Friedrich-Alexander-Universität
Technische Fakultät

Coalgebraic Infinitary Trace Semantics of Nominal Büchi-Automata

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Agenda

Preliminaries I: Fundamentals of Nom and Monads

Definition (*Group*)

A *group* $\mathbf{G} = (G, \cdot, e)$ consists of a set G, a binary operation $\cdot$ on G, and an element e, such that:

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Example: Permutation Group

A *permutation* $\pi\colon X \to X$ on a set X is a bijective map. It gives rise to the permutation group Sym X of X by

$$\mathrm{Sym}\, X := (\{\, \pi\colon X \to X \mid \pi \text{ is bijective} \,\},\ \circ\ ,\ \mathrm{id}_X)\,.$$

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A permutation is called *finite* if the set $\{\, x \in X \mid \pi x \neq x \,\}$ is finite. With this we get the subgroup $\mathrm{Perm}\, X \leqslant \mathrm{Sym}\, X$ of finite permutations of X.

Definition (*Group Actions*)

If X is a set and $\mathbf{G} = (G, \cdot, e)$ is a group, then an *action* of G on X is a function

$$\triangleright : G \times X \to X,\ (g, x) \mapsto g \triangleright x,$$

such that for all $g, h \in G$ and $x \in X$:

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We call the set X together with its action a *G-set*.

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Definition (*Equivariant Functions*)

Let $(X, \triangleright_X)$ and $(Y, \triangleright_Y)$ be G-Sets, then a function $f : X \to Y$ is called *equivariant*, if

$$f(g \triangleright_X x) = g \triangleright_Y fx$$

holds for all $g \in G$ and $x \in X$.

Definition (*Orbits*)

Let $(X, \triangleright)$ be a G-Set for a group G and $x \in X$. Then the *orbit of* x *with respect to* $\triangleright$ is $G \triangleright x := \{ g \triangleright x \mid g \in G \} \subseteq X$. These orbits are the equivalence classes for the equivalence relation

$$x \sim_G y :\Longleftrightarrow \exists g \in G.\, y = g \triangleright x,$$

and we call a G-set *orbit-finite*, if $\# \left({}^{X}/_{\sim_G} \right) < \infty$.

Groups
Orbits and Support

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Definition (*Support*)

Let $(X, \triangleright)$ be a Perm $\mathbb{A}$ set, then $A \subseteq \mathbb{A}$ is a *support* for $x \in X$ if for all $\pi \in \operatorname{Perm} \mathbb{A}$

$$(\forall a \in A.\ \pi a = a) \Rightarrow \pi \triangleright x = x.$$

We then define *the support* $\mathrm{supp}_X\, x$ of a finitely supported x as the least of all finite supports.

Definition (*Uniform Finite Support*)

Let $(X, \triangleright)$ be a finitely supported Perm $\mathbb{A}$ set. A subset $S \subseteq X$ is called *uniformly finitely supported (ufs)* if there exists a finite set $A \subseteq \mathbb{A}$ that supports each $x \in S$.

Remark (*U.F.S. ⇒ F.S.*)

Every ufs subset $S \subseteq X$ is finitely supported by the same subset $A \subseteq \mathbb{A}$. One can also show that in those cases we have

$$A = \bigcup_{x \in S} \operatorname{supp} x.$$

Nominal Sets

Definition (*Category of Nominal Sets*)

A *nominal set* X is a Perm $\mathbb{A}$ set whose elements are all finitely supported. Together with equivariant functions, identiies and compositions as in $\mathbf{Set}$, they form a category $\mathbf{Nom}$.

Remark (*$\mathbf{Nom}$ is a Cartesian Closed Category*)

Since $\mathbf{Nom}$ has finite products and exponentials for every pair X, Y of object of $\mathbf{Nom}$, the category is cartesian closed.

Additionally, $\mathbf{Nom}$ admits arbitrary coproducts.

Nominal Sets

Power Sets

Example: Power Sets

With finitely and uniformly finitely supported subsets we get the following two functors:

$$\begin{array}{rcrcl}
\mathcal{P}_{\mathsf{fs}} & : & \mathbf{Nom} & \to & \mathbf{Nom} \\
 & & X & \mapsto & \{\, S \subseteq X \mid S \text{ is finitely supported.} \,\} \\
 & & f\colon X \to Y & \mapsto & \mathcal{P}_{\mathsf{fs}} f\colon \mathcal{P}_{\mathsf{fs}} X \to \mathcal{P}_{\mathsf{fs}} Y,\ S \mapsto f[S]
\end{array}$$

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Nominal Sets

Freshness and α-Equivalence

Definition (*Freshness*)

A name $a \in \mathbb{A}$ is *fresh* for an element x of a nominal set X if $a \notin \operatorname{supp} x$. We denote this by $a \# x$.

The freshness relation $\#$ is equivariant.

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Lemma (α*-Equivalence*)

Define a binary relation $\approx_\alpha$ on $\mathbb{A} \times X$ by

$$(a, x) \approx_\alpha (b, y) :\Longleftrightarrow (a\ c) \triangleright x = (b\ c) \triangleright y \text{ for some, equivalently all, fresh } c.$$

Then $\approx_\alpha$ is an equivariant equivalence relation, the equivalence class for $(a, x) \in \mathbb{A} \times X$ is denoted $\langle a \rangle\, x$, and called a *name abstraction.*

Nominal Sets

Name Abstractions

Definition (*Abstraction Set*)

We call the quotient set of $\mathbb{A} \times X$ with $\approx_\alpha$ the *nominal set of name abstractions* $[\mathbb{A}]\, X$ together with its action

$$\triangleright : \ \mathsf{Perm}\,\mathbb{A} \times [\mathbb{A}]\, X \to [\mathbb{A}]\, X,\ (\pi,\ \langle a\rangle\, x) \mapsto \langle \pi a\rangle\, (\pi \triangleright_X x)\,.$$

Furthermore, we have $\operatorname{supp} \langle a\rangle\, x = \operatorname{supp} x \setminus \{\, a \,\}$ for all $a \in \mathbb{A}$ and $x \in X$.

