Graph Liftings and Howe's Method

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Abstract GSOS [Turi & Plotkin '97]

Operational rules

$$\frac{p \xrightarrow{a} p'}{p \,||\, q \xrightarrow{a} p' \,||\, q}$$

$\widehat{=}$

GSOS laws: natural transformations

$$\varrho_X\colon \underbrace{\Sigma(X \times BX)}_{\text{premises}} \to \underbrace{B(\Sigma^* X)}_{\text{conclusion}}$$

for functors $\Sigma, B\colon \mathbb{C} \to \mathbb{C}$ representing syntax and behaviour (e.g. $B = \mathcal{P}_{\mathsf{f}}^{L}$).

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for functors $\Sigma, B \colon \mathbb{C} \to \mathbb{C}$ representing syntax and behaviour (e.g. $B = \mathcal{P}_{\mathsf{f}}^{L}$).

- Operational model $\mu\Sigma \to B(\mu\Sigma)$, denotational model $\Sigma(\nu B) \to \nu B$.

  programs

  abstract behaviours

- **Key feature:** compositionality, i.e. bisimilarity is a congruence:

$$p_i \sim q_i \quad (i = 1, \ldots, n) \quad \xRightarrow{f \in \Sigma} \quad \mathsf{f}(p_1, \ldots, p_n) \sim \mathsf{f}(q_1, \ldots, q_n).$$

- **Scope**: first-order (CCS, π-calculus, ...), ~~higher-order~~ (λ-calculus)

The Issue With Higher-Order Languages

HO languages require behaviours like

$$BX = X^X.$$

This is not an endofunctor – but

$$B(X, Y) = Y^X$$

is a bifunctor contravariant in X and covariant in Y.

Key idea for higher-order abstract GSOS

endofunctors $B\colon \mathbb{C} \to \mathbb{C}$ $\quad + \quad$ natural transformations

$$\Downarrow$$

bifunctors $B\colon \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$ $\quad + \quad$ **dinatural** transformations.

Higher-Order Abstract GSOS [POPL'23]

Operational rules

$$\frac{}{(\lambda x.p)\, q \to p[q/x]}$$

$$\frac{p \to p'}{p\, q \to p'\, q}$$

$\widehat{=}$

Higher-order GSOS laws: (di-)natural trf.

$$\varrho_{X,Y} \colon \underbrace{\Sigma(X \times B(X,Y))}_{\text{premises}} \to \underbrace{B(X, \Sigma^*(X+Y))}_{\text{conclusion}}$$

$$\mathbb{C} = \mathsf{Set}^{\mathbb{F}}$$

$$\Sigma X = V + \delta X + X \times X$$

$$\mu\Sigma = \lambda\text{-terms}$$

$$B(X,Y) = \langle X, Y\rangle \times (Y + Y^X + 1)$$

cf. Fiore, Plotkin & Turi '99

Higher-Order Abstract GSOS [POPL'23]

Operational rules

$$\frac{}{(\lambda x.p)\,q \to p[q/x]}$$

$$\frac{p \to p'}{p\,q \to p'\,q}$$

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$$\varrho_{X,Y} : \underbrace{\Sigma(X \times B(X,Y))}_{\text{premises}} \to \underbrace{B(X, \Sigma^*(X+Y))}_{\text{conclusion}}$$

- Operational model $\mu\Sigma \to B(\mu\Sigma, \mu\Sigma)$, ~~denotational model~~ $\nu B(\mu\Sigma, -)$.

  programars

  abstract behaviours

- $\underbrace{\text{\textbf{Key feature:} compositionality, i.e. bisimilarity is a congruence.}}_{\text{Proof: more complex than first-order case + needs mild assumptions.}}$

Strong Applicative Bisimilarity

Coalgebraic bisimilarity on operational model $\mu\Sigma \to B(\mu\Sigma, \mu\Sigma)$

$$=$$

strong applicative bisimilarity.

