The Case of the Missing Fixed Point Calculation

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History 1

1931 It was noted by Kurt Gödel and, independently, by John von Neumann that the Gödel sentence is provably equivalent to the consistency statement.

1955 It was shown by Martin Löb that the Henkin sentence is provably equivalent to $\top$.

1974, 1976 The uniqueness of modalised fixed points is proven, independently, by Dick de Jongh, Giovanni Sambin, and Claudio Bernardi.

1975, 1976 Dick de Jongh and, independently, Giovanni Sambin show that modalised fixed points in provability logic have explicit forms.

... Many many proofs of the de Jongh-Sambin Theorem emerge ...

History 2

See ‘the Henkin Sentence’ by Halbach & Visser in ‘The life and work of Leon Henkin’.

Recently more became known about connections with the μ-calculus. We will not discuss that in our talk.

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Uniqueness 1

Our base system is $\mathsf{LHC}^\flat$. A formula φ is *modalised in p* iff all occurrences of p are in the scope of a $\strictif$.

We will show that over $\mathsf{LHC}^\flat \oplus \mathsf{L}_\Box$ fixed points are unique.

We need two substitution theorems:

- $\mathsf{LHC}^\flat + 4_\Box \vdash \boxdot(\varphi \leftrightarrow \psi) \to (\chi[p:\varphi] \leftrightarrow \chi[p:\psi])$.
- Suppose χ is modalised in p. Then, $\mathsf{LHC}^\flat + 4_\Box \vdash \Box(\varphi \leftrightarrow \psi) \to (\chi[p:\varphi] \leftrightarrow \chi[p:\psi])$.

Uniqueness 2

Theorem (de Jongh)

$\mathsf{LHC}^\flat + \mathsf{L}_\Box \vdash 4_\Box$.

Proof.

Reason in $\mathsf{LHC}^\flat + \mathsf{L}_\Box$: Suppose $\Box\varphi$. Then,

$$\Box(\Box(\varphi \wedge \Box\varphi) \to (\varphi \wedge \Box\varphi)).$$

So, $\Box(\varphi \wedge \Box\varphi)$. Ergo, $\Box\Box\varphi$. ❑

Uniqueness 3

Theorem (Strong Löb Rule)

Let Λ *extend* $\mathsf{LHC}^\flat + \mathsf{L}_\Box$. *Suppose* $\Lambda \vdash \boxdot\chi \to (\Box\varphi \to \varphi)$. *Then,* $\Lambda \vdash \boxdot\chi \to \varphi$.

Proof.

Suppose (a) $\Lambda \vdash \boxdot\chi \to (\Box\varphi \to \varphi)$. It follows that

$$\Lambda \vdash \boxdot\chi \to \Box(\Box\varphi \to \varphi).$$

Hence, (b) $\Lambda \vdash \boxdot\chi \to \Box\varphi$. Combining (a) and (b), we are done. ❑

Uniqueness 4

Theorem (De Jongh-Sambin-Bernardi)

Suppose φ is modalised in p and that q is does not occur in φ.
Then, $\mathsf{LHC}^\flat + \mathsf{L}_\Box \vdash \boxdot((p \leftrightarrow \varphi) \wedge (q \leftrightarrow \varphi[p:q])) \to (p \leftrightarrow q)$.

Proof.

We have:

$$\begin{aligned} \mathsf{LHC}^\flat + \mathsf{L}_\Box \vdash (\boxdot((p \leftrightarrow \varphi) \wedge (q \leftrightarrow \varphi[p:q])) \wedge \Box(p \leftrightarrow q)) \quad &\to \quad (\varphi \leftrightarrow \varphi[p:q]) \\ &\to \quad (p \leftrightarrow q) \end{aligned}$$

By the Strong Löb Rule we are done. ❑

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Lewis to Modalised 1

A formula is *a Lewis formula* if it has $\dashv$ as main operator. A logic Λ *has Lewis fixed points* iff, for every Lewis formula φ and for any p, there is a ψ such that $\Lambda \vdash \psi \leftrightarrow \varphi[p : \psi]$.

We note that we can always assume that all variables in ψ are variables of φ not equal to p.

Theorem (Multiple Fixed Points)

Suppose Λ has Lewis fixed points. Let $\varphi_0, \ldots, \varphi_{n-1}$ be Lewis formulas and consider distinct $p_0, \ldots, p_{n-1}$. Then, for each $i < n$, there are formulas $\psi_0, \ldots, \psi_{n-1}$ such that

$$\Lambda \vdash \psi_i \leftrightarrow \varphi_i[p_0 : \psi_0, \ldots, p_{n-1} : \psi_{n-1}],$$

Here all variables in the ψ_i occur in the φ_j and are distinct from the p_j.

Lewis to Modalised 2

We prove the theorem by induction on n. For $n = 1$ we are done. Suppose $n = k + 1$ ($k \neq 0$) We solve the equations for $\varphi_1, \ldots, \varphi_k$ and $p_1, \ldots, p_k$. Let the solutions be $\alpha_1, \ldots \alpha_k$.

