xrightarrow{a} y'$ and $x'Ry'$
- $y \xrightarrow{a} y'$ implies that there is x' with $x \xrightarrow{a} x'$ and $x'Ry'$

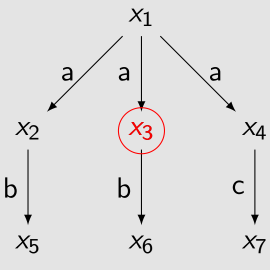
Two states $x, y \in X$ are bisimilar if there is a bisimulation with xRy

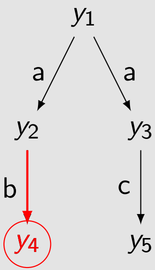
$$
\begin{tikzcd}
& x_1 \arrow[dl, "a"'] \arrow[d, "a"] \arrow[dr, "a"] & \\
x_2 \arrow[d, "b"'] & x_3 \arrow[d, "b"'] & x_4 \arrow[d, "c"'] \\
x_5 & x_6 & x_7
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& y_1 \arrow[dl, "a"'] \arrow[dr, "a"] & \\
y_2 \arrow[d, "b"'] & & y_3 \arrow[d, "c"'] \\
y_4 & & y_5
\end{tikzcd}
$$

$$\begin{tikzcd}
 & x_1 \arrow[dl, "a"'] \arrow[d, "a"] \arrow[dr, "a"] & \\
x_2 \arrow[d, "b"'] & x_3 \arrow[d, "b"'] & x_4 \arrow[d, "c"'] \\
x_5 & x_6 & x_7
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & y_1 \arrow[dl, "a"'] \arrow[dr, "a"] & \\
y_2 \arrow[d, "b"'] & & y_3 \arrow[d, "c"'] \\
y_4 & & y_5
\end{tikzcd}$$

$\xrightarrow{a} x_3$

$$
\begin{tikzcd}
 & x_1 \arrow[dl, "a"'] \arrow[d, "a"] \arrow[dr, "a"] & \\
x_2 \arrow[d, "b"'] & x_3 \arrow[d, "b"'] & x_4 \arrow[d, "c"'] \\
x_5 & x_6 & x_7
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & y_1 \arrow[dl, "a"'] \arrow[d, "a"] \\
y_2 \arrow[d, "b"'] & y_3 \arrow[d, "c"'] \\
y_4 & y_5
\end{tikzcd}
$$





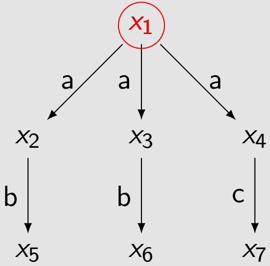
$\xrightarrow{b} y_4$

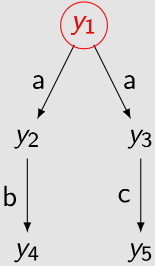
$$\begin{tikzcd}
& x_1 \arrow[dl, "a"'] \arrow[d, "a"] \arrow[dr, "a"] & \\
x_2 \arrow[d, "b"'] & x_3 \arrow[d, "b"'] & x_4 \arrow[d, "c"'] \\
x_5 & x_6 & x_7
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& y_1 \arrow[dl, "a"'] \arrow[dr, "a"] & \\
y_2 \arrow[d, "b"'] & & y_3 \arrow[d, "c"'] \\
y_4 & & y_5
\end{tikzcd}$$

$\xrightarrow{b} x_6$

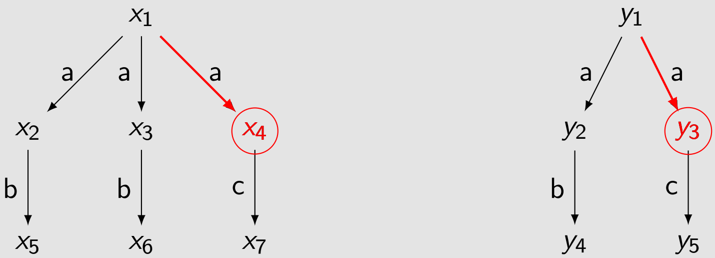
$$\begin{tikzcd}
 & x_1 \arrow[dl, "a"'] \arrow[d, "a"] \arrow[dr, "a"] & \\
x_2 \arrow[d, "b"'] & x_3 \arrow[d, "b"'] & x_4 \arrow[d, "c"'] \\
x_5 & \textcolor{red}{x_6} & x_7
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & y_1 \arrow[dl, "a"'] \arrow[dr, "a"] & \\
y_2 \arrow[d, "b"'] & & y_3 \arrow[d, "c"'] \\
\textcolor{red}{y_4} & & y_5
\end{tikzcd}$$

$$\begin{tikzcd}
 & x_1 \arrow[dl, "a"'] \arrow[d, "a"] \arrow[dr, "a"] & \\
x_2 \arrow[d, "b"'] & x_3 \arrow[d, "b"'] & x_4 \arrow[d, "c"'] \\
x_5 & x_6 & x_7
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & y_1 \arrow[dl, "a"'] \arrow[dr, "a"] & \\
y_2 \arrow[d, "b"'] & & y_3 \arrow[d, "c"'] \\
y_4 & & y_5
\end{tikzcd}$$