Proposition (*Functoriality of* $[\mathbb{A}]\, -$)

The object map $X \mapsto [\mathbb{A}]\, X$ extends to the *abstraction functor* as follows:

$$[\mathbb{A}]\, - : \mathbf{Nom} \to \mathbf{Nom},\ \begin{cases} X & \mapsto \quad [\mathbb{A}]\, X \\ f & \mapsto \quad [\mathbb{A}]\, f : \ \langle a\rangle\, x \mapsto \langle a\rangle\, fx \end{cases}$$

Theorem (*Adjointness of the Abstraction Functor*)

The abstraction functor $[\mathbb{A}]\_$ is both a left and a right adjoint:

$$\_ * \mathbb{A} \dashv [\mathbb{A}]\_ \dashv R\_$$

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Proposition (*Preservation of Exponentials*)

The abstraction functor $[\mathbb{A}]\_$ preserves exponentials:

$$[\mathbb{A}]\,(X \to_{\mathsf{fs}} Y) \cong [\mathbb{A}]\,X \to_{\mathsf{fs}} [\mathbb{A}]\,Y$$

Corollary ($\mathcal{P}_{ufs}$ *distributes over* $[\mathbb{A}]-$)

The functor $\mathcal{P}_{\mathsf{ufs}}$ distributes over the abstraction functor by

$$\varphi_X : \begin{cases} [\mathbb{A}]\,\mathcal{P}_{\mathsf{ufs}}(X) & \to & \mathcal{P}_{\mathsf{ufs}}([\mathbb{A}]\,X) \\ \langle a \rangle\, S & \mapsto & \{\, \langle a \rangle\, x \mid x \in S \,\} \end{cases},$$

$$\psi_X : \begin{cases} \mathcal{P}_{\mathsf{ufs}}([\mathbb{A}]\,X) & \to & [\mathbb{A}]\,\mathcal{P}_{\mathsf{ufs}}(X) \\ S & \mapsto & \langle a \rangle \{\, x \mid \langle a \rangle\, x \in S \,\} \text{ with } a \# S \end{cases}.$$

These morphisms are mutually inverse and natural in X.

Monads

Definition

Definition (*Monads*)

Let $\mathcal{C}$ be a category. A *monad on* $\mathcal{C}$ is a triple $\langle T, \eta, \mu \rangle$, where $T\colon \mathcal{C} \to \mathcal{C}$ is an endofunctor, $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow T$ and $\mu\colon T^2 \Rightarrow T$ are natural transformations, and the following diagrams commute for every object X in $\mathcal{C}$:

$$\begin{array}{ccccc} TX & \xrightarrow{\eta_{TX}} & T^2X & \xleftarrow{T\eta_X} & TX \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \mu_X} & \diagup\!\!\diagup & \\ & & TX & & \end{array} \qquad \begin{array}{ccc} T^3X & \xrightarrow{\mu_{TX}} & T^2X \\ {\scriptstyle T\mu_X}\downarrow & & \downarrow{\scriptstyle \mu_X} \\ T^2X & \xrightarrow[\mu_X]{} & TX \end{array}$$

Definition (*Kleisli Category*)

Let $\langle T, \eta, \mu \rangle$ be a monad on a category $\mathcal{C}$. The *Kleisli Category* $\mathcal{K}\ell_T$ of T has the same objects as $\mathcal{C}$, but arrows $X \nrightarrow Y$ in $\mathcal{K}\ell_T$ are arrows $X \to TY$ in $\mathcal{C}$. The identity in $\mathcal{K}\ell_T$ is given by the unit $\eta_X \colon X \to TX$, and the composition of two arrows $f \colon X \nrightarrow Y$ and $g \colon Y \nrightarrow Z$ in $\mathcal{K}\ell_T$ is written as $g \circledcirc f$ and defined by

$$X \xrightarrow[f]{} TY \xrightarrow[Tg]{} T^2 Z \xrightarrow[\mu_Z]{} TZ. \qquad (g \circledcirc f \colon X \to TZ)$$

Remark (*Canonical Adjunction*)

We have a canonical adjunction

$$\mathcal{C} \underset{U}{\overset{J}{\underset{\perp}{\rightleftarrows}}} \mathcal{K}\ell_T,$$

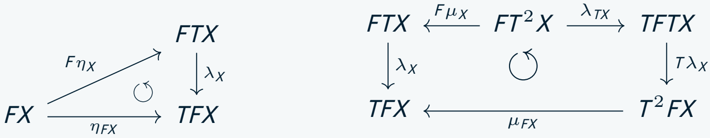
where J is defined by $JX = X$ on objects and $Jf = \eta_{\mathrm{cod}\, f} \circ f$ on arrows, where $\mathrm{cod}\, f$ is the codomain of the arrow f. The functor U is defined by $UX = TX$ on objects and $Uf = \mu_{\mathrm{cod}\, f} \circ Tf$ on arrows.

Monads

Distributive Law

Definition (*Distributive Laws*)

Let $\langle T, \eta, \mu \rangle$ be a monad and $F\colon \mathcal{C} \to \mathcal{C}$ an endofunctor on a category $\mathcal{C}$. A *distributive law* of F over T is a natural transformation $\lambda\colon FT \Rightarrow TF$, such that the following diagramms commute:



Monads

Extensions and Distributive Laws

Proposition (*Correspondence between Extensions and $\mathcal{K}\ell$-Laws*)

Let $\langle T, \eta, \mu \rangle$ be a monad and $F\colon \mathcal{C} \to \mathcal{C}$ an endofunctor on a category $\mathcal{C}$. Then there is a bijective correspondence between distributive laws $\lambda\colon FT \Rightarrow TF$ and extensions of $F\colon \mathcal{C} \to \mathcal{C}$ to a functor $\overline{F}\colon \mathcal{K}\ell_T \to \mathcal{K}\ell_T$, i.e. a functor that makes the diagram on the right commute. Herein, the arrow J is the canonical left adjoint from earlier.

$$\begin{array}{ccc} \mathcal{K}\ell_T & \xrightarrow{\overline{F}} & \mathcal{K}\ell_T \\ {\scriptstyle J}\uparrow & \circlearrowleft & \uparrow{\scriptstyle J} \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{C} \end{array}$$

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Given a distributive law $\lambda\colon FT \Rightarrow TF$ one defines the functor $\overline{F}$ by

$$\overline{F}\colon \mathcal{K}\ell_T \to \mathcal{K}\ell_T, \begin{cases} X & \mapsto \quad FX \\ X \xrightarrow{f} TY & \mapsto \quad FX \xrightarrow{Ff} FTY \xrightarrow{\lambda_Y} TFY \end{cases} .$$

Monads

Extensions and Distributive Laws

Proposition (***Correspondence between Extensions and $\mathcal{K}\ell$-Laws*** **)**

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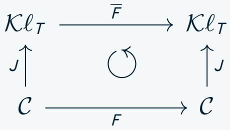

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In the other direction, given $\overline{F}$ one obtains a distributive law by $\lambda_X = \overline{F}(\mathrm{id}_{TX})\colon FTX \to TFX$.