Example: λ-calculus

Greatest relation $\sim \subseteq \Lambda \times \Lambda$ such that for $t_1 \sim t_2$,

closed λ-terms

$$
\begin{aligned}
t_1 \to t_1' &\implies t_2 \to t_2' \quad \wedge\ t_1' \sim t_2'; \\
t_1 = \lambda x.t_1' &\implies t_2 = \lambda x.t_2' \ \wedge\ \forall e \in \Lambda.\ t_1'[e/x] \sim t_2'[e/x];
\end{aligned}
$$

$+$ two symmetric conditions

Applicative Bisimilarity [Abramsky '90]

Weak coalgebraic bisimilarity on operational model $\mu\Sigma \to B(\mu\Sigma, \mu\Sigma)$

$=$

(weak) applicative bisimilarity.

Example: λ-calculus

Greatest relation $\approx \subseteq \Lambda \times \Lambda$ such that for $t_1 \approx t_2$,

$$t_1 \to^\star \lambda x.t_1' \implies t_2 \to^\star \lambda x.t_2' \;\wedge\; \forall e \in \Lambda.\; t_1'[e/x] \approx t_2'[e/x];$$
$$t_2 \to^\star \lambda x.t_2' \implies t_1 \to^\star \lambda x.t_1' \;\wedge\; \forall e \in \Lambda.\; t_1'[e/x] \approx t_2'[e/x].$$

Goal: Compositionality of higher-order GSOS w.r.t. weak bisimilarity.

Proof of compositionality w.r.t. strong applicative bisimilarity

Bisimilarity $\sim$ on $\mu\Sigma \to B(\mu\Sigma, \mu\Sigma)$ is a congruence.

1. Take the closure $\widehat{\sim}$ of $\sim$ under contexts and transitivity:

$$\frac{p \sim q}{p \mathrel{\widehat{\sim}} q} \qquad \frac{p_i \mathrel{\widehat{\sim}} q_i \ (i = 1, \ldots, n)}{f(p_1, \ldots, p_n) \mathrel{\widehat{\sim}} f(q_1, \ldots, q_n)} \qquad \frac{p \mathrel{\widehat{\sim}} q,\ q \mathrel{\widehat{\sim}} r}{p \mathrel{\widehat{\sim}} r}$$

2. Prove that $\widehat{\sim}$ is a bisimulation, e.g. for the λ-calculus:

$$\cdots$$

$$t_1 \mathrel{\widehat{\sim}} t_2 \ \wedge\ t_1 = \lambda x.t_1' \implies t_2 = \lambda x.t_2' \ \wedge\ \forall e \in \Lambda.\ t_1'[e/x] \mathrel{\widehat{\sim}} t_2'[e/x]$$

$$\cdots$$

3. This implies $\widehat{\sim}\ \subseteq\ \sim$, hence the latter is a congruence.

Proof of compositionality w.r.t. weak applicative bisimilarity

Weak bisimilarity $\approx$ on $\mu\Sigma \to B(\mu\Sigma, \mu\Sigma)$ is a congruence.

1. Take the closure $\widehat{\approx}$ of $\approx$ under contexts and transitivity:
$$\frac{p \approx q}{p \mathrel{\widehat{\approx}} q} \qquad \frac{p_i \mathrel{\widehat{\approx}} q_i \ (i = 1, \ldots, n)}{f(p_1, \ldots, p_n) \mathrel{\widehat{\approx}} f(q_1, \ldots, q_n)} \qquad \frac{p \mathrel{\widehat{\approx}} q,\ q \mathrel{\widehat{\approx}} r}{p \mathrel{\widehat{\approx}} r}$$
2. Prove that $\widehat{\approx}$ is a **weak** bisimulation, e.g. for the λ-calculus:
$$\ldots$$
$$t_1 \mathrel{\widehat{\approx}} t_2 \wedge t_1 \to^{\star} \lambda x.t_1' \implies t_2 \to^{\star} \lambda x.t_2' \wedge \forall e \in \Lambda.\ t_1'[e/x] \mathrel{\widehat{\approx}} t_2'[e/x]$$
$$\ldots$$
3. This implies $\widehat{\approx} \ \subseteq \ \approx$, hence the latter is a congruence.

