Graded Monads and (someday) Fixpoints

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Graded Monads and Graded Semantics

Monads

A monad (M, η, μ) consists of

an endofunctor M,

a natural transformation $\eta : Id \to M$, the unit,

and a natural transformation $\mu : MM \to M$, the multiplication.

with the following diagrams commuting:

$$\begin{array}{ccccc} M & \xrightarrow{M\eta} & MM & \xleftarrow{\eta M} & M \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \mu} & \diagup\!\!\diagup & \\ & & M & & \end{array} \qquad \begin{array}{ccc} MMM & \xrightarrow{M\mu} & MM \\ {\scriptstyle \mu M}\downarrow & & \downarrow{\scriptstyle \mu} \\ MM & \xrightarrow[\mu]{} & M \end{array}$$

Complicate things

A graded monad $((M_n)_{n\in\mathbb{N}}, \eta, (\mu^{mk})_{m,k\in\mathbb{N}})$ consists of

- a family of endofunctors M_n, $n \in \mathbb{N}$,
- a natural transformation $\eta : Id \to M_0$, the unit,
- and a family of natural transformations $\mu^{mk} : M_m M_k \to M_{m+k}$, $m, k \in \mathbb{N}$, the multiplication.

with the following diagrams commuting for all $n, k, m \in \mathbb{N}$:

$$\begin{array}{ccc} M_n & \xrightarrow{M_n\eta} & M_n M_0 \\ {\scriptstyle \eta M_n}\downarrow & \searrow^{=} & \downarrow{\scriptstyle \mu^{n0}} \\ M_0 M_n & \xrightarrow[\mu^{0n}]{} & M_n \end{array} \qquad \begin{array}{ccc} M_n M_k M_m & \xrightarrow{M_n\mu^{km}} & M_n M_{k+m} \\ {\scriptstyle \mu^{nk} M_m}\downarrow & & \downarrow{\scriptstyle \mu^{n,k+m}} \\ M_{n+k} M_m & \xrightarrow[\mu^{n+k,m}]{} & M_{n+k+m} \end{array}$$

Simplify things

- Monads on a set X describe a term structure MX.
- η ensures that every element of X can be made into a term.
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- Monads on a set X describe a term structure MX.
- η ensures that every element of X can be made into a term.
- μ describes substitution, where terms of terms are again just terms.
- Then graded monads are like terms with depth.

Graded semantics

A graded semantics $(\alpha, \mathbb{M})$ for an endofunctor G consists of

- a graded monad $\mathbb{M} = ((M_n)_{n \in \mathbb{N}}, \eta, (\mu^{mk})_{m,k \in \mathbb{N}})$,
- and a natural transformation $\alpha : G \to M_1$.

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A G-coalgebra (X, γ) then induces the following sequence of inductively defined maps $\gamma^{(n)} : X \to M_n 1$

$$\gamma^{(0)} : X \xrightarrow{\eta_X} M_0 X \xrightarrow{M_0 !} M_0 1$$

$$\gamma^{(n+1)} : X \xrightarrow{\alpha_X \circ \gamma} M_1 X \xrightarrow{M_1 \gamma^{(n)}} M_1 M_n 1 \xrightarrow{\mu_1^{1n}} M_{n+1} 1$$

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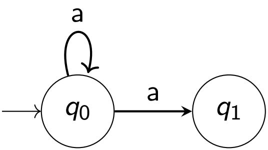
$\gamma^{(n)}(x) \in M_n 1$ is called the n-step $(\alpha, \mathbb{M})$-behaviour of $x \in X$.

An LTS

For the functor $GX = \mathcal{P}(A \times X)$ take the following coalgebra $\gamma : X \to GX$



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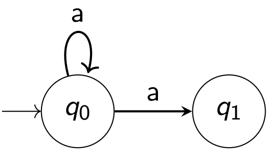

The graded monad $M_n = G^n$ (with α, η and μ^{mk} as all the appropriate identities) corresponds to coalgebraic "step-n behaviour". So if the $\gamma^{(k)}$ images of two states coincide for all $k \leq n$, they are n-step behaviourally equivalent.