Kleisli-Lattices

$\mathcal{K}\ell_{\mathcal{P}_{\mathsf{ufs}}}$ admits Lattices

Proposition (*$\mathcal{K}\ell$-Arrows are a Complete Lattice*)

For any pair X, Y of objects in $\mathbf{Nom}$ the hom-set of the Kleisli category $\mathcal{K}\ell_{\mathcal{P}_{\mathsf{ufs}}}(X, Y)$ of the monad $\mathcal{P}_{\mathsf{ufs}}$ is a complete lattice, where joins and meets are built by taking the union or intersection, respectively:

$$\bigvee_{i \in I} f_i \colon X \nrightarrow Y, \quad x \mapsto \bigcup_{i \in I} f_i(x) \qquad\qquad \bigwedge_{i \in I} f_i \colon X \nrightarrow Y, \quad x \mapsto \bigcap_{i \in I} f_i(x)$$

Remark (*Top and Bottom Element*)

The bottom element $\bot_{X,Y}$ of the lattice $\mathcal{K}\ell_{\mathcal{P}_{\mathsf{ufs}}}(X, Y)$ and the top element $\top_{X,Y}$ are defined by the equivariant functions

$$\bot_{X,Y} \colon X \nrightarrow Y,\ x \mapsto \emptyset \qquad \text{and} \qquad \top_{X,Y} \colon X \nrightarrow Y,\ x \mapsto \{\, y \mid \operatorname{supp} y \subseteq \operatorname{supp} x \,\}.$$

Preliminaries II: Büchi RNNA and Equational Systems

Bar Strings

Definition

We fix a countable infinite set $\mathbb{A}$ of names, and define an extended alphabet $\overline{\mathbb{A}}$ by

$$\overline{\mathbb{A}} := \mathbb{A} \cup \{ \mid a \mid a \in \mathbb{A} \} .$$

Definition (*Bar Strings*)

- A *finite bar string* is a *finite* word over $\overline{\mathbb{A}}$, while an *infinite bar string* is an *infinite* word over $\overline{\mathbb{A}}$. We denote the sets of finite and infinite bar strings by $\overline{\mathbb{A}}^*$ and $\overline{\mathbb{A}}^\omega$, respectively.
- Given a word $w \in \overline{\mathbb{A}}^* \cup \overline{\mathbb{A}}^\omega$ the *set of names* in w is defined by

  $$\mathrm{N}(w) := \{ a \in \mathbb{A} \mid \text{the letter } a \text{ or } \mid a \text{ occurs in } w \} .$$

  An infinite bar string w is *finitely supported* if $\mathrm{N}(w)$ is finite; the set $\overline{\mathbb{A}}^\omega_{\mathsf{fs}} \subseteq \overline{\mathbb{A}}^\omega$ denotes the finitely supported infinite bar strings.
- A name $a \in \mathbb{A}$ occuring in a bar string $w \in \overline{\mathbb{A}}^* \cup \overline{\mathbb{A}}^\omega$ is *free* if it occurs to the left of any occurance of $\mid a$, and *bound* otherwise. We denote the *set of free names* in w by $\mathrm{FN}(w)$.

Bar Strings

α-Equivalence

Definition (*α-Equivalence on Bar Strings*)

- We define α*-equivalence* $\equiv_\alpha$ on finite bar strings as the equivalence generated by
$$w|av \equiv_\alpha w|bu \quad \text{iff} \quad \langle a\rangle\, v = \langle b\rangle\, u \text{ in } [\mathbb{A}]\,\overline{\mathbb{A}}^*.$$
- This then can be extended to an equivalence relation $\equiv_\alpha$ on infinite bar strings by
$$v \equiv_\alpha w \quad \text{iff} \quad v[0:n) \equiv_\alpha w[0:n) \quad \text{for all } n \in \omega.$$
- We write $[w]_\alpha$ for the α-equivalence class of $w \in \overline{\mathbb{A}}^* \cup \overline{\mathbb{A}}^\omega$, and denote by $\overline{\mathbb{A}}^*/{\equiv_\alpha}$ and $\overline{\mathbb{A}}^\omega/{\equiv_\alpha}$ the sets of α-equivalence classes of finite and infinite bar strings, respectively.

Remark (*Right Cancellation Property*)

For all $v, w \in \overline{\mathbb{A}}^*$ and $x \in \overline{\mathbb{A}}^* \cup \overline{\mathbb{A}}^\omega$, we have that $vx \equiv_\alpha wx$ implies $v \equiv_\alpha w$.

Bar Strings
Clean Bar Strings

Definition (*Clean Bar Strings*)

A finite or infinite bar string w is *clean* if for each $a \in \mathsf{FN}(w)$ the letter $|a$ does not occur in w, and for each $a \notin \mathsf{FN}(w)$ the letter $|a$ occurs at most once.

Lemma (*Canonical Form for Bar Strings*)

Given a bar string $w \in \bar{\mathbb{A}}^* \cup \bar{\mathbb{A}}^\omega_{\mathsf{fs}}$ and a set S with $\mathsf{FN}(w) \subseteq S$, there is an α-equivalent clean bar string $\mathrm{nf}(w)$ which is unique with respect to the ordering of the names $\mathbb{A} \setminus S$. Additionally, the mapping nf is equivariant.

Notation (*Prefixes*)

Given two strings $v \in \bar{\mathbb{A}}^n$ and $w \in \bar{\mathbb{A}}^{n+1}$, we write $v \sqsubseteq w$ if $v = w[0:n)$ and write $v \sqsubseteq_\alpha w$ if $v \equiv_\alpha w[0:n)$.

Lemma (α*-Equivalent Prefixes*)

Given two bar strings $v \in \bar{\mathbb{A}}^n$ and $w \in \bar{\mathbb{A}}^{n+1}$ and a finite set $S \subseteq \mathbb{A}$, such that $\mathrm{FN}(v), \mathrm{FN}(w) \subseteq S$, we have that $v \sqsubseteq_\alpha w$ if and only if $\mathrm{nf}(v) \sqsubseteq \mathrm{nf}(w)$.