Proof of compositionality w.r.t. weak applicative bisimilarity

Weak bisimilarity $\approx$ on $\mu\Sigma \to B(\mu\Sigma, \mu\Sigma)$ is a congruence.

1. Take the closure $\widehat{\approx}$ of $\approx$ under contexts and transitivity:

$$\frac{p \approx q}{p \mathrel{\widehat{\approx}} q} \qquad \frac{p_i \mathrel{\widehat{\approx}} q_i \ (i = 1, \ldots, n)}{f(p_1, \ldots, p_n) \mathrel{\widehat{\approx}} f(q_1, \ldots, q_n)} \qquad \frac{p \mathrel{\widehat{\approx}} q,\ q \mathrel{\widehat{\approx}} r}{p \mathrel{\widehat{\approx}} r}$$

2. Prove that $\widehat{\approx}$ is a **weak** bisimulation, e.g. for the λ-calculus:

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$$t_1 \mathrel{\widehat{\approx}} t_2 \wedge t_1 \to^\star \lambda x.t_1' \implies t_2 \to^\star \lambda x.t_2' \wedge \forall e \in \Lambda.\ t_1'[e/x] \mathrel{\widehat{\approx}} t_2'[e/x]$$

$$\ldots$$

3. This implies $\widehat{\approx} \ \subseteq \approx$, hence the latter is a congruence.

. . . **but Step 2 fails ☹**

Proof of compositionality w.r.t. weak applicative bisimilarity

Weak bisimilarity $\approx$ on $\mu\Sigma \to B(\mu\Sigma, \mu\Sigma)$ is a congruence.

1. Take the **Howe closure** $\widehat{\approx}$ of $\approx$:

$$\frac{p \approx q}{p \mathrel{\widehat{\approx}} q} \qquad \frac{p_i \mathrel{\widehat{\approx}} q_i \ (i = 1, \ldots, n)}{f(p_1, \ldots, p_n) \mathrel{\widehat{\approx}} f(q_1, \ldots, q_n)} \qquad \frac{p \mathrel{\widehat{\approx}} q,\ q \approx r}{p \mathrel{\widehat{\approx}} r}$$

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2. Prove that $\widehat{\approx}$ is a **logical weak** bisimulation, e.g. for the λ-calculus:

$$\ldots$$

$$t_1 \mathrel{\widehat{\approx}} t_2 \wedge t_1 \to^\star \lambda x.t_1' \implies t_2 \to^\star \lambda x.t_2' \wedge \forall d \mathrel{\widehat{\approx}} e.\ t_1'[d/x] \mathrel{\widehat{\approx}} t_2'[e/x]$$

$$\ldots$$

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Proof of compositionality w.r.t. weak applicative bisimilarity

Weak bisimilarity $\approx$ on $\mu\Sigma \to B(\mu\Sigma, \mu\Sigma)$ is a congruence.

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$$\ldots$$

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Categorical perspective: Graph liftings of (bi-)functors.

Graphs in a Category

The category $\mathrm{Gra}(\mathbb{C})$ of graphs in $\mathbb{C}$ is given by objects and morphisms

$$\begin{tikzcd} R \arrow[d, shift right, bend right, "\mathrm{outl}_R"'] \arrow[d, shift left, bend left, "\mathrm{outr}_R"] \\ X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R \arrow[r, "h_1"] \arrow[d, shift right, bend right, "\mathrm{outl}_R"'] \arrow[d, shift left, bend left, "\mathrm{outr}_R"] & S \arrow[d, shift right, bend right, "\mathrm{outl}_S"'] \arrow[d, shift left, bend left, "\mathrm{outr}_S"] \\ X \arrow[r, "h_0"] & Y \end{tikzcd}$$

If $\mathbb{C}$ has pullbacks, the projection $(X, R) \to X$ is a bifibration.

