Graded Monads and (someday) Fixpoints

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Graded Monads and Graded Semantics
A monad \((M, \eta, \mu)\) consists of

- an endofunctor \(M\),
- a natural transformation \(\eta : Id \rightarrow M\), the unit,
- and a natural transformation \(\mu : MM \rightarrow M\), the multiplication.

with the following diagrams commuting:
A graded monad \(( (M_n)_{n \in \mathbb{N}}, \eta, (\mu^{mk})_{m, k \in \mathbb{N}} )\) consists of
a family of endofunctors \(M_n, n \in \mathbb{N}\),
a natural transformation \(\eta : \text{Id} \to M_0\), the unit,
and a family of natural transformations
\(\mu^{mk} : M_m M_k \to M_{m+k}, m, k \in \mathbb{N}\), the multiplication.
with the following diagrams commuting for all \(n, k, m \in \mathbb{N}\):
Simplify things

- Monads on a set $X$ describe a term structure $MX$.
- $\eta$ ensures that every element of $X$ can be made into a term.
- $\mu$ describes substitution, where terms of terms are again just terms.
Simplify things

- Monads on a set $X$ describe a term structure $M X$.
- $\eta$ ensures that every element of $X$ can be made into a term.
- $\mu$ describes substitution, where terms of terms are again just terms.
- Then graded monads are like terms with depth.
Graded semantics

A graded semantics \((\alpha, \mathbb{M})\) for an endofunctor \(G\) consists of

- a graded monad \(\mathbb{M} = ((M_n)_{n \in \mathbb{N}}, \eta, (\mu^{mk})_{m,k \in \mathbb{N}})\),
- and a natural transformation \(\alpha : G \to M_1\).

\(\gamma(n) : X \to M^1\)

\(\gamma(0) : X \eta X - \to M^0 X M^0 1\)

\(\gamma(n+1) : X \alpha X \circ \gamma(n) - \to M^1 X M^n 1 \mu^1 n 1 - \to M^{n+1} 1\)

\(\gamma(n)(x) \in M^n 1\) is called the \(n\)-step \((\alpha, \mathbb{M})\)-behaviour of \(x \in X\).
Graded semantics

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- a graded monad \(\mathbb{M} = ((M_n)_{n \in \mathbb{N}}, \eta, (\mu^{mk})_{m, k \in \mathbb{N}})\),
- and a natural transformation \(\alpha : G \rightarrow M_1\).

A \(G\)-coalgebra \((X, \gamma)\) then induces the following sequence of inductively defined maps \(\gamma^{(n)} : X \rightarrow M_n 1\)

\[
\begin{align*}
\gamma^{(0)} : & \quad X \xrightarrow{\eta_X} M_0 X \xrightarrow{M_0!} M_0 1 \\
\gamma^{(n+1)} : & \quad X \xrightarrow{\alpha_X \circ \gamma} M_1 X \xrightarrow{M_1 \gamma^{(n)}} M_1 M_n 1 \xrightarrow{\mu_{1n}} M_{n+1} 1
\end{align*}
\]
A graded semantics \((\alpha, \mathbb{M})\) for an endofunctor \(G\) consists of

- a graded monad \(\mathbb{M} = ((M_n)_{n \in \mathbb{N}}, \eta, (\mu^m)^{m, k \in \mathbb{N}})\),
- and a natural transformation \(\alpha : G \to M_1\).

A \(G\)-coalgebra \((X, \gamma)\) then induces the following sequence of inductively defined maps \(\gamma^{(n)} : X \to M_n1\)

\[
\begin{align*}
\gamma^{(0)} : X &\xrightarrow{\eta_X} M_0X \xrightarrow{M_0!} M_01 \\
\gamma^{(n+1)} : X &\xrightarrow{\alpha_X \circ \gamma} M_1X \xrightarrow{M_1 \gamma^{(n)}} M_1M_n1 \xrightarrow{\mu_1^{1n}} M_{n+1}1
\end{align*}
\]