Proof: We show both implications singularily:

'$\Rightarrow$' The assumption implies, that $u \equiv_\alpha w[0:n)$ and therefore $\mathrm{nf}(u) \equiv_\alpha u \equiv_\alpha w[0:n) \equiv_\alpha \mathrm{nf}(w)[0:n)$, where the last α-equivalence holds because of the *right cancellation property*. However, because of the uniqueness with respect to the ordering of $\mathbb{A} \setminus S$, we have that $\mathrm{nf}(u) = \mathrm{nf}(w)[0:n)$.

Bar Strings

Prefixes

'$\Leftarrow$' We now have $\mathrm{nf}(u) = \mathrm{nf}(w)[0:n)$, i.e.
$u \equiv_\alpha \mathrm{nf}(u) = \mathrm{nf}(w)[0:n) \equiv_\alpha w[0:n)$, where the last α-equivalence holds because of the *right cancellation property*. $\square$

Definition (*Regular Nondeterministic Nominal Automata*)

A *regular nondeterministic nominal automaton (RNNA)* is a tuple $A = (Q, \delta, s, \mathrm{Acc})$ consisting of

- an orbit-finite nominal set Q of states, with an *initial state* $s \in Q$;
- an equivariant subset $\delta \subseteq Q \times \bar{\mathbb{A}} \times Q$, the *transition relation*, where we write $q \xrightarrow{\alpha} q'$ for $(q, \alpha, q') \in \delta$; transitions of type $q \xrightarrow{a} q'$ are called *free*, and those of type $q \xrightarrow{|a} q'$ *bound*;
- an equivariant subset $\mathrm{Acc} \subseteq Q$ of *final* states

such that the following conditions are satisfied:

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such that the following conditions are satisfied:

- The relation δ is α*-invariant*, i.e. closed under α-equivalence of transitions, where transitions $q \xrightarrow{|a} q'$ and $p \xrightarrow{|b} p'$ are α*-equivalent* if $q = p$ and $\langle a \rangle\, q' = \langle b \rangle\, p'$.
- The relation δ is *finitely branching up to* α*-equivalence*, i.e. for each state q the sets
$$\left\{ (a, q') \;\middle|\; q \xrightarrow{a} q' \right\} \text{ and } \left\{ \langle a \rangle\, q' \;\middle|\; q \xrightarrow{|a} q' \right\}$$
are finite or equivalently ufs.

Remark (*RNNAs as Coalgebras*)

Coalgebraically, an RNNA is an orbit-finite coalgebra $\gamma\colon Q \to FQ$ for the functor

$$F = \mathcal{P}_{\mathsf{ufs}}(\mathbb{A} \times - + [\mathbb{A}]\, -),$$

together with an equivariant subset $\mathsf{Acc} \subseteq Q$ of final states and a map $s\colon \mathbb{1} \nrightarrow Q$ in $\mathcal{K}\ell_{\mathcal{P}_{\mathsf{ufs}}}$ for initial states.

Given an RNNA $A = (Q, \delta, s, \mathsf{Acc})$, its equivalent coalgebra is given by

$$\gamma_G\colon \begin{cases} Q & \to \quad \mathcal{P}_{\mathsf{ufs}}\,(\mathbb{A} \times Q + [\mathbb{A}]\, Q) \\ q & \mapsto \quad S_q \end{cases},$$

where $(a, q') \in S_q$ iff $q \xrightarrow{a} q'$, and $\langle a \rangle\, q' \in S_q$ iff $q \xrightarrow{|a} q'$. The map of initial states is given by $s\colon \mathbb{1} \nrightarrow Q,\ * \mapsto \{\, s \,\}$.

Büchi RNNA
Definition

Definition (*Büchi RNNA*)

A *Büchi RNNA* is an RNNA $A = (Q, \delta, q_0, \mathrm{Acc})$, where it *accepts* a run $r \in Q^\omega$, if $\#\{\, i \in \omega \mid r_i \in \mathrm{Acc} \,\} = \omega$. The state $q \in Q$ *accepts* an infinite bar string $w \in \bar{\mathbb{A}}^\omega$, if there is a run for w starting with q. The automaton A *accepts* $w \in \bar{\mathbb{A}}^\omega$, if its initial state q_0 accepts w. We then define by

$$L_{\alpha,\omega}(A) \;:=\; \{\, [w]_\alpha \mid w \in \bar{\mathbb{A}}^\omega,\ A \text{ accepts } w \,\}$$

the *bar ω-language accepted by A.*

Equational Systems

Definition

Definition (*Equational Systems with Two Variables*)

For $i \in \{1, 2\}$ let $f_i : L_1 \times L_2 \to L_i$ be monotone functions where all L_i's are posets. An *equational system* is then a sequence

$$\begin{bmatrix} u_1 =_{\eta_1} f_1(u_1, u_2) \\ u_2 =_{\eta_2} f_2(u_1, u_2) \end{bmatrix},$$

where the u_i's are *variables* and η_i is either ν or μ for all $i \in \{1, 2\}$.

Given all necessary fixed points exist, we can define the *solution* of such a system by the element

$$(h_1, h_2) \in L_1 \times L_2,$$

obtained in the following way:

1) Compute the first 'interim' solution $g_1(u_2) = \eta_1 x_1.\ f_1(u_1, u_2)$.
2) Substitute this solution in the remaining equation, i.e. $u_2 =_{\eta_2} f_2(g_1(u_2), u_2)$, and solve this system to compute h_2, which is used for $h_1 = g_1(h_2)$.

Equational Systems

Solvability

Lemma (*Solvability Criterion*)

Such an equational system for two variables has a solution if each L_i is a complete lattice.

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Coalgebraic Infinitary Trace Semantics of Büchi RNNA

Coalgebraic Modelling
Original Assumptions

Assumption (*Coalgebraic Assumptions after Urabe et al.*)

In what follows a monad T and an endofunctor F, both on a category $\mathcal{C}$, satisfy:

(1) $\mathcal{C}$ has a final object $\mathbb{1}$ and finite coproducts.

(2) F has a final coalgebra $\zeta\colon Z \to FZ$ in $\mathcal{C}$.

(3) There is a distributive law $\lambda\colon FT \Rightarrow TF$, hence $F : \mathcal{C} \to \mathcal{C}$ is lifted to $\overline{F}\colon \mathcal{K}\ell_T \to \mathcal{K}\ell_T$.