- ψ_0 is the solution for p_0 of $\varphi_0[p_1 : \alpha_1, \ldots, p_k : \alpha_k]$.
- $\psi_{i+1} := \alpha_{i+1}[p_0 : \psi_0]$.

We have

$$\begin{aligned} \Lambda \vdash \psi_i \quad & \leftrightarrow \quad \varphi_i[p_1 : \alpha_1, \ldots p_k : \alpha_k][p_0 : \psi_0] \\ & \leftrightarrow \quad \varphi_i[p_0 : \psi_0, p_1 : \alpha_1[p_0 : \psi_0], \ldots, p_k : \alpha_k[p_0 : \psi_0]] \\ & \leftrightarrow \quad \varphi_i[p_0 : \psi_0, \ldots, p_{n-1} : \psi_{n-1}] \end{aligned}$$

Note that for the first equivalence the argument is different for the cases $i = 0$ and $i = j + 1$.

Lewis to Modalised 3

Theorem

Suppose that Λ has Lewis fixed points. Then Λ has modalised fixed points.

Proof.

Suppose φ is modalised in p. We can find formulas $\psi, \theta_0, \ldots \theta_{k-1}$, such that p does not occur in ψ, the θ_i are Lewis and $\varphi = \psi[q_0 : \theta_0, \ldots, q_{k-1} : \theta_{k-1}]$.

Let ν_i be the solutions of the equations for $q_i, \theta_i[p : \psi]$. Let $\chi := \psi[q_0 : \nu_0, \ldots, q_{k-1} : \nu_{k-1}]$. We have:

$$
\begin{array}{lll}
\Lambda \vdash \chi & \leftrightarrow & \psi[q_0 : \nu_0, \ldots, q_{k-1} : \nu_{k-1}] \\
& \leftrightarrow & \psi[q_0 : \theta_0[p : \psi][\vec{q} : \vec{\nu}], \ldots, q_{k-1} : \theta_{k-1}[p : \psi][\vec{q} : \vec{\nu}]] \\
& \leftrightarrow & \psi[q_0 : \theta_0[p : \psi[\vec{q} : \vec{\nu}]], \ldots, q_{k-1} : \theta_{k-1}[p : \psi[\vec{q} : \vec{\nu}]]] \\
& \leftrightarrow & \psi[q_0 : \theta_0, \ldots, q_{k-1} : \theta_{k-1}][p : \psi[\vec{q} : \vec{\nu}]] \\
& \leftrightarrow & \varphi[p : \chi]
\end{array}
$$

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The Logic

The Fixed Point Calculation we discuss in this Section goes back to de Jongh-Visser 1991. Dick first found the fixed points by reasoning model-theoretically and, subsequently, I transformed his idea to a syntactic calculation. An air of mystery remains.

We need the following principles.

4_{a} $\vdash \varphi \multimap \Box\varphi$.

L_{a} $\vdash (\Box\varphi \to \varphi) \multimap \varphi$.

Our system is $\mathsf{iGL}_{\mathsf{a}} := \mathsf{LHC}^{\flat} \oplus \mathsf{L}_{\mathsf{a}}$.

One easily shows that $\mathsf{iGL}_{\mathsf{a}} := \mathsf{LHC}^{\flat} \oplus \mathsf{L}_{\Box} \oplus 4_{\mathsf{a}}$.

In this section all provability will be provability over $\mathsf{iGL}_{\mathsf{a}}$ so we suppress that.

Symmetric Löb's Rule

We define:

sLR$_{\mathsf{a}}$ If $\vdash \Box\varphi \to (\varphi \leftrightarrow \psi)$, then $\vdash \varphi \strictiff \psi$

Theorem

sLR$_{\mathsf{a}}$ *is admissible for* iGL$_{\mathsf{a}}$*.*

Proof.

Suppose $\vdash \Box\varphi \to (\varphi \leftrightarrow \psi)$.

ltr. Then, $\vdash \boxdot\varphi \to \psi$. So $\vdash \boxdot\varphi \strictif \psi$. By 4_{a}, $\vdash \varphi \strictif \psi$.

rtl. Then, $\vdash \psi \to (\Box\varphi \to \varphi)$. So, $\vdash \psi \strictif (\Box\varphi \to \varphi)$. Ergo, $\vdash \psi \strictif \varphi$.

❑

We note that closure under sLR$_{\mathsf{a}}$ immediately implies L$_{\mathsf{a}}$.

The Fixed Point Theorem 1

We fix a variable p and write $\varphi\psi$ for $\varphi[p:\psi]$. We note that $(\varphi\psi)\chi = \varphi(\psi\chi)$, so we write $\varphi\psi\chi$.

Theorem

Let $\varphi := (\psi \strictif \chi)$ *and* $\theta := (\psi\Box\chi\top \strictif \chi\top)$. *Then,* $\vdash \theta \leftrightarrow \varphi\theta$.

