Hennessy-Milner Theorems for Graded Quantitative Semantics

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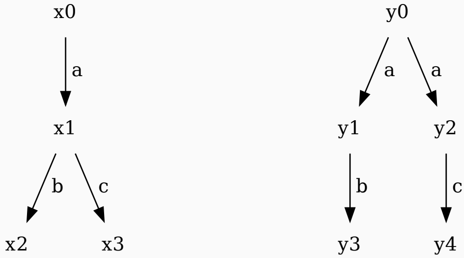
Motivation

Motivation I - Linear-Time Branching-Time

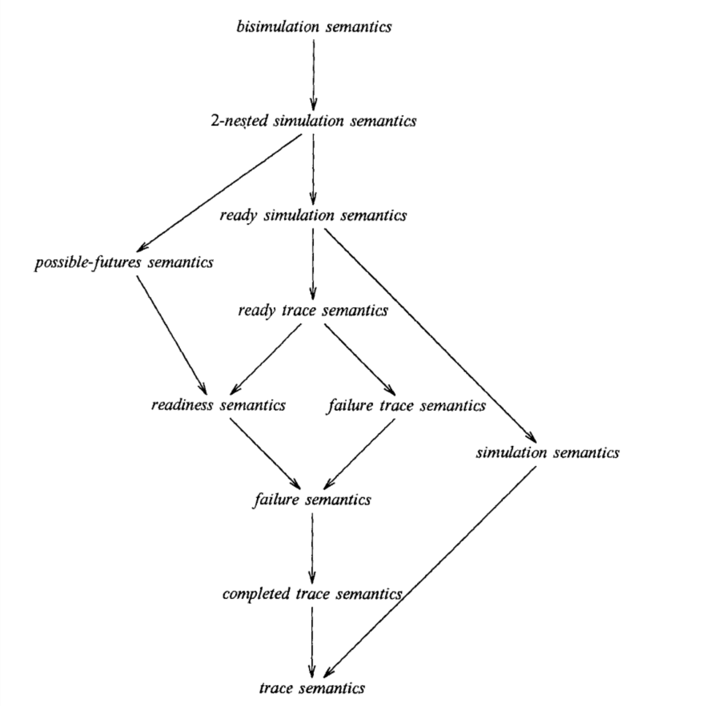
What does it mean for states to behave the same way?

- Behavioural equivalence / Bisimilarity?
- Trace equivalence?
- Something in between?

Dependent on the observers ability to interact with the system.



⇒ *Linear-Time Branching Time Spectrum*


bisimulation semantics
2-nested simulation semantics
ready simulation semantics
possible-futures semantics
ready trace semantics
readiness semantics
failure trace semantics
simulation semantics
failure semantics
completed trace semantics
trace semantics

Motivation II

This spectrum is captured categorically via graded semantics

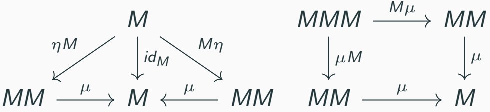
- Sometimes we care not whether states are equal but about "how equal"
  - probabilities
  - delay
  - physical distance
- Move to the category **Met**
- Study expressive quantitative modal logics

Preliminaries

A *monad* on a category C consists of:

- a functor M : C $\rightarrow$ C
- a natural transformation η : $Id \rightarrow M$
- a natural transformations μ : $M\ M \rightarrow M$

such that the following diagrams commute:



Graded Monads

A *graded monad* on a category C consists of:

- a family of functors $M_n : \mathsf{C} \to \mathsf{C}$ for $n \in \mathbb{N}$
- a natural transformation $\eta : Id \to M_0$
- a family of natural transformations $\mu^{nk} : M_n M_k \to M_{n+k}$

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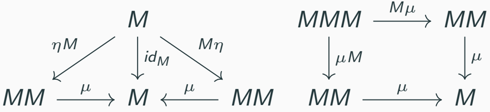

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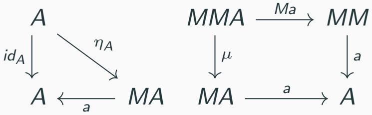
such that the following diagrams commute:

$$
\begin{array}{ccccc}
 & & M_n & & \\
 & \swarrow^{\eta M_n} & \downarrow^{id_{M_n}} & \searrow^{M_n \eta} & \\
M_0 M_n & \xrightarrow{\mu^{0n}} & M_n & \xleftarrow{\mu^{n0}} & M_n M_0
\end{array}
\qquad
\begin{array}{ccc}
M_n M_k M_m & \xrightarrow{M_n \mu^{km}} & M_n M_{k+m} \\
\downarrow^{\mu^{nk} M_m} & & \downarrow^{\mu^{n,k+m}} \\
M_{n+k} M_m & \xrightarrow{\mu^{n+k,m}} & M_{n+k+m}
\end{array}
$$