(4) For every pair X, Y of objects in $\mathcal{K}\ell_T$, the hom-set $\mathcal{K}\ell_T(X, Y)$ carries an order $\preccurlyeq_{X,Y}$ and is a complete lattice.

(5) Kleisli composition $\odot$ and cotupling $[-,-]$ are monotone with respect to the order $\preccurlyeq$.

(6) The lifting $\overline{F}$ is *locally monotone*, i.e. for $f, g \in \mathcal{K}\ell_T(X, Y)$, $f \preccurlyeq_{X,Y} g$ implies $\overline{F}f \preccurlyeq_{\overline{F}X,\overline{F}Y} \overline{F}g$.

Coalgebraic Modelling

Büchi RNNAs satisfy the Assumptions

Büchi RNNAs

The category $\mathbf{Nom}$, the ufs powerset monad $\mathcal{P}_{\mathsf{ufs}}$ and the functor

$$F = \mathbb{A} \times - + [\mathbb{A}] -$$

satisfy the assumptions.

Proof: Since most of the points have already been proven, we will only look at the final coalgebra of F: We prove that the map

$$\zeta \colon (\overline{\mathbb{A}}^\omega/{\equiv_\alpha})_{\mathsf{fs}} \to G((\overline{\mathbb{A}}^\omega/{\equiv_\alpha})_{\mathsf{fs}}),\ [w]_\alpha \mapsto \begin{cases} (a, [w']_\alpha) & \text{if } [w]_\alpha = [aw']_\alpha \\ \langle a \rangle\, [w']_\alpha & \text{if } [w]_\alpha = [|aw']_\alpha \end{cases} \tag{1}$$

is the final coalgebra for the functor G by use of Adámek's Lemma for final coalgebras. (The ω^{op}-limit of the chain $G^n\mathbb{1}$ carries the structure of a final coalgebra, if G preserves that limit)

We then prove that $G^n\mathbb{1} \cong \overline{\mathbb{A}}^n/{\equiv_\alpha}$ holds by induction over $n \in \omega$:

Base Case ($n = 0$): Obviously this holds, since
$G^0\mathbb{1} = \mathbb{1} \cong \{ [\varepsilon]_\alpha \} = \overline{\mathbb{A}}^0/{\equiv_\alpha}$.

Step Case ($n \to n+1$): Suppose now that $G^n\mathbb{1} \cong \overline{\mathbb{A}}^n/{\equiv_\alpha}$ holds for n, then we have

$$\begin{aligned} G^{n+1}\mathbb{1} &= G\left(G^n\mathbb{1}\right) \overset{\text{I.H.}}{\cong} G\left(\overline{\mathbb{A}}^n/{\equiv_\alpha}\right) \\ &= \mathbb{A} \times \overline{\mathbb{A}}^n/{\equiv_\alpha} + [\mathbb{A}]\,\overline{\mathbb{A}}^n/{\equiv_\alpha} \overset{(*)}{\cong} \overline{\mathbb{A}}^{n+1}/{\equiv_\alpha}. \end{aligned}$$

The last isomorphism $(*)$ is given by

$$(a, [w]_\alpha) \mapsto [aw]_\alpha \qquad \text{and} \qquad \langle a \rangle\, [w]_\alpha \mapsto [|aw]_\alpha$$

for $a \in \mathbb{A}$ and $[w]_\alpha \in \overline{\mathbb{A}}^n/{\equiv_\alpha}$. It should be obvious to see that this mapping is an isomorphism.

Coalgebraic Modelling

Büchi RNNAs satisfy the Assumptions

With this, we only have to prove that
$\left(\left(\overline{\mathbb{A}}^\omega/{\equiv_\alpha}\right)_{\mathsf{fs}}, \varphi_n\colon \left(\overline{\mathbb{A}}^\omega/{\equiv_\alpha}\right)_{\mathsf{fs}} \to \overline{\mathbb{A}}^n/{\equiv_\alpha},\ w \mapsto w[0:n)\right)$ is indeed the limit cone for the ω^{op} chain. Indeed, our candidate is a cone since the following diagram obviously commutes:

$$\begin{array}{ccc} & \left(\overline{\mathbb{A}}^\omega/{\equiv_\alpha}\right)_{\mathsf{fs}} & \\ \swarrow^{\varphi_n} & & \searrow^{\varphi_{n-1}} \\ \overline{\mathbb{A}}^n/{\equiv_\alpha} & \xrightarrow[w \,\mapsto\, w[0:n-1)]{} & \overline{\mathbb{A}}^{n-1}/{\equiv_\alpha} \end{array}$$

So suppose $\left(K, \psi_i\colon K \to \overline{\mathbb{A}}^i/{\equiv_\alpha}\right)$ is another cone. This means that for each $k \in K$ we have a finitely supported family $\psi_i(k)$ of finitely supported bar strings that is compatible, i.e. $\psi_i(k) \sqsubseteq_\alpha \psi_{i+1}(k)$ for all $i \in \omega$.

Since the family $\{\ \psi_i(k)\ \}_{i\in\omega}$ is finitely supported, the set $S := \bigcup_{i\in\omega} \mathsf{FN}(w_i)$ is finite. We therefore define

$$a_K : K \to \overline{\mathbb{A}}^\omega/{\equiv_\alpha},\ k \mapsto \left[w_k : \omega \to \overline{\mathbb{A}},\ i \mapsto \mathrm{nf}(\psi_{i+1}(k))(i)\right]_\alpha$$

and prove below that it is the unique arrow between K and our limit candidate. It is well-defined in the sense that we have $\mathrm{nf}(\psi_i(k)) \sqsubseteq \mathrm{nf}(\psi_{i+1}(k))$ for each $i \in \omega$. The mapping is also equivariant. Additionally, this a_K fulfills the limit equations, i.e. we have for every $k \in K$ and $n \in \omega$, that $\psi_n(k) = \varphi_n(a_K(k))$.