\(\gamma^{(n)}(x) \in M_n1\) is called the \(n\)-step \((\alpha, \mathbb{M})\)-behaviour of \(x \in X\).
For the functor $GX = \mathcal{P}(A \times X)$ take the following coalgebra $\gamma : X \to GX$

```
$$
\begin{array}{c}
q_0 \\
\uparrow \\
\downarrow \\
q_1
\end{array}
$$
```

```
a
```

```
a
```

```
a
```

```
\gamma
```

```
\gamma(k)
```

The graded monad $M_n = G_n$ (with $\alpha$, $\eta$, and $\mu$ as all the appropriate identities) corresponds to coalgebraic "step-behaviour". So if the $\gamma(k)$ images of two states coincide for all $k \leq n$, they are $n$-step behaviourally equivalent.
An LTS

For the functor $GX = \mathcal{P}(A \times X)$ take the following coalgebra $\gamma : X \to GX$

![Diagram](image)

The graded monad $M_n = G^n \ (\text{with } \alpha, \eta \text{ and } \mu^{mk} \text{ as all the appropriate identities})$ corresponds to coalgebraic "step-$n$ behaviour". So if the $\gamma^{(k)}$ images of two states coincide for all $k \leq n$, they are $n$-step behaviourally equivalent.
Trace Semantics

To obtain trace semantics, define $M$ as

$$
M \colon X := P (A \times X),
$$

$$
\eta(x) := \{ (\epsilon, x) \},
$$

$$
\mu_{m_k}(S) := \{ (vw, W) \mid (v, V) \in S, (w, W) \in V \},
$$

and $\alpha$ as $\text{Id}$.

Again consider the example:

$$
\begin{align*}
\gamma^0(q_0) &= \{ \epsilon \} \\
\gamma^1(q_0) &= \{ \gamma^0(q_0) \} = \{ \gamma^0(q_1) \} = \mu_{0, 0} \{ (a, \{ \epsilon \}), (a, \{ \epsilon \}) \} = \{ a \} \\
\gamma^2(q_0) &= \{ \gamma^1(q_0) \} = \{ \gamma^1(q_1) \} = \{ a a \}
\end{align*}
$$

Note that $\gamma^{n+1}(q_1) = \emptyset$. 
Trace Semantics

To obtain trace semantics, define $\mathcal{M}$ as

- $M_nX := \mathcal{P}(A^n \times X)$,
- $\eta(x) := \{(\epsilon, x)\}$,
- $\mu^{mk}(S) := \{(vw, W) | (v, V) \in S, (w, W) \in V\}$

and $\alpha$ as $\text{Id}$. Note that $\gamma(n+1)(q_1) =$ ∅
Trace Semantics

To obtain trace semantics, define $\mathcal{M}$ as

- $M_nX := \mathcal{P}(A^n \times X)$,
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and $\alpha$ as $\text{Id}$. Again consider the example:

$$
\gamma^{(0)}(q_0) = \{\epsilon\}
$$
$$
\gamma^{(1)}(q_0) = \mu^{1,0}\{(a, \gamma^{(0)}(q_0)), (a, \gamma^{(0)}(q_1))\}
= \mu^{1,0}\{(a, \{\epsilon\}), (a, \{\epsilon\})\}
= \{a\}
$$
$$
\gamma^{(2)}(q_0) = \{aa\}
$$

Note that $\gamma^{(n+1)}(q_1) = \emptyset$
The ugly truth

To obtain trace semantics, define $M$ as

- $M_nX := \mathcal{P}(A^n \times X)$,
- $\eta(x) := \{(\epsilon, x)\}$,
- $\mu^{mk}(S) := \{(vw, W) \mid (v, V) \in S, (w, W) \in V\}$

and $\alpha$ as Id. Again consider the example:

\[\gamma^{(0)}(q_0) = \{(\epsilon, *)\}\]
\[\gamma^{(1)}(q_0) = \mu^{1,0}\{(a, \gamma^{(0)}(q_0)), (a, \gamma^{(0)}(q_1))\}\]
\[= \mu^{1,0}\{(a, \{(\epsilon, *)\}), (a, \{(\epsilon, *)\})\}\]
\[= \{(a\epsilon, *)\}\]
\[\gamma^{(2)}(q_0) = \{(aa\epsilon, *)\}\]