An *M -Algebra* (A , a) consists of

- a **C**-object A
- a morphism $a : M A \to A$

such that the following diagrams commute:



Graded Algebras

A *graded M_n-Algebra* $((A_k)_{k \leq n}, (a^{mk})_{m+k \leq n})$ consists of

- a family of **C**-objects A_i
- a family of morphisms $a^{mk} : M_m A_k \to A_{m+k}$

such that the following diagrams commute:

$$\begin{array}{ccc} A & & \\ {\scriptstyle id_A}\downarrow & \searrow^{\eta_A} & \\ A & \xleftarrow[a]{} & MA \end{array} \qquad \begin{array}{ccc} MMA & \xrightarrow{Ma} & MM \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle a} \\ MA & \xrightarrow{a} & A \end{array}$$

Graded Algebras

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- a family of **C**-objects A_i
- a family of morphisms $a^{mk} : M_m A_k \to A_{m+k}$

such that the following diagrams commute:

$$\begin{array}{ccc} A_m & & \\ {\scriptstyle id_{A_m}}\downarrow & \searrow^{\eta_{A_m}} & \\ A_m & \xleftarrow[a^{0m}]{} & M_0 A_m \end{array} \qquad \begin{array}{ccc} M_m M_r A_k & \xrightarrow{M_m a^{rk}} & M_m M_{r+k} \\ \downarrow{\scriptstyle \mu^{mr}_{A_k}} & & \downarrow{\scriptstyle a^{m,r+k}} \\ M_{m+r} A_k & \xrightarrow{a^{m+r,k}} & A_{m+r+k} \end{array}$$

Let $(-)_i : \mathrm{Alg}_1(\mathbb{M}) \to \mathrm{Alg}_0(\mathbb{M})$, with $i \in \{0,1\}$ be the functor taking a M_1-algebra $((A_k)_{k\leq 1}, (a^{mk})_{m+k\leq 1})$ to the M_0 algebra (A_i, a^{0i}).

Canonical Algebra

An M_1-algebra A is *canonical* if it is free over $(-)_0$.

$$A_0 \xrightarrow{f_0} B_0$$

$$\begin{array}{ccc} M_1 A_0 & \xrightarrow{M_1 f_0} & M_1 B_0 \\ \downarrow{\scriptstyle a^{10}} & & \downarrow{\scriptstyle b^{10}} \\ A_1 & \dashrightarrow^{f_1} & B_1 \end{array}$$

Canonical M_1 Algebras

Alternative Characterisation

An M_1-algebra A is canonical iff the following diagram is a coequalizer in C^{M_0}

$$M_1 M_0 A_0 \underset{\mu^{10}}{\overset{M_1 a^{00}}{\rightrightarrows}} M_1 A_0 \xrightarrow{a^{10}} A_1$$

We use a system of quantitative equational reasoning with equations of the form $s =_\epsilon t$

Mardare, Panagaden, Plotkin '16

Ingredients for Quantitative Theories

- Σ algebraic similarity type with a depth for each operation
- V a fixed set of variables
- E set of Axioms of the form $\Gamma \vdash t =_\epsilon s$ where s and t are Σ-terms of the same uniform depth over V where Γ prescribes distances of variables $x =_\epsilon y$ with $x, y \in V$

Quantitative Equational Reasoning

Rules of equational reasoning:

$$(\text{refl})\ \frac{}{s =_0 s} \qquad (\text{sym})\ \frac{t =_\epsilon s}{s =_\epsilon t} \qquad (\text{triang})\ \frac{t =_\epsilon s \quad s =_{\epsilon'} u}{t =_{\epsilon+\epsilon'} u}$$

$$(\text{wk})\ \frac{t =_\epsilon s}{t =_{\epsilon'} s}\ (\epsilon' \geq \epsilon) \qquad (\text{arch})\ \frac{t =_{\epsilon'} s \mid \epsilon' > \epsilon}{t =_\epsilon s}$$

$$(\text{nexp})\ \frac{t_1 =_\epsilon s_1 \quad \dots \quad t_n =_\epsilon s_n}{f(t_1, \dots, t_n) =_\epsilon f(s_1, \dots, s_n)}$$

$$(\text{ax})\ \frac{\Gamma\sigma}{t\sigma =_\epsilon s\sigma}\ ((\Gamma \vdash t =_\epsilon s) \in E)$$

$$(\text{assn})\ \frac{}{\phi}\ (\phi \in \Gamma_0)$$

Theories induce graded monads and vice versa.