We show that $\varphi_n(a_K(k)) = \mathrm{nf}(\psi_n(k))$ by induction over $n \in \omega$:

Base Case ($n = 0$): Obviously this holds, since

$$\mathrm{nf}(\psi_0(k)) = [\varepsilon]_\alpha = \varphi_0(a_K(k)).$$

Step Case ($n \to n+1$): Suppose now that $\mathrm{nf}(\psi_n(k)) = \varphi_n(a_K(k))$ holds for n. Let w_k be the representant of $a_K(k)$, then we have

$$\begin{aligned}
\varphi_{n+1}(w_k) &= \varphi_n(w_k)\,(\mathrm{nf}(\psi_{n+1}(k))(n)) \\
&\overset{I.H.}{=} \mathrm{nf}(\psi_n(k))\,(\mathrm{nf}(\psi_{n+1}(k))(n)) \\
&\overset{(\Delta)}{=} \mathrm{nf}(\psi_{n+1}(k))[0:n)\,(\mathrm{nf}(\psi_{n+1}(k))(n)) \\
&= \mathrm{nf}(\psi_{n+1}(k)),
\end{aligned}$$

where the step (Δ) holds because of the prefix properties of the canonical form.

The uniqueness of the mapping a_K is easy to prove since the cone projections are jointly monic:

- Let $g, h\colon X \to (\bar{\mathbb{A}}^\omega/{\equiv_\alpha})_{\mathsf{fs}}$ be two maps with $\varphi_i \circ g = \varphi_i \circ h$ for all $i \in \omega$.
- This means that for every $x \in X$, putting $[w]_\alpha = g(x)$ and $[w']_\alpha = h(x)$, we have $w[0:i) \equiv_\alpha w'[0:i)$ for every $i \in \omega$.
- But this means $w {\equiv_\alpha} w'$ and therefore $g(x) = h(x)$.

With this in mind, we see that for every other map f between K and $(\bar{\mathbb{A}}^\omega/{\equiv_\alpha})_{\mathsf{fs}}$ with $\varphi_i \circ f = \psi_i$ we have $\varphi_i \circ f = \varphi_i \circ a_K$ and thus $f = a_K$. Hereby, uniqueness is shown, and because

$$a_G\colon G((\bar{\mathbb{A}}^\omega/{\equiv_\alpha})_{\mathsf{fs}}) \to (\bar{\mathbb{A}}^\omega/{\equiv_\alpha})_{\mathsf{fs}}, \begin{cases} (a, [w]_\alpha) & \mapsto & [aw]_\alpha \\ \langle a \rangle\, [w]_\alpha & \mapsto & [|aw]_\alpha \end{cases}$$

is the unique limit mapping between $G((\bar{\mathbb{A}}^\omega/{\equiv_\alpha})_{\mathsf{fs}})$ and $(\bar{\mathbb{A}}^\omega/{\equiv_\alpha})_{\mathsf{fs}}$, we see that its inverse must be the final coalgebra of G.

Definition (*Büchi (T, F)-System*)

A *Büchi (T, F)-System* is given by a triple $\mathcal{X} = \left((X_1, X_2), c \colon X \nrightarrow \overline{F}X, s \colon \mathbb{1} \nrightarrow X\right)$, where X is defined as the coproduct $X_1 + X_2$ in $\mathcal{C}$, the *state objects* with their *priorities*, meaning that X_1 encodes the non-final, and X_2 the final states of the Büchi automaton. Additionally, $c \colon X \nrightarrow \overline{F}X$ is an arrow in $\mathcal{K}\ell_T$, the *dynamics*, and $s \colon \mathbb{1} \nrightarrow X$ an arrow in $\mathcal{K}\ell_T$ providing *initial states*. We define for each $i = 1, 2$ the arrow $c_i \colon X_i \nrightarrow \overline{F}X$ to be the restriction $c \circ \kappa_i \colon X_i \nrightarrow \overline{F}X$ along the coproduct injections $\kappa_i \colon X_i \to X$.

Definition (*Trace Semantics of Büchi (T, F)-Systems*)

Let $\mathcal{X} = \big((X_1, X_2), c\colon X \nrightarrow \overline{F}X, s\colon \mathbb{1} \nrightarrow X\big)$ be a Büchi (T, F)-System. It induces the following equational system $E_{\mathcal{X}}$, where $\zeta\colon Z \to FZ$ is the final coalgebra of F in $\mathcal{C}$. Herein, the variable u_i ranges over the poset $\mathcal{K}\ell_T(X_i,\ Z)$:

$$E_{\mathcal{X}} := \begin{bmatrix} u_1 & =_\mu & (J\zeta)^{-1} \odot \overline{F}[u_1, u_2] \odot c_1 \\ u_2 & =_\nu & (J\zeta)^{-1} \odot \overline{F}[u_1, u_2] \odot c_2 \end{bmatrix}$$

(T, F) consitutes a *Büchi trace situation*, if $E_{\mathcal{X}}$ has a solution for any Büchi (T, F)-System $\mathcal{X}$, denoted by $\mathrm{trace}_i^{\mathrm{b}}(\mathcal{X})\colon X_i \nrightarrow Z$ for $i \in \{1, 2\}$. The composite

$$\mathrm{trace}^{\mathrm{b}}(\mathcal{X}) := \Big(\mathbb{1} \xrightarrow{s} X_1 + X_2 \xrightarrow{[\mathrm{trace}_1^{\mathrm{b}}(\mathcal{X}), \mathrm{trace}_2^{\mathrm{b}}(\mathcal{X})]} Z\Big)$$

is called the *trace semantics* of the Büchi (T, F)-System $\mathcal{X}$.

Theorem (*Coincidence with RNNAs*)

Every Büchi RNNA System $\mathcal{A} = ((Q \setminus \mathsf{Acc}, \mathsf{Acc}), c_A \colon Q \nrightarrow FQ, s \colon \mathbb{1} \nrightarrow Q)$ consitutes a Büchi trace situation, where the trace mappings are given by:

$$\mathsf{trace}_1^{\mathsf{b}}(\mathcal{A}) \colon Q \setminus \mathsf{Acc} \nrightarrow \overline{\mathbb{A}}^{\omega}/\equiv_\alpha, q \mapsto L_{\alpha,\omega}(q) \quad \text{and}$$

$$\mathsf{trace}_2^{\mathsf{b}}(\mathcal{A}) \colon \mathsf{Acc} \nrightarrow \overline{\mathbb{A}}^{\omega}/\equiv_\alpha, q \mapsto L_{\alpha,\omega}(q)$$

Additionally its trace semantics is given by

$$\mathsf{trace}^{\mathsf{b}}(\mathcal{A}) \colon \mathbb{1} \nrightarrow \overline{\mathbb{A}}^{\omega}/\equiv_\alpha, * \mapsto L_{\alpha,\omega}(A).$$

