Flat Heyting-Lewis Calculus and Algebraic Countermodels

Tadeusz Litak (FAU Erlangen-Nuremberg)
joint lecture with Albert Visser
course *Lewis meets Brouwer: Constructive strict implication*

ESSLLI 2021, day IV

What have we seen Mon and Tue?

Axioms and rules of the minimal system $\mathsf{HLC}^\flat$:

Those of IPC plus:

$$\mathsf{Tra} \quad (\varphi \prec\!\!\!\!\!\!\;\;\supset\; \psi) \wedge (\psi \dashv\!\!\!\!\supset \chi) \to (\varphi \dashv\!\!\!\!\supset \chi)$$

"syntactic transitivity" of $\dashv\!\!\!\!\supset$

$$\mathsf{K_a} \quad (\varphi \dashv\!\!\!\!\supset \psi) \wedge (\varphi \dashv\!\!\!\!\supset \chi) \to (\varphi \dashv\!\!\!\!\supset (\psi \wedge \chi))$$

normality=normality in the second coördinate

$$\mathsf{N_a} \ \frac{\varphi \to \psi}{\varphi \dashv\!\!\!\!\supset \psi.}$$

binary generalization of necessitation

not only implies congruentiality, but also anti-monotonicity in the first coördinate

Axioms and rules of the full system $\mathsf{HLC}^\sharp$:

All the axioms and rules of IPC and $\mathsf{HLC}^\flat$ and

$$\mathsf{Di} \quad ((\varphi \dashv\!\!\!\!\supset \chi) \wedge (\psi \dashv\!\!\!\!\supset \chi)) \to ((\varphi \vee \psi) \dashv\!\!\!\!\supset \chi).$$

should implication be anti-multiplicative in the first coördinate?

Running this axiom system via the AAL machinery yields:

The class of flat Heyting-Lewis algebras:
Heyting algebras plus:

$$\mathsf{CTra} \quad (\varphi \dashv\!\!3\, \psi) \wedge (\psi \dashv\!\!3\, \chi) \leq \varphi \dashv\!\!3\, \chi$$

$$\mathsf{CK_a} \quad (\varphi \dashv\!\!3\, \psi) \wedge (\varphi \dashv\!\!3\, \chi) = \varphi \dashv\!\!3\, (\psi \wedge \chi)$$

$$\mathsf{CId} \qquad \varphi \dashv\!\!3\, \varphi = \top$$

The class of sharp Heyting-Lewis algebras:

all the equalities above plus

$$\mathsf{CDi} \quad (\varphi \dashv\!\!3\, \chi) \wedge (\psi \dashv\!\!3\, \chi) = (\varphi \vee \psi) \dashv\!\!3\, \chi.$$

In fact, we haven't discussed algebras properly.
We will do so today.

- We haven't seen any examples of sharp Lewis applications of SQEMA via GMT either

  Will be fixed later today

- We haven't seen any examples of sharp Lewis applications of SQEMA via GMT either

  Will be fixed later today

- But even having a complete Kripke semantics for the sharp variant of a given logic does clarify which derivations essentially require sharpness (i.e., Di)

- We haven't seen any examples of sharp Lewis applications of SQEMA via GMT either

  Will be fixed later today

- But even having a complete Kripke semantics for the sharp variant of a given logic does clarify which derivations essentially require sharpness (i.e., Di)
- Or, indeed, if Di itself is derivable. It is forced in each and every Kripke structure

- We haven't seen any examples of sharp Lewis applications of SQEMA via GMT either

  Will be fixed later today
- But even having a complete Kripke semantics for the sharp variant of a given logic does clarify which derivations essentially require sharpness (i.e., Di)
- Or, indeed, if Di itself is derivable. It is forced in each and every Kripke structure
- Here are some examples of such questions

Example 1: Collapse to $\Box$?

- In the first lecture, we have shown that over $\mathsf{HLC}^\sharp$, i.e., in the presence of Di

$$\varphi \prec\!\!\!\!- \psi$$

is equivalent to

$$(\varphi \vee \neg\varphi) \prec\!\!\!\!- (\varphi \to \psi).$$

Example 1: Collapse to $\Box$?