Depth-1 graded Algebra

A graded M-Algebra in some category C is *depth-1* if the following diagram is an object wise coequalizer diagram in $\mathbf{C}^{M_0}$:

$$M_1 M_0 M_n \underset{\mu^{10} M_n}{\overset{M_1 \mu^{0n}}{\rightrightarrows}} M_1 M_n \xrightarrow{\mu^{1n}} M_{1+n}$$

and all multiplications μ^{ij} are epi.

Dorsch, Milius, Schröder '19

Lemma

This is the case iff all operations and all axioms $\Gamma \vdash s =_\epsilon t$ have uniform depth at most 1.

Free Algebras are Canonical

M_1-Algebras of the form $(M_n X, M_{n+1} X, \mu^{0n,}, \mu^{0n+1}, \mu^{1n})$ are canonical.

Graded Semantics

Ingredients of Graded Semantics

- Coalgebra functor G : **Met** $\to$ **Met**
- Graded monad $\mathbb{M}$
- Natural transformation $\alpha : G \to M_1$

Depth n observable behaviour of a G coalgebra (X, γ):

$$\gamma^{(0)} : (X \xrightarrow{\eta} M_0 X \xrightarrow{M_0 !} M_0 1)$$

$$\gamma^{(n+1)} : X \xrightarrow{\alpha \circ \gamma} M_1 X \xrightarrow{M_1 \gamma^{(n)}} M_1 M_n 1 \xrightarrow{\mu^{1n}} M_{n+1} 1$$

Syntax of Graded Logics

- Truth constants Θ
- Propositional operators $\mathcal{O}$
- Modal operators Λ

Formulae of uniform depth 0 are given by the grammar

$$\phi ::= p(\phi_1, \ldots, \phi_k) \mid c \quad (p_{/k} \in \mathcal{O}, c \in \Theta)$$

and formulae of uniform depth $n + 1$ by the grammar

$$\phi ::= p(\phi_1, \ldots, \phi_k) \mid L(\psi_1, \ldots, \psi_j) \quad (p_{/k} \in \mathcal{O}, L_{/j} \in \Lambda)$$

where all ϕ_i are of depth $n + 1$ and all ψ_i are of depth n

Graded Logics - Semantics I

Semantics of formulae over:

For an M_0-algebra of truth values (Ω, o)

- For $c \in \Theta$ we have $\hat{c} : 1 \to \Omega$
- For $p \in \mathcal{O}$ n-ary we have an M_0 algebra homomorphism $[\![p]\!] : \Omega^n \to \Omega$
- For $L \in \Lambda$ n-ary we have an M_1-algebra $((\Omega^n, \Omega), (o^n, o, [\![L]\!]))$

We generally use $\Omega = [0, 1]$

Graded Logics - Semantics II

For a G-coalgebra (X, γ) we get a morphism evaluating states by

$$X \xrightarrow{\gamma^n} M_n 1 \xrightarrow{[\![\phi]\!]} \Omega$$

- for $c \in \Theta$ we have $\hat{c} : 1 \to \Omega$, then

$$[\![c]\!] = M_0 1 \xrightarrow{M_0 \hat{c}} M_0 \Omega \xrightarrow{o} \Omega$$

- for $p \in \mathcal{O}$ k-ary we have

$$[\![p(\phi_1, \dots, \phi_k)]\!] = [\![p]\!] \circ \langle [\![\phi_1]\!], \dots, [\![\phi_k]\!] \rangle$$

- for $L \in \Lambda$ k-ary we have

$$[\![L(\phi_1, \dots, \phi_k)]\!] = [\![L]\!](\langle [\![\phi_1]\!], \dots, [\![\phi_k]\!] \rangle)$$

  where $[\![L]\!](f)$ for $f : M_n 1 \to \Omega^n$ is the unique morphism $M_{n+1} 1 \to \Omega$

Behavioural and Logical Distance

The *depth-n behavioural distance* is defined as

$$d^{\alpha,n}(x,y) = d_{M_n 1}(\gamma^n(x), \gamma^n(y))$$

The *depth-n logical distance* of x and y for a logic $\mathcal{L}$ is given by

$$d^{\mathcal{L},n}(x,y) = \sup\{d_\Omega([\![\phi]\!](\gamma^n(x)), [\![\phi]\!](\gamma^n(y))) \mid \phi \in \mathcal{L}_n\}$$

.