Fibres of the category of graphs

$$\mathrm{Gra}_X(\mathbb{C}) \;=\; \text{graphs with vertices } X \text{ and morphisms } (\mathrm{id}_X, \cdot)$$

Preorder on $\mathrm{Gra}_X(\mathbb{C})$

$$(X, R) \leq (X, S) \iff (X, R) \xrightarrow{\exists} (X, S) \quad \text{in } \mathrm{Gra}_X(\mathbb{C}).$$

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Opcartesian lift of $f \colon X \to Y$

$$f_\star \colon \mathsf{Gra}_X(\mathbb{C}) \to \mathsf{Gra}_Y(\mathbb{C}), \qquad \begin{tikzcd} R \arrow[d, bend right, "\mathsf{outl}_R"'] \arrow[d, bend left, "\mathsf{outr}_R"] \\ X \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} R \arrow[d, bend right, "\mathsf{outl}_R"'] \arrow[d, bend left, "\mathsf{outr}_R"] & \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Graphs in a Category

The category of graphs in $\mathbb{C}$ is given by objects and morphisms

$$
\begin{tikzcd}
R \arrow[d, bend right, "\mathsf{outl}_R"'] \arrow[d, bend left, "\mathsf{outr}_R"] \\
X
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
R \arrow[rr, "h_1"] \arrow[d, bend right, "\mathsf{outl}_R"'] \arrow[d, bend left, "\mathsf{outr}_R"] & & S \arrow[d, bend right, "\mathsf{outl}_S"'] \arrow[d, bend left, "\mathsf{outr}_S"] \\
X \arrow[rr, "h_0"] & & Y
\end{tikzcd}
$$

Composite $(X, R)\,;(X, R')$ of graphs (X, R) and (X, R')

$$
\begin{tikzcd}
& & R\,;R' \arrow[dl, "\pi_{R;R',R}"'] \arrow[dr, "\pi_{R;R',R'}"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] \arrow[ddll, bend right=60, "\mathsf{outl}_{R;R'}"'] \arrow[ddrr, bend left=60, "\mathsf{outr}_{R;R'}"] & & \\
& R \arrow[dl, "\mathsf{outl}_R"'] \arrow[dr, "\mathsf{outr}_R"] & & R' \arrow[dl, "\mathsf{outl}_{R'}"'] \arrow[dr, "\mathsf{outr}_{R'}"] & \\
X & & X & & X
\end{tikzcd}
$$

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Canonical graph lifting of endofunctor $F\colon \mathbb{C} \to \mathbb{C}$

$$\begin{tikzcd} \mathrm{Gra}(\mathbb{C}) \arrow[r, "\overline{F}"] \arrow[d] & \mathrm{Gra}(\mathbb{C}) \arrow[d] \\ \mathbb{C} \arrow[r, "F"] & \mathbb{C} \end{tikzcd} \quad \text{given by} \quad \begin{tikzcd} R \arrow[d, shift right, "\mathrm{outl}_R"'] \arrow[d, shift left, "\mathrm{outr}_R"] \\ X \end{tikzcd} \mapsto \begin{tikzcd} FR \arrow[d, shift right, "F\mathrm{outl}_R"'] \arrow[d, shift left, "F\mathrm{outr}_R"] \\ FX \end{tikzcd}$$

Proof of compositionality w.r.t. weak applicative bisimilarity

Weak bisimilarity $\approx$ on $\mu\Sigma \to B(\mu\Sigma, \mu\Sigma)$ is a congruence.

1. Take the **Howe closure** $\widehat{\approx}$ of $\approx$:

$$\frac{p \approx q}{p \mathrel{\widehat{\approx}} q} \qquad \frac{p_i \mathrel{\widehat{\approx}} q_i \ (i = 1, \ldots, n)}{f(p_1, \ldots, p_n) \mathrel{\widehat{\approx}} f(q_1, \ldots, q_n)} \qquad \frac{p \mathrel{\widehat{\approx}} q,\ q \approx r}{p \mathrel{\widehat{\approx}} r}$$

2. Prove that $\widehat{\approx}$ is a **logical weak** bisimulation, e.g. for the λ-calculus:

$$\ldots$$

$$t_1 \mathrel{\widehat{\approx}} t_2 \wedge t_1 \to^\star \lambda x.t_1' \implies t_2 \to^\star \lambda x.t_2' \wedge \forall d \mathrel{\widehat{\approx}} e.\ t_1'[d/x] \mathrel{\widehat{\approx}} t_2'[e/x]$$

$$\ldots$$

3. This implies $\widehat{\approx} \subseteq \approx$, hence the latter is a congruence.

Categorical perspective: Graph liftings of (bi-)functors.