Note that $\gamma^{(n+1)}(q_1) = \emptyset$
To obtain completed trace semantics, define $M$ as
$$M_n := \mathcal{P}(A_n \times X + A_{<n})$$ (especially $M_1 := \mathcal{P}(A_n \times X + 1)$),

$$\eta(x) := \{ (\epsilon, x) \},$$

$$\mu_{mk}(S) := \{ (vw, W) | (v, V) \in S, (w, W) \in V \} \cup \{ vw \star | (v, V) \in S, w \star \in V \} \cup \{ v \star \in S \}$$

and

$$\alpha : G \to M_1$$ (so $\mathcal{P}(A \times X) \to \mathcal{P}(A \times X + 1)$) as
$$\alpha(\emptyset) := \{ \star \},$$

$$\alpha(S) := S \subseteq \mathcal{P}(A \times X + 1) \ (S \neq \emptyset).$$
Completed Trace Semantics

To obtain completed trace semantics, define $\overline{M}$ as

$M_nX := P(A^n \times X + A^{<n})$ (especially $M_1 := P(A^n \times X + 1)$),
Completed Trace Semantics

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Completed Trace Semantics

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- $M^n_X := \mathcal{P}(A^n \times X + A^{<n})$ (especially $M_1 := \mathcal{P}(A^n \times X + 1)$),
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- $\mu^{mk}(S) := \{ (vw, W) \mid (v, V) \in S, (w, W) \in V \}$
  $\cup \{ vw \star \mid (v, V) \in S, w \star \in V \}$
  $\cup \{ v \star \in S \}$
Completed Trace Semantics

To obtain completed trace semantics, define $\mathbb{M}$ as

- $M_nX := \mathcal{P}(A^n \times X + A^{\leq n})$ (especially $M_1 := \mathcal{P}(A^n \times X + 1)$),
- $\eta(x) := \{ (\epsilon, x) \}$,
- $\mu^{mk}(S) := \{ (vw, W) \mid (v, V) \in S, (w, W) \in V \}
  \cup \{ vw \star \mid (v, V) \in S, w \star \in V \}
  \cup \{ v \star \in S \}$

and $\alpha : G \rightarrow M_1$ (so $\mathcal{P}(A \times X) \rightarrow \mathcal{P}(A \times X + 1)$) as

\[
\alpha(\emptyset) := \{ \star \}, \quad (\star \in 1)
\]

\[
\alpha(S) := S \subseteq \mathcal{P}(A \times X + 1), \quad (S \neq \emptyset)
\]
\( \gamma^{(1)}(q_1) = \alpha(\emptyset) = \{\star\} \)
\( \gamma^{(1)}(q_0) = \{(a\epsilon, \star)\} \)

\( \gamma^{(2)} : q_0 \xrightarrow{\gamma} \{(a, q_0), (a, q_1)\} \)
\( \xrightarrow{\alpha} \{(a, q_0), (a, q_1)\} \)
\( \xrightarrow{M_1 \gamma^{(1)}} \{(a, \gamma^{(1)}(q_0)), (a, \gamma^{(1)}(q_1))\} \)
\( = \{(a, \{(a\epsilon, \star)\}), (a, \{\star\})\} \)
\( \xrightarrow{\mu} \{(aa\epsilon, \star), a\star\} \)

\( \gamma^{(3)}(q_0) = \{(aaa\epsilon, \star), aa\star, a\star\} \)
\( \gamma^{(4)}(q_0) = \ldots \)
\gamma^{(1)}(q_1) = \alpha(\emptyset) = \{\star\}
\gamma^{(1)}(q_0) = \{a\}

\gamma^{(2)} : q_0 \xrightarrow{\gamma} \{(a, q_0), (a, q_1)\}
\xrightarrow{\alpha} \{(a, q_0), (a, q_1)\}

M_1 \gamma^{(1)} \xrightarrow{} \{(a, \gamma^{(1)}(q_0)), (a, \gamma^{(1)}(q_1))\}
= \{(a, \{a\}), (a, \{\star\})\}
\xrightarrow{\mu} \{aa, a\star\}