- In the first lecture, we have shown that over $\mathsf{HLC}^\sharp$, i.e., in the presence of Di

$$\varphi \multimap \psi$$

is equivalent to

$$(\varphi \vee \neg\varphi) \multimap (\varphi \to \psi).$$

- How can we show this equivalent does not hold over Di?

Example 1: Collapse to $\Box$?

- In the first lecture, we have shown that over $\mathsf{HLC}^\sharp$, i.e., in the presence of Di

$$\varphi \dashv\!\!\!\!\exists\ \psi$$

is equivalent to

$$(\varphi \vee \neg\varphi) \dashv\!\!\!\!\exists\ (\varphi \to \psi).$$

- How can we show this equivalent does not hold over Di?
- Albert: via arithmetical interpretations, of course

Example 1: Collapse to $\Box$?

- In the first lecture, we have shown that over $\mathsf{HLC}^\sharp$, i.e., in the presence of Di
$$\varphi \rhd \psi$$
is equivalent to
$$(\varphi \vee \neg\varphi) \rhd (\varphi \to \psi).$$
- How can we show this equivalent does not hold over Di?
- Albert: via arithmetical interpretations, of course
- Otherwise, the study of classical interpretability logics would collapse to the study of their provability fragments

Example 1: Collapse to $\Box$?

- In the first lecture, we have shown that over $\mathsf{HLC}^\sharp$, i.e., in the presence of Di

$$\varphi \rhd \psi$$

is equivalent to

$$(\varphi \vee \neg\varphi) \rhd (\varphi \to \psi).$$

- How can we show this equivalent does not hold over Di?
- Albert: via arithmetical interpretations, of course
- Otherwise, the study of classical interpretability logics would collapse to the study of their provability fragments
- But what if you want to have a simple finite countermodel?

Example 2a: Outrageous Arrows, No Choice

- Consider the axioms of (the Curry-Howard) logic of Haskell/Hughes arrows $\mathsf{S}^\flat$:

Example 2a: Outrageous Arrows, No Choice

- Consider the axioms of (the Curry-Howard) logic of Haskell/Hughes arrows $\mathsf{S}^\flat$:
  - the baseline flat calculus $\mathsf{HLC}^\flat$

Example 2a: Outrageous Arrows, No Choice

- Consider the axioms of (the Curry-Howard) logic of Haskell/Hughes arrows $\mathsf{S}^\flat$:
  - the baseline flat calculus $\mathsf{HLC}^\flat$
  - the strength axiom

$$\varphi \to \Box\varphi \qquad \text{or} \qquad (\varphi \to \psi) \to (\varphi \multimap \psi)$$

Example 2a: Outrageous Arrows, No Choice

- Consider the axioms of (the Curry-Howard) logic of Haskell/Hughes arrows $\mathsf{S}^\flat$:
  - the baseline flat calculus $\mathsf{HLC}^\flat$
  - the strength axiom

$$\varphi \to \Box\varphi \qquad \text{or} \qquad (\varphi \to \psi) \to (\varphi \prec\!\!\!\to \psi)$$

- Is there a finite countermodel showing that Choice (i.e., Di) isn't derivable?

Haskell arrows as proposed by John Hughes

```
class Arrow a where
  arr    :: (b -> c) -> a b c
  (>>>)  :: a b c ->  a c d -> a b d
  first  :: a b c ->  a (b, d) (c, d)
```

$$\mathsf{S_a} \qquad (\beta \to \gamma) \to (\beta \multimap \gamma)$$

$$\mathsf{Tra} \quad (\beta \multimap \gamma) \wedge (\gamma \multimap \delta) \to (\beta \multimap \delta)$$

$$\mathsf{K'_a} \qquad (\beta \multimap \gamma) \to ((\beta \wedge \delta) \multimap (\gamma \wedge \delta))$$

Example 2b: Monads Are Promiscuous with Choice

- For contrast, extend $\mathsf{S}^\flat$ with your preferred axioms for monads

$$\Box\varphi \multimap \Box\varphi \quad \text{or} \quad ((\beta \multimap \gamma)\wedge\beta) \multimap \gamma \quad \text{or} \quad (\varphi \to \Box\psi) \to (\varphi \multimap \psi)$$

Example 2b: Monads Are Promiscuous with Choice

- For contrast, extend $\mathsf{S}^\flat$ with your preferred axioms for monads

$$\Box\varphi \multimap \Box\varphi \quad \text{or} \quad ((\beta \multimap \gamma)\wedge\beta) \multimap \gamma \quad \text{or} \quad (\varphi \to \Box\psi) \to (\varphi \multimap \psi)$$