Howe closure, categorically

Howe closure $\widehat{R}$ of a relation $R \subseteq X \times X$ w.r.t. Σ-algebra $\xi \colon \Sigma X \to X$:

$$\frac{p \; R \; q}{p \; \widehat{R} \; q} \qquad \frac{p_i \; \widehat{R} \; q_i \; (i = 1, \ldots, n)}{f(p_1, \ldots, p_n) \; \widehat{R} \; f(q_1, \ldots, q_n)} \qquad \frac{p \; \widehat{R} \; q, \; q \; R \; r}{p \; \widehat{R} \; r}$$

Equivalently, $\widehat{R}$ is the least fixed point of the following operator on $\mathrm{Rel}(X)$:

$$S \;\; \mapsto \;\; R \cup \Sigma(S) \, ; R.$$

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Equivalently, $\widehat{R}$ is the least fixed point of the following operator on $\mathrm{Rel}(X)$:

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Howe closure of graph $(X, R) \in \mathrm{Gra}(\mathbb{C})$ w.r.t. Σ-algebra $\xi \colon \Sigma X \to X$

Initial algebra of the functor $\overline{\Sigma}_{R,\xi} \colon \mathrm{Gra}_X(\mathbb{C}) \to \mathrm{Gra}_X(\mathbb{C})$ given by

$$(X, S) \quad \mapsto \quad (X, R) + (\xi_\star \overline{\Sigma}(X, S)) \,;\, (X, R).$$

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2. Prove that $\widehat{\approx}$ is a **logical weak** bisimulation, e.g. for the λ-calculus:

$$\dots$$

$$t_1 \mathrel{\widehat{\approx}} t_2 \wedge t_1 \to^\star \lambda x.t_1' \implies t_2 \to^\star \lambda x.t_2' \wedge \forall d \mathrel{\widehat{\approx}} e.\ t_1'[d/x] \mathrel{\widehat{\approx}} t_2'[e/x]$$

$$\dots$$

3. This implies $\widehat{\approx} \subseteq \approx$, hence the latter is a congruence.

Categorical perspective: Graph liftings of (bi-)functors.

Bisimulations

Bisimulation on coalgebra $c\colon X \to FX$ = graph (X, R) such that

$$\begin{tikzcd} R \arrow[r, dashed, "\exists c_R"] \arrow[d, shift right, "\mathrm{outl}_R"'] \arrow[d, shift left, "\mathrm{outr}_R"] & FR \arrow[d, shift right, "F\mathrm{outl}_R"'] \arrow[d, shift left, "F\mathrm{outr}_R"] \\ X \arrow[r, "c"] & FX \end{tikzcd}$$

Equivalently, in terms of the canonical lifting $\overline{F}\colon \mathrm{Gra}(\mathbb{C}) \to \mathrm{Gra}(\mathbb{C})$:

$$c_\star(X, R) \leq \overline{F}(X, R).$$

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Equivalently, in terms of the canonical lifting $\overline{F}\colon \mathrm{Gra}(\mathbb{C}) \to \mathrm{Gra}(\mathbb{C})$:

$$c_\star(X, R) \leq \overline{F}(X, R).$$

Key step towards logical bisimulations

To abstractly express properties like

$$t_1 \mathrel{\widehat{\approx}} t_2 \wedge t_1 = \lambda x.t_1' \implies t_2 = \lambda x.t_2' \wedge \forall d \mathrel{\widehat{\approx}} e.\ t_1'[d/x] \mathrel{\widehat{\approx}} t_2'[e/x],$$

need to lift $B\colon\ \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{C}$ to $\overline{B}\colon \mathrm{Gra}(\mathbb{C})^{\mathrm{op}} \times \mathrm{Gra}(\mathbb{C}) \to \mathrm{Gra}(\mathbb{C})$.