\gamma^{(3)}(q_0) = \{aaa, aa\star, a\star\}
\gamma^{(4)}(q_0) = \ldots
Trace Semantics

q_0 \xrightarrow{a} q_1

...
Trace Semantics

\[ \gamma(0)(q_0) = \{ \epsilon \} \]

\[ \gamma(1)(q_0) = \{ \epsilon \} \]

\[ \gamma(2)(q_0) = \{ \epsilon \} \]

\[ \gamma(3)(q_0) = \{ \epsilon \} \]
Trace Semantics

\[ \gamma(0)(q_0) = \{ \epsilon \} \]
\[ \gamma(1)(q_0) = \{ a \} \]
Trace Semantics

\[ \gamma^0(q_0) = \{ \epsilon \} \]
\[ \gamma^1(q_0) = \{ a \} \]
\[ \gamma^2(q_0) = \{ aa \} \]
Trace Semantics

\[
\begin{align*}
\gamma^{(0)}(q_0) &= \{ \epsilon \} \\
\gamma^{(1)}(q_0) &= \{ a \} \\
\gamma^{(2)}(q_0) &= \{ aa \} \\
\gamma^{(3)}(q_0) &= \{ aaa \} \\
&\ldots \\
\end{align*}
\]
Complete Trace Semantics

\[ \gamma^0(q_0) = \{ \epsilon \} \]
\[ \gamma^1(q_0) = \{ a \} \]
\[ \gamma^2(q_0) = \{ aa, a^* \} \]
\[ \gamma^3(q_0) = \{ aaa, aa^*, a^* \} \]

...
Graded Monads and Graded Theories
A signature is a set $\Sigma$ of operations $f$ with finite arity $\text{ar}(f)$. 
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For a set $X$ of variables, the set of $\Sigma$-terms $T_\Sigma(X)$ is defined inductively as:

$$x \in T_\Sigma(X) \quad (x \in X)$$

$$f(t_1, \ldots, t_{\text{ar}(f)}) \in T_\Sigma(X) \quad (f \in \Sigma, t_1, \ldots, t_{\text{ar}(f)} \in T_\Sigma(X))$$
Algebraic Theories

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  $$
  \begin{align*}
  x & \in T_\Sigma(X) & (x \in X) \\
  f(t_1, \ldots, t_{\text{ar}(f)}) & \in T_\Sigma(X) & (f \in \Sigma, t_1, \ldots, t_{\text{ar}(f)} \in T_\Sigma(X))
  \end{align*}
  $$

- A $\Sigma$-theory $E$ is a set of equations $s = t$, such that $s, t \in T_\Sigma(X)$. 
Monads and Theories

- Monads correspond to algebraic theories, i.e. the quotient $T_{\Sigma}(X)/\sim$ of $T_{\Sigma}$ modulo the congruence $\sim$ generated by $E$ is in bijection to $MX$.
- Monads can thus be induced by an algebraic theory (and vice versa).
An Example for Monads

Take $MX := \mathcal{P}(\{a, b\}^* \times X)$. 
An Example for Monads

Take $MX := \mathcal{P}(\{a, b\}^* \times X)$. $MX$ corresponds to the following theory:

$$\Sigma := \{\bot/0, \lor/2, a/1, b/1\}$$

$$E := \begin{cases} 
\begin{align*}
    x \lor x &= x, \\
    x \lor (y \lor z) &= (x \lor y) \lor z, \\
    x \lor y &= y \lor x, \\
    x \lor \bot &= x, \\
    c(x \lor y) &= c(x) \lor c(y), \\
    c(\bot) &= \bot
\end{align*}
\end{cases} \quad (c \in \{a, b\})$$
An Example for Monads

Take $MX := \mathcal{P}(\{a, b\}^* \times X)$. $MX$ corresponds to the following theory:

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    x \lor x = x, \\
    x \lor (y \lor z) = (x \lor y) \lor z, \\
    x \lor y = y \lor x, \\
    x \lor \bot = x, \\
    c(x \lor y) = c(x) \lor c(y), & (c \in \{a, b\}) \\
    c(\bot) = \bot & (c \in \{a, b\}) 
\end{cases}$$

Like this:

$$a(b(x)) \lor b(x) \lor b(x) \lor y = a(b(x)) \lor y \lor b(x)$$

$$\cong \{(ab, x), (b, x), (\epsilon, y)\}$$
A graded signature is a set $\Sigma$ of operations $f$ with finite arity $\text{ar}(f)$ and finite depth $d(f)$.
Graded Algebraic Theories

- A graded signature is a set $\Sigma$ of operations $f$ with finite arity $\text{ar}(f)$ and finite depth $d(f)$.