- All equivalent above $\mathsf{S}^\flat$

Example 2b: Monads Are Promiscuous with Choice

- For contrast, extend $\mathsf{S}^\flat$ with your preferred axioms for monads

  $\Box\varphi \multimap \Box\varphi \quad \text{or} \quad ((\beta \multimap \gamma)\wedge\beta) \multimap \gamma \quad \text{or} \quad (\varphi \to \Box\psi) \to (\varphi \multimap \psi)$

- All equivalent above $\mathsf{S}^\flat$
- Di is derivable now, as $\varphi \multimap \psi$ is decomposable as $(\varphi \to \Box\psi)$

Monads

(recall they allow decomposing $\beta \multimap \gamma$ as $\beta \to \Box\gamma$)

```
class Arrow  => ArrowApply a where
app :: a (a b c, b) c
```

$$\mathsf{App_a} \qquad ((\beta \multimap \gamma) \wedge \beta) \multimap \gamma$$

Recall it's precisely Lewis's $\mathsf{B7}$!
The only $\mathsf{S2}$ axiom underivable in $\mathsf{HLC}^\sharp$...

...actually, can you see what is its corresponding semantic condition?

Example 3a: Strange Siblings of Transitivity

- Recall the transitivity axiom for $\sqsubset$

$$\mathsf{P} \qquad (\varphi \multimap \psi) \to \Box(\varphi \multimap \psi)$$

Example 3a: Strange Siblings of Transitivity

- Recall the transitivity axiom for $\sqsubset$

$$\mathsf{P} \qquad (\varphi \dashv\!\!\!\!\mathsf{3}\, \psi) \to \Box(\varphi \dashv\!\!\!\!\mathsf{3}\, \psi)$$

- It plays a rôle in arithmetical interpretations. It also has some weird siblings:

Example 3a: Strange Siblings of Transitivity

- Recall the transitivity axiom for $\sqsubset$

$$\mathsf{P} \qquad (\varphi \dashv\!3\, \psi) \to \Box(\varphi \dashv\!3\, \psi)$$

- It plays a rôle in arithmetical interpretations. It also has some weird siblings:
- $\mathsf{4}_\Box \qquad \Box\varphi \to \Box\Box\varphi$

Example 3a: Strange Siblings of Transitivity

- Recall the transitivity axiom for $\sqsubset$

$$\mathsf{P} \qquad (\varphi \dashv\!3\, \psi) \to \Box(\varphi \dashv\!3\, \psi)$$

- It plays a rôle in arithmetical interpretations. It also has some weird siblings:
- $\mathsf{4}_{\Box} \qquad \Box\varphi \to \Box\Box\varphi$
- $\mathsf{4}_{\mathsf{a}} \qquad \varphi \dashv\!3\, \Box\varphi$

Example 3a: Strange Siblings of Transitivity

- Recall the transitivity axiom for $\sqsubset$

$$\mathsf{P} \qquad (\varphi \dashv\!\!3\, \psi) \to \Box(\varphi \dashv\!\!3\, \psi)$$

- It plays a rôle in arithmetical interpretations. It also has some weird siblings:
- $4_\Box$ $\qquad \Box\varphi \to \Box\Box\varphi$
- 4_{a} $\qquad \varphi \dashv\!\!3\, \Box\varphi$
- 4_{a}° $\qquad (\varphi \dashv\!\!3\, \psi) \to (\varphi \dashv\!\!3\, (\varphi \dashv\!\!3\, \psi))$

Example 3a: Strange Siblings of Transitivity

- Recall the transitivity axiom for $\sqsubset$

$$\mathsf{P} \qquad (\varphi \dashv\!\!3\, \psi) \to \Box(\varphi \dashv\!\!3\, \psi)$$

- It plays a rôle in arithmetical interpretations. It also has some weird siblings:
- $\mathsf{4}_{\Box} \qquad \Box\varphi \to \Box\Box\varphi$
- $\mathsf{4}_{\mathsf{a}} \qquad \varphi \dashv\!\!3\, \Box\varphi$
- $4^{\circ}_{\mathsf{a}} \qquad (\varphi \dashv\!\!3\, \psi) \to (\varphi \dashv\!\!3\, (\varphi \dashv\!\!3\, \psi))$
- $44^{\circ}_{\mathsf{a}} \quad (\varphi \dashv\!\!3\, (\psi \dashv\!\!3\, \chi)) \to (\varphi \dashv\!\!3\, (\psi \dashv\!\!3\, (\varphi \dashv\!\!3\, (\psi \dashv\!\!3\, \chi))))$.