$B(X, Y) = Y^X$ $\qquad$ $\overline{B}((X, R), (Y, S)) = (Y, S)^{(X,R)}$

Graph Liftings of Bifunctors

Canonical lifting $\overline{B}\colon \mathsf{Gra}(\mathbb{C})^{\mathsf{op}} \times \mathsf{Gra}(\mathbb{C}) \to \mathsf{Gra}(\mathbb{C})$ of $B\colon\ \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$:

$$
\begin{tikzcd}
R \arrow[d, bend right, "\mathsf{outl}_R"'] \arrow[d, bend left, "\mathsf{outr}_R"] \\
X
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
T_{R,S} \arrow[d, bend right, "\mathsf{outl}_{R,S}"'] \arrow[d, bend left, "\mathsf{outr}_{R,S}"] \\
B(X,Y)
\end{tikzcd}
$$

with $T_{R,S}$ defined via the following triple-pullback diagram:

$$
\begin{tikzcd}
 & T_{R,S} \arrow[ddl, bend right, "p_{R,S}"'] \arrow[ddr, bend left, "q_{R,S}"] & \\
 & B(R,S) \arrow[d, shift right, "B(R{,}\mathsf{outl}_S)"'] \arrow[d, shift left, "B(R{,}\mathsf{outr}_S)"] & \\
\overleftarrow{T}_{R,S} \arrow[ur, bend left, "\overleftarrow{q}_{R,S}"] \arrow[dr, bend right, "\overleftarrow{p}_{R,S}"'] & B(R,Y) & \vec{T}_{R,S} \arrow[ul, bend right, "\vec{p}_{R,S}"'] \arrow[dl, bend left, "\vec{q}_{R,S}"] \\
 & B(X,Y) \arrow[u, shift right, "B(\mathsf{outl}_R{,}Y)"] \arrow[u, shift left, "B(\mathsf{outr}_R{,}Y)"'] &
\end{tikzcd}
$$

Logical Bisimulations for Bifunctors

$$\begin{array}{ccccc} \mathrm{Gra}(\mathbb{C})^{\mathrm{op}} \times \mathrm{Gra}(\mathbb{C}) \to \mathrm{Gra}(\mathbb{C}) & \xrightarrow{\overline{B}} & \mathrm{Gra}(\mathbb{C}) \\ \downarrow & & \downarrow \\ \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{C} & \xrightarrow{\qquad B \qquad} & \mathbb{C} \end{array}$$

Logical bisimulation on $c\colon X \to B(X, X)$ = graph (X, R) such that

$$c_\star(X, R) \le \overline{B}((X, R), (X, R)).$$

This captures properties like

$$t_1 \mathrel{\widehat{\approx}} t_2 \wedge t_1 = \lambda x.t_1' \implies t_2 = \lambda x.t_2' \wedge \forall d \mathrel{\widehat{\approx}} e.\ t_1'[d/x] \mathrel{\widehat{\approx}} t_2'[e/x],$$

Summary and Ongoing Work

- Howe's method and logical bisimulations, categorically.
- **Key technique:** Graph liftings of (bi-)functors.
- First application: a generalized version of our [POPL'23] result.

Theorem (Compositionality of Higher-Order Abstract GSOS)

Suppose that the following conditions hold:

$$\dots$$

Then for every higher-order abstract GSOS law, bisimilarity on the canonical model $\mu\Sigma \to B(\mu\Sigma, \mu\Sigma)$ is a congruence.

Proof: Howe's method and primitive recursion (not coinduction).

- **Next step:** Extension to **weak** bisimilarity.