- For a set $X$ of variables, the sets of $\Sigma$-terms $T_{\Sigma,n}(X)$ of uniform depth $n$ for $n \in \mathbb{N}$ are inductively defined as:

  $$x \in T_{\Sigma,0}(X) \quad (x \in X)$$  

  $$f(t_1, \ldots, t_{\text{ar}(f)}) \in T_{\Sigma,n+k}(X) \quad (f \in \Sigma, d(f) = n, t_1, \ldots, t_{\text{ar}(f)} \in T_{\Sigma,k}(X))$$
Graded Algebraic Theories

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- For a set $X$ of variables, the sets of $\Sigma$-terms $T_{\Sigma,n}(X)$ of uniform depth $n$ for $n \in \mathbb{N}$ are inductively defined as:

$$
\begin{align*}
\text{x} & \in T_{\Sigma,0}(X) & (\text{x} \in X) \\
\text{f}(t_1, \ldots, t_{\text{ar}(f)}) & \in T_{\Sigma,n+k}(X) & (f \in \Sigma, d(f) = n, t_1, \ldots, t_{\text{ar}(f)} \in T_{\Sigma,k}(X))
\end{align*}
$$

- A $\Sigma$-theory $E$ is a set of equations $s = t$, such that $s, t \in T_{\Sigma,n}(X)$ for $n \in \mathbb{N}$. 

Graded monads correspond to graded algebraic theories, i.e. the quotient $T\Sigma(X),n/\sim$ of $T\Sigma,n$ modulo the congruence $\sim$ generated by $E$ is in bijection to $M_nX$ for every $n \in \mathbb{N}$.

Graded monads can thus be induced by graded algebraic theories (and vice versa).
Adding depth

Σ := \{⊥ \lor 0, \lor 2, a \lor 1, b \lor 1\},

d(\lor) = d(\bot) = 0,

d(a) = d(b) = 1.

E :=

\begin{align*}
x \lor x & = x, \\
x \lor (y \lor z) & = (x \lor y) \lor z, \\
x \lor y & = y \lor x, \\
x \lor \bot & = x \lor \bot, \\
c(x \lor y) & = c(x) \lor c(y). \\
\end{align*}

(\text{for } c \in \{a, b\})

Corresponds to $M_{\mathcal{X}} = P(^n\mathcal{X})$. 
Adding depth

$$\Sigma := \{\bot/0, \lor/2, a/1, b/1\},$$
$$d(\lor) = d(\bot) = 0,$$
$$d(a) = d(b) = 1,$$

$$E := \begin{cases} 
x \lor x = x, \\
x \lor (y \lor z) = (x \lor y) \lor z, \\
x \lor y = y \lor x, \\
x \lor \bot = x, \\
\forall c \in \{a, b\}: c(x \lor y) = c(x) \lor c(y), \\
\forall c \in \{a, b\}: c(\bot) = \bot 
\end{cases}$$
Adding depth

$$\Sigma := \{\bot/0, \lor/2, a/1, b/1\},$$
$$d(\lor) = d(\bot) = 0,$$
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  x \lor (y \lor z) = (x \lor y) \lor z, \\
  x \lor y = y \lor x, \\
  x \lor \bot = x, \\
  c(x \lor y) = c(x) \lor c(y), \quad (c \in \{a, b\}) \\
  c(\bot) = \bot \quad (c \in \{a, b\})
\end{cases}$$

Corresponds to $$M_n X = \mathcal{P}(\{a, b\}^n \times X).$$
Adding depth

\[\Sigma := \{\bot/0, \lor/2, a/1, b/1, \star/0\},\]
\[d(\lor) = d(\bot) = 0,\]
\[d(a) = d(b) = d(\star) = 1,\]
\[E := \begin{cases}
x \lor x = x, \\
x \lor (y \lor z) = (x \lor y) \lor z, \\
x \lor y = y \lor x, \\
x \lor \bot = x, \\
c(x \lor y) = c(x) \lor c(y), & (c \in \{a, b\}) \\
c(\bot) = \bot & (c \in \{a, b\})
\end{cases}\]

Corresponds to \[M_nX = \mathcal{P}(\{a, b\}^n \times X + 1).\]
Abstract Foreshadowing

- When a graded algebraic theory only consists of operations and equations up to depth 1, we call it a depth-1 theory.
- If a graded monad is induced by a depth-1 theory, it is also called depth-1.
Graded Monads and Graded Algebras
Why even bother?