We call $\mathcal{L}$ *expressive* for α if $d^{\alpha,n} = d^{\mathcal{L},n}$ for all $n \in \mathbb{N}$

$\bar{\mathcal{P}}_\omega : \mathbf{Met} \to \mathbf{Met}$ finite powerset functor with Hausdorff distance for finite $A, B \subseteq X$ defined as

$$d_{\bar{\mathcal{P}}_\omega X}(A, B) = \max\{\sup_{a\in A}\inf_{b\in B} d_X(a, b), \sup_{b\in B}\inf_{a\in A} d_X(a, b)\}$$

Quantitative Trace semantics

- $G : \mathbf{Met} \to \mathbf{Met}$ defined as $G = \bar{\mathcal{P}}_\omega(S \times -)$.
- $M_n = \bar{\mathcal{P}}_\omega(S^n \times -)$.
- α is the obvious natural transformation.

Equivalent to semantics in the literature

Fahrenberg, Legay, Thrane '11

M_n Algebraically

Signature $\Sigma = \underbrace{\{0_{/0}, +_{/2}\}}_{\text{Depth 0}} \cup \underbrace{\{s_{/1} \mid s \in S\}}_{\text{depth 1}}$

Axioms:

$$\vdash x + 0 =_0 x \qquad \vdash x + x =_0 x$$

$$\vdash x + y =_0 y + x$$

$$\vdash (x + y) + z =_0 x + (y + z)$$

$$\vdash s(0) =_0 0 \qquad \vdash s(x + y) =_0 s(x) + s(y)$$

$$x =_\epsilon y \vdash s(x) =_{\max\{\epsilon, d_S(s,t)\}} t(y)$$

Characteristic Trace Logic

Syntax of $\mathcal{L}^{Trace}$:

$\Theta = \{\top\}, \qquad \mathcal{O} = \emptyset \qquad \Lambda = \{s_{/1} \mid s \in S\}$

Semantics of $\mathcal{L}^{Trace}$ over the Truth object $([0,1], \max)$:

- $\hat{\top}$ defined as $* \mapsto 1$
- $[\![\langle s \rangle]\!](x) = \sup_{(t,v)\in x} \min\{1 - d(s,t), v\}$

Separation via Eilenberg-Moore Algebras

Modalities as morphisms in Eilenberg-Moore

- Canonical Algebras are unique up to isomorphism
- Taking M_0-algebras to their canonical M_1-algebra defines a functor E
- We define a functor $\overline{M}_1$

$$\overline{M}_1 = (\mathsf{Alg}_0(\mathbb{M}) \xrightarrow{E} \mathsf{Alg}_1(\mathbb{M}) \xrightarrow{(-)_1} \mathsf{Alg}_0(\mathbb{M}))$$

This implies that

$$\overline{M}_1(M_n X, \mu^{0n}) = (M_{n+1} X, \mu^{0n+1})$$

and in particular that

$$U\overline{M}_1 F = M_1$$

Lemma

Modal operators are $\mathsf{Alg}_0(\mathbb{M})$ morphisms $\overline{M}_1 \Omega^n \to \Omega$

Probabilistic Trace logic

- $\mathcal{D}$: **Met** $\to$ **Met** finite distributions with Kantorvich distance.
- $G = \mathcal{D}(A \times -)$
- $M_n = \mathcal{D}(A^n \times -)$
- $(\Omega, o) = ([0,1], \mathbb{E})$

Since $([0,1], \mathbb{E}) = (M_0 2, \mu^{00})$, unary modalities correspond precisely to nonexpansive convex maps $f : \mathcal{D}(A \times \{0,1\}) \to [0,1]$

The corresponding modal operator $[\![L]\!] : \mathcal{D}(A \times [0,1]) \to [0,1]$ is given by

$$[\![L]\!](\pi) = \sum_{a \in A, v \in [0,1]} (v\pi(a,v)h(a,1)) + ((1-v)\pi(a,v)h(a,0))$$

Proposition

There is no expressive unary modal logic for probabilistic trace semantics.