Example 3a: Strange Siblings of Transitivity

- Recall the transitivity axiom for $\sqsubset$

$$\mathsf{P} \qquad (\varphi \dashv\!\!3\; \psi) \to \Box(\varphi \dashv\!\!3\; \psi)$$

- It plays a rôle in arithmetical interpretations. It also has some weird siblings:
- $\mathsf{4}_\Box \qquad \Box\varphi \to \Box\Box\varphi$
- $\mathsf{4_a} \qquad \varphi \dashv\!\!3\; \Box\varphi$
- $\mathsf{4_a^\circ} \qquad (\varphi \dashv\!\!3\; \psi) \to (\varphi \dashv\!\!3\; (\varphi \dashv\!\!3\; \psi))$
- $\mathsf{44_a^\circ} \quad (\varphi \dashv\!\!3\; (\psi \dashv\!\!3\; \chi)) \to (\varphi \dashv\!\!3\; (\psi \dashv\!\!3\; (\varphi \dashv\!\!3\; (\psi \dashv\!\!3\; \chi))))$.
- Incidentally, note that the S axiom implies all of them!

Example 3a: Strange Siblings of Transitivity

- Recall the transitivity axiom for $\sqsubset$

$$\mathsf{P} \qquad (\varphi \dashv\!\!\!3\ \psi) \to \Box(\varphi \dashv\!\!\!3\ \psi)$$

- It plays a rôle in arithmetical interpretations. It also has some weird siblings:
- $\mathsf{4}_{\Box}$ $\qquad \Box\varphi \to \Box\Box\varphi$
- $\mathsf{4}_{\mathsf{a}}$ $\qquad \varphi \dashv\!\!\!3\ \Box\varphi$
- 4°_{a} $\qquad (\varphi \dashv\!\!\!3\ \psi) \to (\varphi \dashv\!\!\!3\ (\varphi \dashv\!\!\!3\ \psi))$
- 44°_{a} $\quad (\varphi \dashv\!\!\!3\ (\psi \dashv\!\!\!3\ \chi)) \to (\varphi \dashv\!\!\!3\ (\psi \dashv\!\!\!3\ (\varphi \dashv\!\!\!3\ (\psi \dashv\!\!\!3\ \chi))))$.
- Incidentally, note that the S axiom implies all of them!
- Even on Kripke frames, what do they all mean?

Example 3b: Unrelated Transitive Siblings?

- In our paper with Albert underlying his next lecture *Lewisian Fixed Points I: Two Incomparable Constructions* `https://arxiv.org/abs/1905.09450` we showed (Theorem 32) that

$$4^{\circ}_{\mathsf{a}} \vdash^{\sharp} 44^{\circ}_{\mathsf{a}}$$

i.e.,

$$(\varphi \dashv\!\!\!3\, \psi) \to (\varphi \dashv\!\!\!3\, (\varphi \dashv\!\!\!3\, \psi)) \vdash^{\sharp}$$
$$(\varphi \dashv\!\!\!3\, (\psi \dashv\!\!\!3\, \chi)) \to (\varphi \dashv\!\!\!3\, (\psi \dashv\!\!\!3\, (\varphi \dashv\!\!\!3\, (\psi \dashv\!\!\!3\, \chi))))$$

Example 3b: Unrelated Transitive Siblings?

- In our paper with Albert underlying his next lecture *Lewisian Fixed Points I: Two Incomparable Constructions* `https://arxiv.org/abs/1905.09450` we showed (Theorem 32) that

$$4^{\circ}_{\mathsf{a}} \vdash^{\sharp} 44^{\circ}_{\mathsf{a}}$$

i.e.,

$$(\varphi \strictif \psi) \to (\varphi \strictif (\varphi \strictif \psi)) \vdash^{\sharp}$$
$$(\varphi \strictif (\psi \strictif \chi)) \to (\varphi \strictif (\psi \strictif (\varphi \strictif (\psi \strictif \chi))))$$

- But we care mostly for the flat base because arithmetic

Example 3b: Unrelated Transitive Siblings?