Trace Semantics

Trace Semantics

To obtain trace semantics, define $\mathbb{M}$ as

- $M_n X := \mathcal{P}(A^n \times X)$,
- $\eta(x) := \{(\epsilon, x)\}$,
- $\mu^{mk}(S) := \{(vw, W) \mid (v, V) \in S, (w, W) \in V\}$

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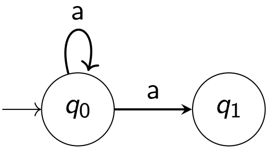

$$\begin{aligned}
\gamma^{(0)}(q_0) &= \{\epsilon\} \\
\gamma^{(1)}(q_0) &= \mu^{1,0}\{(a, \gamma^{(0)}(q_0)), (a, \gamma^{(0)}(q_1))\} \\
&= \mu^{1,0}\{(a, \{\epsilon\}), (a, \{\epsilon\})\} \\
&= \{a\} \\
\gamma^{(2)}(q_0) &= \{aa\} \\
&\dots
\end{aligned}$$

Note that $\gamma^{(n+1)}(q_1) = \emptyset$

The ugly truth

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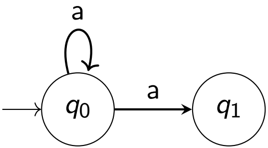

$$\begin{aligned}
\gamma^{(0)}(q_0) &= \{(\epsilon, *)\} \\
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&= \mu^{1,0}\{(a, \{(\epsilon, *)\}), (a, \{(\epsilon, *)\})\} \\
&= \{(a\epsilon, *)\} \\
\gamma^{(2)}(q_0) &= \{(aa\epsilon, *)\} \\
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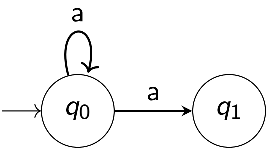
and $\alpha : G \to M_1$ (so $\mathcal{P}(A \times X) \to \mathcal{P}(A \times X + \mathbf{1})$) as

$$\alpha(\emptyset) := \{\star\}, \qquad (\star \in \mathbf{1})$$
$$\alpha(S) := S \subseteq \mathcal{P}(A \times X + \mathbf{1}) \qquad (S \neq \emptyset)$$

Verbose and ugly

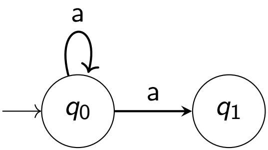


$$\gamma^{(1)}(q_1) = \alpha(\emptyset) = \{\star\}$$
$$\gamma^{(1)}(q_0) = \{(a\epsilon, *)\}$$

$$\begin{aligned}
\gamma^{(2)} : q_0 &\overset{\gamma}{\mapsto} \{(a, q_0), (a, q_1)\} \\
&\overset{\alpha}{\mapsto} \{(a, q_0), (a, q_1)\} \\
&\xmapsto{M_1\gamma^{(1)}} \{(a, \gamma^{(1)}(q_0)), (a, \gamma^{(1)}(q_1))\} \\
&= \{(a, \{(a\epsilon, *)\}), (a, \{\star\})\} \\
&\overset{\mu}{\mapsto} \{(aa\epsilon, *), a\star\}
\end{aligned}$$

$$\gamma^{(3)}(q_0) = \{(aaa\epsilon, *), aa\star, a\star\}$$
$$\gamma^{(4)}(q_0) = \dots$$

Just verbose



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$$\gamma^{(3)}(q_0) = \{aaa, aa\star, a\star\}$$
$$\gamma^{(4)}(q_0) = \ldots$$

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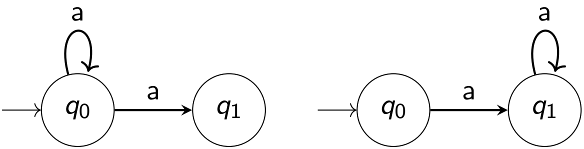

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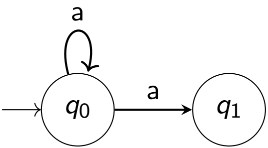

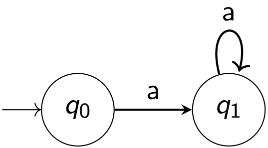

$\gamma^{(0)}(q_0)$ $\{\epsilon\}$ $\{\epsilon\}$

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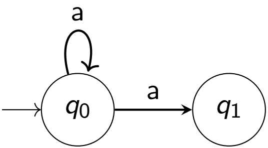

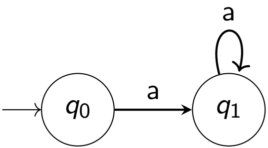

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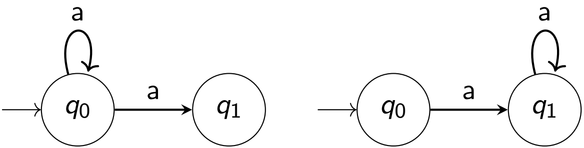

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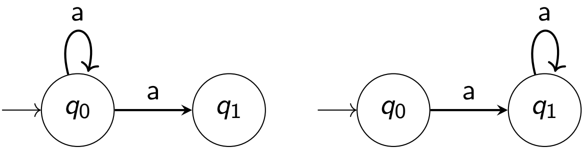

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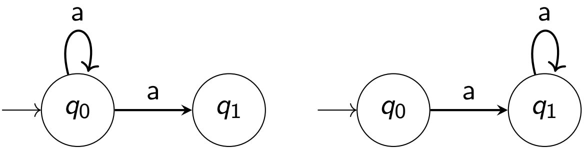
...	...	...