Criteria for Expressivity

Φ-Type Separation

Initial cones

A cone F of morphisms $f_i : A \to B_i$ is *initial* if for all $x, y \in A$

$$d_A(x, y) = \sup_{f_i \in F} d_{B_i}(f_i(x), f_i(y))$$

Φ-Type Separation

$\mathcal{L}$ is *Φ-type depth-0 separating* if the family of maps $[\![c]\!] : M_0 1 \to \Omega$ for $c \in \Theta$ has property Φ.

$\mathcal{L}$ is *Φ-type depth-1 separating* if, whenever A is an M_1-algebra of the form $(M_n X, M_{n+1} X, \mu^{0n}, \mu^{0n+1}, \mu^{1n})$ and $\mathfrak{A}$ cone of M_0-homomorphisms $M_n X \to \Omega$ with property Φ, closed under propositional operators in $\mathcal{O}$, then the set

$$\Lambda(\mathfrak{A}) := \{[\![L]\!](\langle f_1, \ldots, f_n \rangle) : M_{n+1} X \to \Omega \mid L_{/n} \in \Lambda, f_i \in \mathfrak{A}\}$$

has property Φ.

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has property Φ.

Expressivity

Theorem

If a graded logic $\mathcal{L}$ is Φ-type depth-0 separating, Φ-type depth-1 separating and any cone with property Φ is initial, then $\mathcal{L}$ is expressive.

Failure of Weak-Type Separation

Functor $F : \mathbf{Met} \to \mathbf{Met}$ defined as $FX = S \times X$

Syntax of $\mathcal{L}^{Stream}$:

- $\Theta = \{\top\}$
- $\mathcal{O} = \emptyset$
- $\Lambda = \{\langle s \rangle_{/1} \mid s \in S\}$.

Monad $M_n X = S^n \times X$ with $\alpha : F \to M_1$ the obvious natural tranformation.

$\hat{\top} : 1 \to \Omega$ is $* \mapsto 1$.

$[\![\langle s \rangle]\!]((t, v)) = \min\{1 - d_S(s, t), v\}$ for all $s \in S$ and $(t, v) \in M_1\Omega$.

Strong-Type Separation

Normed Isometry

We call a set F of morphisms $X \to [0,1]$ *normed isometric* if for all $x, y \in X$ and $\epsilon > 0$ there is a $f \in F$ such that $f(x) > 1 - \epsilon$ and $f(y) < 1 - d_X(x,y) + \epsilon$

$\mathcal{L}^{Stream}$ **is normed isometric-type depth-1 separating but not initial-type depth-1 separating.**

$\Rightarrow \mathcal{L}^{Stream}$ is expressive

Similarly $\mathcal{L}^{Trace}$ is expressive.

The proof Separates the elements of $\overline{M}_1 A_0$

Stronger Notions of Expressivity

Strong Expressivity

We require formulae to witness distances.

Stronger: Formulae can describe all properties of the semantics.

- More in line with previous works
- Fits naturally with the classical definition of separation

Definition

$\mathcal{L}$ is *strongly expressive* if $[\![\mathcal{L}_n]\!]$ is dense in $\mathrm{Alg}_0(\mathbb{M})(M_n 1, \Omega)$

Under which conditions can we get strong expressivity from initiality?

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Under which conditions can we get strong expressivity from initiality?

$\Rightarrow$ Stone-Weierstrass Theorems

Stone-Weierstrass for different

Stone-Weierstrass Property in Eilenberg-Moore

We say that a monad M on $\mathcal{C}$ has the *$\mathcal{O}$-SW property* with respect to a class C of of algebras, if for all M-algebras $(A, a) \in C$ and all $\mathfrak{A} \subseteq \mathcal{C}^M((A, a), (\Omega, o))$ such that $\mathfrak{A}$ is initial as a set of $\mathcal{C}$-morphisms, the closure of $\mathfrak{A}$ under $\mathcal{O}$ is dense in $\mathcal{C}^M((A, a), (\Omega, o))$.

$\mathcal{L}$ is strongly expressive if it is expressive, $(M_n 1, \mu^{0n}) \in C$ and M_0 has the $\mathcal{O}$-SW property.

M	$\mathcal{C}$	C	SW?
Id	**Set**	Finite	Yes
Id	**Met**	Totally bounded	Yes
$\mathcal{P}_\omega$	**Set**	Finite	Yes
$\overline{\mathcal{P}}_\omega$	**Met**	Finite	No

Future Work

- Characterize the poset of graded semantics (algebraically)
- Find more examples
- Study applications
  - compute behavioural distances in a system
  - minimization up to ϵ
- Genaralised (quantale valued?) metrics