- Recall that the behaviour of a coalgebra is represented as sequence in $(M_n1)_{n \in \mathbb{N}}$
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- The motivation is to define the semantics of formulas of depth $n$ for a graded logic as maps $M_n1 \to \{\bot, \top\}$.
Why even bother?

- Recall that the behaviour of a coalgebra is represented as sequence in \((M_n1)_{n \in \mathbb{N}}\).
- The motivation is to define the semantics of formulas of depth \(n\) for a graded logic as maps \(M_n1 \to \{\bot, \top\}\).
- A Hennessy-Milner logic formula like \(\varphi := \square \diamond \top\) will be interpreted as a map \([\varphi] : M_21 \to \{\bot, \top\}\).
Why even bother?

- Recall that the behaviour of a coalgebra is represented as sequence in $(M_n 1)_{n \in \mathbb{N}}$.
- The motivation is to define the semantics of formulas of depth $n$ for a graded logic as maps $M_n 1 \rightarrow \{\bot, \top\}$.
- A Hennessy-Milner logic formula like $\varphi := \square \lozenge \top$ will be interpreted as a map $\llbracket \varphi \rrbracket : M_2 1 \rightarrow \{\bot, \top\}$.
- What about $\lozenge \varphi$?
Why even bother?

- Recall that the behaviour of a coalgebra is represented as sequence in \((M_n1)_{n \in \mathbb{N}}\).
- The motivation is to define the semantics of formulas of depth \(n\) for a graded logic as maps \(M_n1 \to \{\bot, \top\}\).
- A Hennessy-Milner logic formula like \(\varphi := \Box \Diamond \top\) will be interpreted as a map \(\llbracket \varphi \rrbracket : M_21 \to \{\bot, \top\}\).
- What about \(\Diamond \varphi\)?
- The semantics for a modality like \(\Diamond\) will be defined by a map \(M_1\{\bot, \top\} \to \{\bot, \top\}\), the semantics \(\llbracket \Diamond \varphi \rrbracket : M_31 \to \{\bot, \top\}\) shall be derivable from that.
For a monad $M$, an Eilenberg-Moore algebra $(A, a)$ consists of a carrier set $A$ and a structure map $a : MA \rightarrow A$ such that the following diagrams commute:
For a graded monad \( \mathbb{M} \), a graded \( M_n \)-algebra \( ((A_k)_{k \leq n}, (a^{mk})_{m+k \leq n}) \) consists of a carrier sets \( A_k \) and structure maps \( a^{mk} : M_mA_k \rightarrow A_{m+k} \) such that the following diagrams commute for all \( r, m, k \) with \( r + m + k \leq n \):

\[
\begin{align*}
A_m \xrightarrow{\eta_{Am}} & \quad M_0A_m \\
\downarrow \mu_{Am} & \quad \downarrow a^{0m} \\
A_m & \quad A_m
\end{align*}
\]

\[
\begin{align*}
M_rM_kA_m \xrightarrow{Mr^ka^km} & \quad M_rA_{k+m} \\
\downarrow \mu_{Am} & \quad \downarrow a^{r+k+m} \\
M_{r+k}A_m & \quad A_{r+k+m}
\end{align*}
\]
$M_0$ and $M_1$

- $M_0$-algebras are just Eilenberg-Moore algebras for $(M_0, \eta, \mu^{00})$. 
$M_0$ and $M_1$

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- E.g. $(M_n X, \mu^{0n})$ is an $M_0$-algebra, as $\mu^0_X : M_0 M_n X \to M_n X$. 