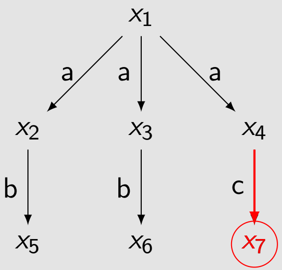
$$\xrightarrow{a} x_2 \quad \Leftrightarrow \quad \xrightarrow{a} y_2$$
$$\xrightarrow{a} x_3 \quad \Leftrightarrow \quad \xrightarrow{a} y_2$$
$$\xrightarrow{a} x_4 \quad \Leftrightarrow \quad \xrightarrow{a} y_3$$

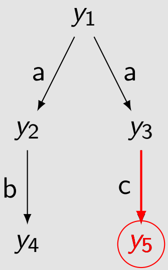


$\xrightarrow{a} x_4 \quad \Leftrightarrow \quad \xrightarrow{a} y_3$

$$
\begin{tikzcd}
 & x_1 \arrow[dl, "a"'] \arrow[d, "a"] \arrow[dr, "a"] & \\
x_2 \arrow[d, "b"'] & x_3 \arrow[d, "b"'] & \color{red}{x_4} \arrow[d, "c"'] \\
x_5 & x_6 & x_7
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & y_1 \arrow[dl, "a"'] \arrow[dr, "a"] & \\
y_2 \arrow[d, "b"'] & & \color{red}{y_3} \arrow[d, "c"'] \\
y_4 & & y_5
\end{tikzcd}
$$

$\xrightarrow{c} x_7 \quad \Leftrightarrow \quad \xrightarrow{c} y_5$





$$\xrightarrow{c} x_7 \quad \Leftrightarrow \quad \xrightarrow{c} y_5$$

$$\begin{tikzcd}
 & x_1 \arrow[dl, "a"'] \arrow[d, "a"] \arrow[dr, "a"] & \\
x_2 \arrow[d, "b"'] & x_3 \arrow[d, "b"'] & x_4 \arrow[d, "c"'] \\
x_5 & x_6 & x_7
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & y_1 \arrow[dl, "a"'] \arrow[dr, "a"] & \\
y_2 \arrow[d, "b"'] & & y_3 \arrow[d, "c"'] \\
y_4 & & y_5
\end{tikzcd}$$

$\emptyset$

$$\begin{tikzcd} & x_1 \arrow[dl, "a"'] \arrow[d, "a"] \arrow[dr, "a"] & \\ x_2 \arrow[d, "b"'] & x_3 \arrow[d, "b"'] & x_4 \arrow[d, "c"'] \\ x_5 & x_6 & {\color{red}x_7} \end{tikzcd}
\qquad
\begin{tikzcd} & y_1 \arrow[dl, "a"'] \arrow[dr, "a"] & \\ y_2 \arrow[d, "b"'] & & y_3 \arrow[d, "c"'] \\ y_4 & & {\color{red}y_5} \end{tikzcd}$$

Local bisimulation

A *local bisimulation* at (x, y) is a relation $R \subseteq X \times X$ such that

- $x \xrightarrow{a} x'$ implies that there is y' with $y \xrightarrow{a} y'$ and $x'Ry'$
- $y \xrightarrow{a} y'$ implies that there is x' with $x \xrightarrow{a} x'$ and $x'Ry'$

Game variant

To proof bisimilarity of (x, y)

1. Duplicator plays a local bisimulation R at (x, y)
2. Spoiler picks an element $(x', y') \in R$ as a new position.
3. Goto step 1.

A player that can not move loses, infinite plays are won by Duplicator.

Setting up the Game

How to Play

Duplicator wants to show that an edge e holds in the behaviour of $(M_0 X, \gamma^{\#})$.

	Spoiler	Duplicator
Position	Set Z of edges in $M_0 X$	A single edge e in $M_0 X$
Move	An edge $e \in Z$	An admissible set Z of edges

Admissible???

Algebraically

A set of edges Z is admissible at $e(f)$ if $Z \vdash_1 e(\gamma^\# \cdot f)$

Assume $Z \supseteq E(M_0 X)$.

Categorically

The reflector $r\colon X \to RX$ closes Objects $X \in \mathbf{Str}(\Pi)$ under axioms in $\mathcal{A}$.
Define the morphism

$$\bar{Z}\colon (M_0 X \xrightarrow{\gamma^\#} \bar{M}_1 M_0 X \xrightarrow{\bar{M}_1 \iota} \bar{M}_1(|M_0 X|, Z) \xrightarrow{\bar{M}_1 r_X} \bar{M}_1 R(|M_0 X|, Z))$$

Then Z is admissible at $e(f)$ if $e(\bar{Z} \cdot f) \in \bar{M}_1(|M_0 X|, Z)$

Calling the Bluff

Additional Condition

Let $e(f)$ be the position after n rounds. Duplicator wins the n-round equivalence game if

$$e(M_0! \cdot f) \in E(M_0 1)$$

Terminal object in $\mathbf{Str}(\Pi, \mathcal{A})$

Theorems (Eventually)

Assumptions

Let $(\alpha, \mathbb{M})$ be a depth-1 graded semantics for a functor G, such that $\bar{M}_1$ preserves monomorphisms, and let (X, γ) be a G-coalgebra.

Future Theorem 1

For every $n \in \mathbb{N}$, we have $e(f) \in E^{\alpha,n}(X)$ iff Duplicator wins the n-round game at $e(\eta_X \cdot f)$

Future Theorem 2

The infinite depth (in)equivalence $e(f)$ in X holds iff Duplicator wins the infinite depth game in position $e(\eta_X \cdot f)$.

Conclusion

Future Work

- Finish this work
- Work out examples
- Extend to topological functors ($\mathbf{Clat}_\sqcap$-fibrations)

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