- In our paper with Albert underlying his next lecture *Lewisian Fixed Points I: Two Incomparable Constructions* https://arxiv.org/abs/1905.09450 we showed (Theorem 32) that

$$4^{\circ}_{\mathsf{a}} \vdash^{\sharp} 44^{\circ}_{\mathsf{a}}$$

i.e.,

$$(\varphi \multimap \psi) \to (\varphi \multimap (\varphi \multimap \psi)) \vdash^{\sharp}$$
$$(\varphi \multimap (\psi \multimap \chi)) \to (\varphi \multimap (\psi \multimap (\varphi \multimap (\psi \multimap \chi))))$$

- But we care mostly for the flat base because arithmetic
- How do we show that $4^{\circ}_{\mathsf{a}} \nvdash_{\flat} 44^{\circ}_{\mathsf{a}}$?

Flat semantic worlds

- First of all, $\mathsf{HLC}^\flat$ is an extension of Weiss' recent ICK

  Basic intuitionistic conditional logic, JPL 2019

Flat semantic worlds

- First of all, $\mathsf{HLC}^\flat$ is an extension of Weiss' recent ICK

  Basic intuitionistic conditional logic, JPL 2019

- This allows intutionistic variants of Chellas' semantics

  Weiss himself, other developed by my coauthor Sedlár

Flat semantic worlds

- First of all, $\mathsf{HLC}^\flat$ is an extension of Weiss' recent ICK

  Basic intuitionistic conditional logic, JPL 2019

- This allows intutionistic variants of Chellas' semantics

  Weiss himself, other developed by my coauthor Sedlár

- But we can also generalize Routley-Meyer semantics of relevance logics

  instead of ternary relation on states, multi-sorted, with neighbourhoods on one coördinate

Flat semantic worlds

- First of all, $\mathsf{HLC}^\flat$ is an extension of Weiss' recent ICK

  Basic intuitionistic conditional logic, JPL 2019

- This allows intutionistic variants of Chellas' semantics

  Weiss himself, other developed by my coauthor Sedlár

- But we can also generalize Routley-Meyer semantics of relevance logics

  instead of ternary relation on states, multi-sorted, with neighbourhoods on one coördinate

- We can also generalize Veltman semantics that Albert was talking about

  with Igor Sedlár, we show that this generalized Veltman semantics is a functional variant of generalized RM

Flat semantic worlds

- First of all, $\mathsf{HLC}^\flat$ is an extension of Weiss' recent ICK
  Basic intuitionistic conditional logic, JPL 2019
- This allows intutionistic variants of Chellas' semantics
  Weiss himself, other developed by my coauthor Sedlár
- But we can also generalize Routley-Meyer semantics of relevance logics
  instead of ternary relation on states, multi-sorted, with neighbourhoods on one coördinate
- We can also generalize Veltman semantics that Albert was talking about
  with Igor Sedlár, we show that this generalized Veltman semantics is a functional variant of generalized RM
- Starting from IPC, we have semantics discussed by G. Bezhanishvili and W.H. Holliday
  Beth frames, FM frames, Dragalin frames, nuclear frames . . .

But we can do with just algebra only

- Some people are still mislead by Johan van Benthem's provocative "syntax in disguise" quote

  He delegates the blame to some other unspecified people (Grayson?) and claims duality changes the picture

But we can do with just algebra only

- Some people are still mislead by Johan van Benthem's provocative "syntax in disguise" quote

  He delegates the blame to some other unspecified people (Grayson?) and claims duality changes the picture
- My take: syntax always comes with a notion of identity decidable in polynomial time

But we can do with just algebra only

- Some people are still mislead by Johan van Benthem's provocative "syntax in disguise" quote

  He delegates the blame to some other unspecified people (Grayson?) and claims duality changes the picture
- My take: syntax always comes with a notion of identity decidable in polynomial time
- It is computationally trivial to compare two strings

But we can do with just algebra only

- Some people are still mislead by Johan van Benthem's provocative "syntax in disguise" quote