$M_0$ and $M_1$

- $M_0$-algebras are just Eilenberg-Moore algebras for $(M_0, \eta, \mu^{00})$.
- E.g. $(M_nX, \mu^{0n})$ is an $M_0$-algebra, as $\mu^{0n}_X : M_0M_nX \to M_nX$.
- $M_1$-algebras are a 5-tuple $(A_0, A_1, a^{00}, a^{01}, a^{10})$, where $(A_0, a^{00})$ and $(A_1, a^{01})$ are both $M_0$-algebras, and the structure map $a^{10}$ satisfies:

\[ M_0M_1A_0 \xrightarrow{M_0a^{10}} M_0A_1 \]
\[ M_1A_0 \xrightarrow{a^{10}} A_1 \]
\[ M_1M_0A_0 \xrightarrow{M_1a^{00}} M_1A_0 \]
\[ A_1 \]

(homomorphism)

(coequalization)
Canonical $M_1$-Algebra

If $(A_1, a^{10})$ is the coequalizer in the category of $M_0$-algebras, then $A := (A_0, A_1, a^{00}, a^{01}, a^{10})$ is called a canonical $M_1$-algebra.
Canonical $M_1$-Algebra

If $(A_1, a^{10})$ is the coequalizer in the category of $M_0$-algebras, then $A := (A_0, A_1, a^{00}, a^{01}, a^{10})$ is called a canonical $M_1$-algebra.

For canonical $M_1$-algebra $A$, we get for any $M_1$-algebra $B$ that any $M_0$-algebra homomorphism $A_0 \to B_0$ extends (freely) to a unique $M_1$-algebra homomorphism $A \to B$, i.e. a map $A_1 \to B_1$ joins in.
Canonical $M_1$-Algebra

$$M_1 M_0 A_0 \xrightarrow{M_1 a^{00}} M_1 A_0 \xrightarrow{a^{10}} A_1$$

- If $(A_1, a^{10})$ is the coequalizer in the category of $M_0$-algebras, then $A := (A_0, A_1, a^{00}, a^{01}, a^{10})$ is called a canonical $M_1$-algebra.

- For canonical $M_1$-algebra $A$, we get for any $M_1$-algebra $B$ that any $M_0$-algebra homomorphism $A_0 \to B_0$ extends (freely) to a unique $M_1$-algebra homomorphism $A \to B$, i.e. a map $A_1 \to B_1$ joins in.

- In short, if the left algebra is canonical, we get this:

$$
\begin{array}{c}
M_1 M_n 2 \xrightarrow{M_1[\varphi]} M_1 2 \\
\mu_2^{1n} \downarrow \quad \quad \quad \quad \quad \quad \downarrow [\Diamond] \\
M_{n+1} 2 \xrightarrow{[\Diamond \varphi]} 2
\end{array}
$$
Depth-1ness Returns

- For that to hold, all pairs of $M_0$-algebras $(M_n, \mu^{0n})$ and $(M_{n+1}, \mu^{0,n+1})$ have to form a canonical $M_1$-algebra with $\mu^{1n}$. 

E.g. so that we can have this:

$$\begin{array}{c}
M_1 \\
M_0 \\
M_1 \\
M_1 \\
M_1 \\
\end{array}$$

$$\begin{array}{c}
\bot \\
\top \\
\\ \\
\\ \\
\\ \\
\end{array}$$

$$\begin{array}{c}
M_1 \\
M_1 \\
M_1 \\
\bot \\
\top \\
\end{array}$$

$$\begin{array}{c}
\diamond \\
\square \\
\\ \\
\\ \\
\\ \\
\end{array}$$
Depth-1ness Returns

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- E.g. so that we can have this:

\[
\begin{array}{ccc}
M_1 M_0 1 & \xrightarrow{M_1[\top]} & M_1 \{\bot, \top\} \\
\mu_1^{10} & \downarrow & \downarrow [\diagup] \\
M_1 1 & \xrightarrow{[\diamond \top]} & \{\bot, \top\}
\end{array}
\quad
\begin{array}{ccc}
M_1 M_1 1 & \xrightarrow{M_1[\diamond \top]} & M_1 \{\bot, \top\} \\
\mu_1^{11} & \downarrow & \downarrow [\square] \\
M_2 1 & \xrightarrow{[\square \diamond \top]} & \{\bot, \top\}
\end{array}
\]
For that to hold, all pairs of $M_0$-algebras $(M_n, \mu^{0n})$ and $(M_{n+1}, \mu^{0,n+1})$ have to form a canonical $M_1$-algebra with $\mu^{1n}$.