Complete Trace Semantics



$\gamma^{(0)}(q_0)$	$\{\epsilon\}$	$\{\epsilon\}$
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Graded Monads and Graded Theories

Algebraic Theories

- A signature is a set Σ of operations f with finite arity $\mathrm{ar}(f)$.

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- For a set X of variables, the set of Σ-terms $T_\Sigma(X)$ is defined inductively as:

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& x \in T_\Sigma(X) && (x \in X) \\
& f(t_1, \ldots, t_{\mathrm{ar}(f)}) \in T_\Sigma(X) && (f \in \Sigma, t_1, \ldots, t_{\mathrm{ar}(f)} \in T_\Sigma(X))
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- A Σ-theory E is a set of equations $s = t$, such that $s, t \in T_\Sigma(X)$.

Monads and Theories

- Monads correspond to algeraic theories, i.e. the quotient $T_\Sigma(X)/\sim$ of T_Σ modulo the congruence $\sim$ generated by E is in bijection to MX.
- Monads can thus be induced by an algebraic theory (and vice versa).

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Like this:

$$\begin{aligned} a(b(x)) \vee b(x) \vee b(x) \vee y &= a(b(x)) \vee y \vee b(x) \\ &\mathrel{\widehat{=}} \{(ab, x), (b, x), (\epsilon, y)\} \end{aligned}$$

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Corresponds to $M_n X = \mathcal{P}(\{a, b\}^n \times X + \mathbf{1})$.

Abstract Foreshadowing

- When a graded algebraic theory only consists of operations and equations up to depth 1, we call it a depth-1 theory.
- If a graded monad is induced by a depth-1 theory, it is also called depth-1.

Graded Monads and Graded Algebras

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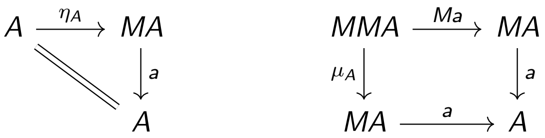
- A Hennessy-Milner logic formula like $\varphi := \Box\Diamond\top$ will be interpreted as a map $[\![\varphi]\!] : M_2 1 \to \{\bot, \top\}$.
- What about $\Diamond\varphi$?
- The semantics for a modality like $\Diamond$ will be defined by a map $M_1\{\bot, \top\} \to \{\bot, \top\}$, the semantics $[\![\Diamond\varphi]\!] : M_3 1 \to \{\bot, \top\}$ shall be derivable from that.

Giving Terms Value

For a monad M, an Eilenberg-Moore algebra (A, a) consists of a carrier set A and a structure map $a : MA \to A$ such that the following diagrams commute:



Giving Graded Terms Value

For a graded monad $\mathbb{M}$, a graded M_n-algebra $((A_k)_{k \leq n}, (a^{mk})_{m+k \leq n})$ consists of a carrier sets A_k and structure maps $a^{mk} : M_m A_k \to A_{m+k}$ such that the following diagrams commute for all r, m, k with $r + m + k \leq n$:

$$
\begin{array}{ccc}
A_m & \xrightarrow{\eta_{A_m}} & M_0 A_m \\
 & \searrow^{=} & \downarrow{\scriptstyle a^{0m}} \\
 & & A_m
\end{array}
\qquad\qquad
\begin{array}{ccc}
M_r M_k A_m & \xrightarrow{M_r a^{km}} & M_r A_{k+m} \\
{\scriptstyle \mu^{rk}_{A_m}} \downarrow & & \downarrow {\scriptstyle a^{r,k+m}} \\
M_{r+k} A_m & \xrightarrow{a^{r+k,m}} & A_{r+k+m}
\end{array}
$$

M_0 and M_1

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- M_1-algebras are a 5-tuple $(A_0, A_1, a^{00}, a^{01}, a^{10})$, where (A_0, a^{00}) and (A_1, a^{01}) are both M_0-algebras, and the structure map a^{10} satisfies:

$$\begin{array}{ccc} M_0 M_1 A_0 & \xrightarrow{M_0 a^{10}} & M_0 A_1 \\ {\scriptstyle \mu^{01}_{A_0}} \downarrow & & \downarrow {\scriptstyle a^{01}} \\ M_1 A_0 & \xrightarrow[a^{10}]{} & A_1 \end{array}$$