Proof.

a. We have $\vdash \Box\chi\top \to \theta$. So, $\vdash \Box\chi\top \to \boxdot(\top \leftrightarrow \theta)$. Thus, $\Box\chi\top \vdash \chi\top \leftrightarrow \chi\theta$. So, by $\mathrm{sLR_a}$, we find $\vdash \chi\top \strictiff \chi\theta$.

b. We have $\vdash \Box\psi\Box\chi\top \to (\Box\chi\top \leftrightarrow \theta)$. So, $\vdash \Box\psi\Box\chi\top \to (\psi\Box\chi\top \leftrightarrow \psi\theta)$. So by $\mathrm{sLR_a}$, $\vdash \psi\Box\chi\top \strictiff \psi\theta$

It follows by (a) and (b) that:

$$\begin{aligned} \vdash \theta &\leftrightarrow (\psi\Box\chi\top \strictif \chi\top) \\ &\leftrightarrow (\psi\theta \strictif \chi\theta) \end{aligned}$$

❑

The Fixed Point Theorem 2

Modulo $\mathsf{iGL_a}$-provable equivalence, we can also take as fixed point $\varphi\Box\chi\top$.

Caveat emptor: This does not seem to have the same reverse mathematics.

From JV to $\mathsf{L_a}$

Let JV be the set of de Jongh-Visser Fixed point equivalences. We have:

Theorem

$\mathsf{LHC}^\flat \oplus \mathsf{JV}$ *implies* $\mathsf{L_a}$.

Proof.

The JV-fixed point of $\Box(p \to \varphi)$, where p not in φ, is $\Box(\top \to \varphi)$. This gives $\mathsf{L}_\Box$. So we have unique modalised fixed points. Consider $(p \to \varphi) \dashv_3 \varphi)$, where p not in φ. On the one hand, $\top$ is a fixed point by $\mathsf{LHC}^\flat$-reasoning; on the other hand, $(\Box\varphi \to \varphi) \dashv_3 \varphi$ is one by JV. ❑

So, over $\mathsf{LHC}^\flat$, we find that $\mathsf{L_a}$ *is* JV.

Can you find a shorter/simpler proof?

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JS1

We define JS as the following scheme:

- $\vdash \varphi\top \leftrightarrow \varphi\varphi\top$, for φ a Lewis formula.

In our paper, we prove the hygienic result that $\mathsf{LHC}^\flat \oplus \mathsf{JS}$ is an extension stable logic.

No nice principle $\mathfrak{X}$ analogous to $\mathsf{L_a}$ is known such $\mathsf{LHC}^\flat \oplus \mathsf{JS}$ is $\mathsf{LHC}^\flat \oplus \mathfrak{X}$. So, in spite of its apparent greater simplicity JS is an some sense more complicated than JV.

Many logics prove JS. We zoom in on one example.

JS2

We define:

P $\vdash (\varphi \dashv\!\!3\, \psi) \to \Box(\varphi \dashv\!\!3\, \psi)$.

- iGLP$^-$ is LHC$^\flat \oplus$ L$_\Box \oplus$ P.

Theorem

iGLP$^- \vdash$ JS.

Proof.

Let φ be a Lewis formula. Reason in iGLP$^-$.

ltr. Suppose $\varphi\top$. Then $\boxdot(\varphi\top \leftrightarrow \top)$. So, $\varphi\varphi\top$.

rtl. Suppose $\varphi\varphi\top$ and $\Box\varphi\top$. Then $\Box(\varphi\top \leftrightarrow \top)$. Hence, $\varphi\top$. Applying Löb's Rule (with a small twist), we can drop the assumption $\Box\varphi\top$.

❑

JS3

Interpreting $\varphi \strictif \psi$ as $\Box(\varphi \to \psi)$ in GL yields an interpretation of iGLP^-. However, we do not have $\mathrm{GL} \vdash \Box(\varphi \to \Box\varphi)$. So, we do not interpret 4_a, and, *a fortiori*, we do not interpret JV. So:

Theorem

$\mathrm{LHC}^\flat \oplus \mathrm{JS} \nvdash \mathrm{JV}$.

On the other hand, considering the fixed point of $p \strictif \bot$, we see that $\mathrm{LHC}^\flat \oplus \mathrm{JS} \vdash (\Box\bot \strictif \bot) \to \Box\bot$. Now consider the following model.

$$\begin{tikzcd}
 & c \\
a \arrow[r, rightsquigarrow] & b \arrow[u, rightsquigarrow] \arrow[u, bend right]
\end{tikzcd}$$

This model is upward well founded and gathering. So it satisfies $\mathrm{iGL}_\mathrm{a}^-$. However in node a we do not have $(\Box\bot \strictif \bot) \to \Box\bot$.

JS4

Ergo,

Theorem

$\mathsf{LHC}^{\flat} \oplus \mathsf{JV} = \mathsf{iGL}^{-}_{\mathsf{a}} \nvdash \mathsf{JS}$.

Mysteries

What is the underlying fixed point calculation that unifies JS and JV as being specialisations of it? What is the appropriate natural base theory?

Or are these things illusions? Should we just be content to study the structure of Lewisian fixed point calculations?

The work of Lorenzo Sacchetti and the subsequent work of Taishi Kurahashi and Yuya Okawa suggests this last to be the case. However, I think we should do something to block Sacchetti-style examples. We at least want the fixed points on the box fragment to be those of iGL.

Thank You