E.g. so that we can have this:

$$
\begin{align*}
M_1 M_0 1 & \xrightarrow{M_1[\top]} M_1 \{\bot, \top\} \\
\mu_1^{10} & \downarrow \quad \quad \quad \downarrow [\lozenge] \\
M_1 1 & \xrightarrow{[\lozenge\top]} \{\bot, \top\}
\end{align*}

\begin{align*}
M_1 M_1 1 & \xrightarrow{M_1[\lozenge\top]} M_1 \{\bot, \top\} \\
\mu_1^{11} & \downarrow \quad \quad \quad \downarrow [[\Box]] \\
M_2 1 & \xrightarrow{[[\Box\lozenge\top]]} \{\bot, \top\}
\end{align*}

Luckily, depth-1 monads are characterised by

$$
\begin{align*}
M_1 M_0 M_n & \xrightarrow{M_1\mu^{10}} M_1 M_n \\
\mu^{1n} & \xrightarrow{} M_{n+1}
\end{align*}
$$

being a coequalizer diagram in $M_0$-algebras.
Graded Logics

Fix an $M_0$-algebra $(\Omega, o)$ of truth values.
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- Truth constants $c \in \Theta$, which are understood as maps $\hat{c} : 1 \to \Omega$
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- Truth constants $c \in \Theta$, which are understood as maps $\hat{c} : 1 \rightarrow \Omega$
- $k$-ary propositional operators $p \in \mathcal{O}$, which are understood as $M_0$-morphisms $\llbracket p \rrbracket : \Omega^k \rightarrow \Omega$
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- Modal operators $L \in \Lambda$, which are understood as $M_1$-algebras (with $(\Omega, o)$ as both $M_0$-algebras and $\llbracket L \rrbracket : M_1\Omega \to \Omega$ as the structure morphism between them).
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The semantics of $\varphi$ is an $M_0$-morphism $\llbracket \varphi \rrbracket : M_n1 \to \Omega$ recursively defined by:

\[
\begin{align*}
\llbracket c \rrbracket & : M_01 \xrightarrow{M_0\hat{c}} M_0\Omega \xrightarrow{o} \Omega, \\
\llbracket p(\varphi_1, ..., \varphi_k) \rrbracket & = \llbracket p \rrbracket (\llbracket \varphi_1 \rrbracket, ..., \llbracket \varphi_k \rrbracket), \\
\llbracket L\varphi \rrbracket & = \llbracket L \rrbracket (\llbracket \varphi \rrbracket)
\end{align*}
\]
Graded Monads and Fixpoints
Where do they live?
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- We think it’s here:

\[ \prod_{n \in \mathbb{N}} M_n 1 \]
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- So the semantics of a formula \( \varphi \) is supposed to be a map:
  \[ [\varphi] : (\prod_{n \in \mathbb{N}} M_n 1) \rightarrow \{ \bot, \top \} \]
Where do they live?

- We think it’s here:
  \[ \prod_{n \in \mathbb{N}} M_n \mathbf{1} \]

- So the semantics of a formula \( \varphi \) is supposed to be a map:
  \[ \llbracket \varphi \rrbracket : \left( \prod_{n \in \mathbb{N}} M_n \mathbf{1} \right) \to \{ \bot, \top \} \]

- The \( M_1 \)-part of the canonical \( M_1 \)-algebra is the codomain of the coequalizer

\[
\begin{array}{c}
M_1 M_0 \Pi_n M_n \\
\text{\(M_1 \langle M_0 \pi_n \rangle\)} \\
M_1 \Pi_n M_0 M_n
\end{array}
\xrightarrow{\mu^{10}}
\begin{array}{c}
M_1 \prod_n M_n \\
M_1 \Pi_n M_n \mu^{0n}
\end{array}
\xrightarrow{c}
\begin{array}{c}
C
\end{array}
\]

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Graded Monads and (someday) Fixpoints

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Thank you!