Proof: Since every Kleisli hom-set $\mathcal{K}\ell_{\mathcal{P}_{\mathsf{ufs}}}(Y, Z)$ is a complete lattice, it is obvious that every Büchi RNNA System consitutes a Büchi trace situation. For this prove, we will calculate the solution of the following equational system:

$$E_{\mathcal{A}} := \left[\begin{array}{lll} u_1 & =_\mu & (J\zeta)^{-1} \circledcirc \overline{F}[u_1, u_2] \circledcirc c_1 \\ u_2 & =_\nu & (J\zeta)^{-1} \circledcirc \overline{F}[u_1, u_2] \circledcirc c_2 \end{array} \right]$$

Herein, the state set Q is divided into $Q_1 := Q \setminus \mathrm{Acc}$ and $Q_2 := \mathrm{Acc}$, the mapping $c_i\colon Q_i \nrightarrow FQ$ is the restriction of the coalgebra along the coproduct injections, the functor $F = \mathbb{A} \times - + [\mathbb{A}] -$ is the Büchi RNNA functor, while

$$\zeta\colon \left(\overline{\mathbb{A}}^\omega/{\equiv_\alpha}\right)_{\mathsf{fs}} \to F\left(\left(\overline{\mathbb{A}}^\omega/{\equiv_\alpha}\right)_{\mathsf{fs}}\right),\ [w]_\alpha \mapsto \begin{cases} (a, [w']_\alpha) & \text{if } [w]_\alpha = [aw']_\alpha \\ \langle a \rangle\, [w']_\alpha & \text{if } [w]_\alpha = [|aw']_\alpha \end{cases}$$

is the final coalgebra for F.

Notation (*Paths in Büchi RNNAs*)

Given some $q, q' \in Q$ and $v \in \overline{\mathbb{A}}^*/{\equiv_\alpha}$, we write $q \xrightarrow{v}^* q'$ if there is a v-labeled path from $q \to q'$, and $q \overset{v}{\Rightarrow}^* q'$ if, additionally, all intermediate states on the path are from Q_1. Note, that q and q' may still be elements of Q_2.

We will then solve this system just like it was mentioned earlier:

Step 1 For every fixed $u_2 \colon Q_2 \nrightarrow (\overline{\mathbb{A}}^\omega/\equiv_\alpha)_{\mathsf{fs}}$, define the interim solution $l_1^{(1)}$ by

$$l_1^{(1)}(u_2) := \mu\, \mathfrak{n}_1.\, (J\zeta)^{-1} \circledcirc \overline{F}[\mathfrak{n}_1, u_2] \circledcirc c_1$$

and solve this by using Kleene. To make the notation less convoluted, we define the ‘helper function’ $\mathfrak{f}_1$ to be

$$\mathfrak{f}_1 : \begin{cases} \mathcal{K}\ell_{\mathcal{P}_{\mathsf{ufs}}}\left(Q_1,\ (\overline{\mathbb{A}}^\omega/\equiv_\alpha)_{\mathsf{fs}}\right) \to \mathcal{K}\ell_{\mathcal{P}_{\mathsf{ufs}}}\left(Q_1,\ (\overline{\mathbb{A}}^\omega/\equiv_\alpha)_{\mathsf{fs}}\right), \\ \mathfrak{n}_1 \mapsto (J\zeta)^{-1} \circledcirc \overline{F}[\mathfrak{n}_1, u_2] \circledcirc c_1. \end{cases}$$

We claim, that for all $k \in \omega$ and $q \in Q_1$, we have

$$\mathfrak{f}^k(\bot)(q) = \left\{ [vw]_\alpha \;\middle|\; \begin{array}{l} v \in \overline{\mathbb{A}}^{\leqslant k}, w \in \overline{\mathbb{A}}^\omega \\ \exists q' \in Q_2.\ q \overset{v}{\Rightarrow}{}^{*} q' \wedge [w]_\alpha \in u_2(q'). \end{array} \right\}.$$

Herein, $\mathfrak{f}_1^k$ denotes the k-fold application of $\mathfrak{f}_1$. We prove this claim per induction over $k \in \omega$:

Base Case ($k = 0$): For $k = 0$ the claim obviously holds: Since $f_1^0(\bot)(q) = \bot(q) = \emptyset$ by definition and $q \in Q_1$, we do not have $q \xrightarrow{\varepsilon} q'$ for any $q' \in Q_2$.

Step Case ($k \to k+1$): Suppose now that the claim holds for some $k \in \omega$. Let, furthermore, $[u]_\alpha = [aw]_\alpha \in \left(\overline{\mathbb{A}}^\omega/{\equiv_\alpha}\right)_{\mathsf{fs}}$, where $a \in \overline{\mathbb{A}}$ and $w \in \overline{\mathbb{A}}^\omega$. Then, the following statements are equivalent:

(i) $[u]_\alpha \in \mathfrak{f}_1(\mathfrak{f}_1^k(\bot))(q)$.

(ii) There is a $q_1 \in Q_1$, such that $q \xrightarrow{a} q_1$ and $[w]_\alpha \in \mathfrak{f}_1^k(\bot)(q_1)$, or a $q_2 \in Q_2$, such that $q \xrightarrow{a} q_2$ and $[w]_\alpha \in u_2(q_2)$.

(iii) There is a $q_1 \in Q_1$, $q_2 \in Q_2$, $v \in \overline{\mathbb{A}}^{\leqslant k}$, and $w' \in \left(\overline{\mathbb{A}}^\omega/{\equiv_\alpha}\right)_{\mathsf{fs}}$, such that

$$[w]_\alpha = [vw']_\alpha, q \xrightarrow{a} q_1 \overset{v}{\Longrightarrow}{}^* q_2, \text{ and } [w']_\alpha \in u_2(q_2),$$

or a $q_2 \in Q_2$, such that $q \xrightarrow{a} q_2$ and $[w]_\alpha \in u_2(q_2)$.