Logical Bisimulations for Bifunctors

$$
\begin{tikzcd}
\Sigma(\mu\Sigma \times B(\mu\Sigma,\mu\Sigma)) \arrow[r, equal] \arrow[dr, bend left, "\Sigma(\mu\Sigma\times B(h,\mu\Sigma))"] \arrow[ddr, bend left, "\Sigma(\mathrm{it}\,a\times B(h,\mu\Sigma))"] \arrow[dddd, "\Sigma(\mathrm{it}\,a\times B(h,\mu\Sigma))"] & \Sigma(\mu\Sigma \times B(\mu\Sigma,\mu\Sigma)) \arrow[rr, "\varrho_{\mu\Sigma,\mu\Sigma}"] \arrow[dr, phantom, "\text{dinat. } \varrho_{-,\mu\Sigma}"] & & B(\mu\Sigma, \Sigma^*(\mu\Sigma+\mu\Sigma)) \arrow[rr, "B(\mu\Sigma,\Sigma^*\nabla)"] \arrow[dl, "B(\mu\Sigma,\Sigma^*(\mathrm{it}\,a+\mu\Sigma))"] \arrow[dd, "B(h,\Sigma^*(\mathrm{it}\,a+\mu\Sigma))"] \arrow[ddr, "B(h,\Sigma^*(\mu\Sigma+\mu\Sigma))"] & & B(\mu\Sigma,\Sigma^*(\mu\Sigma)) \arrow[r, "B(\mu\Sigma,\hat{\imath})"] \arrow[dd, "B(h,\Sigma^*(\mu\Sigma))"] & B(\mu\Sigma,\mu\Sigma) \arrow[dddd, "B(h,\mu\Sigma)"] \\
 & \Sigma(\mu\Sigma \times B(A,\mu\Sigma)) \arrow[u, "\Sigma(\mu\Sigma\times B(\mathrm{it}\,a,\mu\Sigma))"] \arrow[d, "\Sigma(\mathrm{it}\,a\times B(A,\mu\Sigma))"] & B(\mu\Sigma,\Sigma^*(A+\mu\Sigma)) \arrow[dr, "B(h,\Sigma^*(A+\mu\Sigma))"] & & & & \\
 & \Sigma(A \times B(A,\mu\Sigma)) \arrow[r, "\varrho_{A,\mu\Sigma}"] & B(A,\Sigma^*(A+\mu\Sigma)) \arrow[u, "B(\mathrm{it}\,a,\Sigma^*(A+\mu\Sigma))"] \arrow[r, "B(r,\Sigma^*(A+\mu\Sigma))"] \arrow[d, phantom, "\text{dinat. } \varrho_{-,\mu\Sigma}"] & B(A,\Sigma^*(A+\mu\Sigma)) \arrow[r, "B(A,\Sigma^*(h+\mu\Sigma))"] & B(A,\Sigma^*(\mu\Sigma+\mu\Sigma)) \arrow[r, "B(A,\Sigma^*\nabla)"] & B(A,\Sigma^*(\mu\Sigma)) \arrow[ddr, "B(A,\hat{\imath})"] \arrow[dd, phantom, "\text{morph. } h"] & \\
 & \Sigma(A \times B(A,\mu\Sigma)) \arrow[u, "\Sigma(r\times B(A,\mu\Sigma))"] \arrow[dl, "\Sigma(A\times B(r,\mu\Sigma))"] & \arrow[d, phantom, "\text{nat. } \varrho_{A,-}"] & B(A,\Sigma^*(A+\mu\Sigma)) \arrow[u, "B(A,\Sigma^*(r+\mu\Sigma))"] \arrow[d, "B(A,\Sigma^*(A+\mathrm{it}\,a))"] & & & \\
\Sigma(A \times B(A,\mu\Sigma)) \arrow[rrru, "\varrho_{A,\mu\Sigma}"] \arrow[rr, "\Sigma(A\times B(A,\mathrm{it}\,a))"] & & \Sigma(A \times B(A,A)) \arrow[r, "\varrho_{A,A}"] & B(A,\Sigma^*(A+A)) \arrow[r, "B(A,\Sigma^*\nabla)"] \arrow[uur, bend right, "B(A,\Sigma^*(h+h))"] & B(A,\Sigma^* A) \arrow[r, "B(A,\hat{a})"] \arrow[uur, bend right, "B(A,\Sigma^* h)"] & B(A,A) \arrow[r, "B(A,h)"] & B(A,\mu\Sigma)
\end{tikzcd}
$$