(homomorphy)

$$M_1 M_0 A_0 \underset{\mu^{10}_{A_0}}{\overset{M_1 a^{00}}{\rightrightarrows}} M_1 A_0 \xrightarrow{a^{10}} A_1$$

(coequalization)

Canonical M_1-Algebra

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- In short, if the left algebra is canonical, we get this:

$$\begin{array}{ccc} M_1 M_n 2 & \xrightarrow{M_1 [\![\varphi]\!]} & M_1 2 \\ {\scriptstyle \mu^{1n}_2} \downarrow & & \downarrow {\scriptstyle [\![\Diamond]\!]} \\ M_{n+1} 2 & \underset{[\![\Diamond\varphi]\!]}{\dashrightarrow} & 2 \end{array}$$

Depth-1ness Returns

- For that to hold, all pairs of M_0-algebras (M_n, μ^{0n}) and $(M_{n+1}, \mu^{0,n+1})$ have to form a canonical M_1-algebra with μ^{1n}.

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- E.g. so that we can have this:

$$\begin{array}{ccc} M_1 M_0 1 & \xrightarrow{M_1 [\![\top]\!]} & M_1\{\bot, \top\} \\ {\scriptstyle \mu_1^{10}} \big\downarrow & & \big\downarrow {\scriptstyle [\![\Diamond]\!]} \\ M_1 1 & \dashrightarrow_{[\![\Diamond \top]\!]} & \{\bot, \top\} \end{array} \qquad \begin{array}{ccc} M_1 M_1 1 & \xrightarrow{M_1 [\![\Diamond \top]\!]} & M_1\{\bot, \top\} \\ {\scriptstyle \mu_1^{11}} \big\downarrow & & \big\downarrow {\scriptstyle [\![\Box]\!]} \\ M_2 1 & \dashrightarrow_{[\![\Box \Diamond \top]\!]} & \{\bot, \top\} \end{array}$$

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- Luckily, depth-1 monads are characterised by

$$M_1 M_0 M_n \underset{\mu^{10} M_n}{\overset{M_1 \mu^{10}}{\rightrightarrows}} M_1 M_n \xrightarrow{\mu^{1n}} M_{n+1}$$

being a coequalizer diagram in M_0-algebras.

Graded Logics

Fix an M_0-algebra (Ω, o) of truth values.

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The semantics of φ is an M_0-morphism $[\![\varphi]\!] : M_n 1 \to \Omega$ recursively defined by:

$$[\![c]\!] : M_0 1 \xrightarrow{M_0 \hat{c}} M_0 \Omega \xrightarrow{o} \Omega,$$

$$[\![p(\varphi_1, \dots, \varphi_k)]\!] = [\![p]\!]([\![\varphi_1]\!], \dots, [\![\varphi_k]\!]),$$

$$[\![L\varphi]\!] = [\![L]\!]([\![\varphi]\!])$$

Graded Monads and Fixpoints

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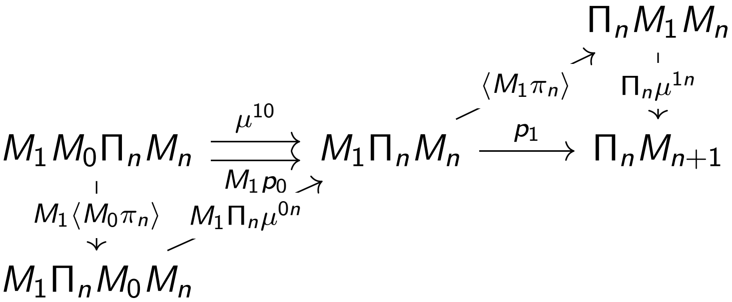
- The M_1-part of the canonical M_1-algebra is the codomain of the coequalizer

$$M_1 M_0 \Pi_n M_n \underset{M_1 p_0}{\overset{\mu^{10}}{\rightrightarrows}} M_1 \Pi_n M_n \xrightarrow{\;c\;} C$$

$$M_1 M_0 \Pi_n M_n \xrightarrow{M_1 \langle M_0 \pi_n \rangle} M_1 \Pi_n M_0 M_n \xrightarrow{M_1 \Pi_n \mu^{0n}} M_1 \Pi_n M_n$$

Conjecture


$\Pi_n M_1 M_n$
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Thank you!