(iv) There is a $q_2 \in Q_2$, $v \in \overline{\mathbb{A}}^{\leqslant k+1}$, and $w' \in \left(\overline{\mathbb{A}}^\omega/{\equiv_\alpha}\right)_{\mathsf{fs}}$, such that

$$[u]_\alpha = [vw']_\alpha, \; q \overset{v}{\Longrightarrow}{}^* q_2, \text{ and } [w']_\alpha \in u_2(q_2).$$

Because the function f_1 is clearly ω-continuous, the interim solution $l_1^{(1)}(u_2)$ is obtained by taking the supremum of the Kleene chain. Therfore, we get the explicit description

$$l_1^{(1)}(u_2)(q) = \left\{ [vw]_\alpha \;\middle|\; v \in \overline{\mathbb{A}}^+,\; w \in \overline{\mathbb{A}}^\omega, \exists q' \in Q_2 . q \stackrel{v}{\Rightarrow}^* q' \wedge [w]_\alpha \in u_2(q') \right\}$$

of our interim solution.

Step 2 Define the iterim solution $l_2^{(2)}$ by

$$l_2^{(2)} := \nu\, \mathfrak{n}_2.\, (J\zeta)^{-1} \circledcirc \overline{F}\left[l_1^{(1)}(\mathfrak{n}_2), \mathfrak{n}_2\right] \circledcirc c_2.$$

Again, to make the notation less convoluted, we define the 'helper function' $\mathfrak{f}_2$ to be

$$\mathfrak{f}_2 \colon\, \mathfrak{n}_2 \mapsto (J\zeta)^{-1} \circledcirc \overline{F}\left[l_1^{(1)}(\mathfrak{n}_2), \mathfrak{n}_2\right] \circledcirc c_2.$$

Similar to Step 1, $\mathfrak{f}_2$ is given by

$$\mathfrak{f}_2(u_2)(q) = \left\{ [vw]_\alpha \;\middle|\; v \in \overline{\mathbb{A}}^+,\ w \in \overline{\mathbb{A}}^\omega, \exists q' \in Q_2.q \overset{v}{\Rightarrow}{}^* q' \wedge [w]_\alpha \in u_2(q$$

We then claim that $l_2^{(2)}(q) = L^2_{\alpha,\omega}(q)$. Here, $L^2_{\alpha,\omega}$ is the restriction of the language mapping $L_{\alpha,\omega}$ to Q_2. Since $L^2_{\alpha,\omega}$ is obviously a fixed point of $\mathfrak{f}_2$, we have $L^2_{\alpha,\omega}(q) \subseteq l_2^{(2)}(q)$. It remains to prove $l_2^{(2)}(q) \subseteq L^2_{\alpha,\omega}(q)$. Let $[w]_\alpha \in l_2^{(2)}(q) = \mathfrak{f}_2(l_2^{(2)})(q)$ and $w \in \bar{\mathbb{A}}^\omega$ be a representant of $[w]_\alpha$. We shall construct infinite sequences of states $q_0,\ q_1,\ \dots \in Q_2$ and non-empty words $v_1,\ v_2,\ \dots \in \bar{\mathbb{A}}^+$, such that

(i) $q = q_0$ and $q_i \xrightarrow{v_{i+1}}{}^{*} q_{i+1}$ holds for all $i \in \omega$;

(ii) for each $k \in \omega$ the word $v_1 \cdots v_k$ is a prefix of w, i.e. $w = v_1 \cdots v_k w'$ for some $w' \in \bar{\mathbb{A}}^\omega$ and the equivalence class $[w']_\alpha$ of the suffix w' lies in $l_2^{(2)}(q_k)$.

Given this, **(ii)** implies that $w = v_1 v_2 \cdots$, while **(i)** implies that w has an accepting run from q. Therefore, we can conclude that $[w]_\alpha \in L^2_{\alpha,\omega}(q)$.

We construct this sequence recursively. Obviously, we fix $q_0 = q$. Moreover, fix $k \in \omega$ and suppose that we already defined $q0,\ \dots,\ q_k$ and $v_1,\ \dots,\ v_k$, such that

(i') $q = q_0 \xrightarrow{v_1}{}^{*} q_1 \xrightarrow{v_2}{}^{*} \cdots \xrightarrow{v_k}{}^{*} q_k$;

(ii') the word $v_1 \cdots v_k$ is a prefix of w, i.e. $w = v_1 \cdots v_k w'$ for some $w' \in \bar{\mathbb{A}}^\omega$ and the equivalence class $[w']_\alpha$ of the suffix w' lies in $l_2^{(2)}(q_k)$.

Because of **(ii')**, we have that $w' \in l_2^{(2)}(q_k) = \mathfrak{f}_2(l_2^{(2)})(q_k)$. Therefore, there are $v' \in \bar{\mathbb{A}}^+$, $w'' \in \bar{\mathbb{A}}^\omega$, and $q' \in Q_2$, such that $w' \equiv_\alpha v' w''$, $q_k \overset{v'}{\Longrightarrow}{}^* q'$ and $[w'']_\alpha \in l_2^{(2)}(q')$. Thus, $v_{k+1} = v'$ and $q_{k+1} = q'$ fulfill all desired properties.

Step 3 Lastly, we calculate the trace mappings. Obviously, $\mathrm{trace}_2^{\mathrm{b}}(\mathcal{A}) \colon \mathrm{Acc} \nrightarrow \left(\bar{\mathbb{A}}^\omega / \equiv_\alpha\right)_{\mathrm{fs}}, q \mapsto L_{\alpha,\omega}(q)$ holds, since $\mathrm{trace}_2^{\mathrm{b}}(\mathcal{A}) = l_2^{(2)}$. Moreover, we get the trace map for Q_1 by $l_1^{(1)}(l_2^{(2)})$. Thus, for any $q \in Q_1$, we have

$$l_1^{(1)}(L^2_{\alpha,\omega})(q) = \left\{ [vw]_\alpha \;\middle|\; v \in \bar{\mathbb{A}}^+,\ w \in \bar{\mathbb{A}}^\omega,\ \text{there is a } q' \in Q_2,\ \text{s.t. } q \overset{v}{\Rightarrow}{}^* q',\ \text{and } [w]_\alpha \in L^2_{\alpha,\omega}(q') \right\}.$$

This is clearly equal to $L^1_{\alpha,\omega}$, the restriction of $L_{\alpha,\omega}$ to Q_1.

This concludes the proof that the trace mappings are given by the language mappings. It is obvious, that the composite $\left[\mathsf{trace}_1^{\mathsf{b}}(\mathcal{A}), \mathsf{trace}_2^{\mathsf{b}}(\mathcal{A})\right] \odot s$ maps the singular element $* \in \mathbb{1}$ to the accepted bar ω-language by the Büchi RNNA *A*.

Thank you for your attention!