  He delegates the blame to some other unspecified people (Grayson?) and claims duality changes the picture
- My take: syntax always comes with a notion of identity decidable in polynomial time
- It is computationally trivial to compare two strings
- The Lindenbaum-Tarski equivalence relation of meaningful logics almost never satisfies this criterion

But we can do with just algebra only

- Some people are still mislead by Johan van Benthem's provocative "syntax in disguise" quote

  He delegates the blame to some other unspecified people (Grayson?) and claims duality changes the picture
- My take: syntax always comes with a notion of identity decidable in polynomial time
- It is computationally trivial to compare two strings
- The Lindenbaum-Tarski equivalence relation of meaningful logics almost never satisfies this criterion
- Bezhanishvili & Holliday offer another take

This is a legitimate objection if all one means by "giving algebraic semantics" is to translate the axioms of IPC *into equations defining a class of algebras and then observe that* IPC *is sound and complete with respect to such algebras. In this case, soundness and completeness is hardly illuminating. By contrast, it is quite illuminating to know that* IPC *is sound and complete with respect to Heyting algebras defined order-theoretically*

G. Bezhanishvili & W.H. Holliday, *A Semantic Hierarchy for Intuitionistic Logic*

Heyting-Lewis algebras, once again

- Structures of the form $\mathscr{A} = (A, \multimap, \to, \wedge, \vee, \top, \bot)$

Heyting-Lewis algebras, once again

- Structures of the form $\mathscr{A} = (A, \dashv\!3, \to, \wedge, \vee, \top, \bot)$
- The $\dashv\!3$-free reduct $(A, \to, \wedge, \vee, \top, \bot)$ is a Heyting algebra

  Bounded distributive lattice with $\to$ adjoint to/residual of $\wedge$

Heyting-Lewis algebras, once again

- Structures of the form $\mathscr{A} = (A, \dashv\!3, \to, \wedge, \vee, \top, \bot)$
- The $\dashv\!3$-free reduct $(A, \to, \wedge, \vee, \top, \bot)$ is a Heyting algebra

  Bounded distributive lattice with $\to$ adjoint to/residual of $\wedge$
- The laws for $\dashv\!3$ (flat and sharp) already discussed on three different occasions

Heyting-Lewis algebras, once again

- Structures of the form $\mathscr{A} = (A, \multimap\!\!\!\!3, \to, \wedge, \vee, \top, \bot)$
- The $\multimap\!\!\!\!3$-free reduct $(A, \to, \wedge, \vee, \top, \bot)$ is a Heyting algebra
  Bounded distributive lattice with $\to$ adjoint to/residual of $\wedge$
- The laws for $\multimap\!\!\!\!3$ (flat and sharp) already discussed on three different occasions
- A Heyting-Lewis-Kripke frame can be always turned into a Heyting-Lewis algebra

Heyting-Lewis algebras, once again

- Structures of the form $\mathscr{A} = (A, \dashv\!3, \to, \wedge, \vee, \top, \bot)$
- The $\dashv\!3$-free reduct $(A, \to, \wedge, \vee, \top, \bot)$ is a Heyting algebra
  Bounded distributive lattice with $\to$ adjoint to/residual of $\wedge$
- The laws for $\dashv\!3$ (flat and sharp) already discussed on three different occasions
- A Heyting-Lewis-Kripke frame can be always turned into a Heyting-Lewis algebra
- $\mathfrak{F} = (X, \preceq, \sqsubset)$, $\mathfrak{F}^+$ has $\mathit{up}(X, \preceq)$ as its carrier

Heying-Lewis algebras, once again

- Structures of the form $\mathscr{A} = (A, \dashv\!\!3, \to, \wedge, \vee, \top, \bot)$
- The $\dashv\!\!3$-free reduct $(A, \to, \wedge, \vee, \top, \bot)$ is a Heyting algebra
  Bounded distributive lattice with $\to$ adjoint to/residual of $\wedge$
- The laws for $\dashv\!\!3$ (flat and sharp) already discussed on three different occasions
- A Heyting-Lewis-Kripke frame can be always turned into a Heyting-Lewis algebra
- $\mathfrak{F} = (X, \preceq, \sqsubset)$, $\mathfrak{F}^{+}$ has $up(X, \preceq)$ as its carrier
- The lattice part is interpreted set-theoretically

Heyting-Lewis algebras, once again

- Structures of the form $\mathscr{A} = (A, \strictif, \to, \wedge, \vee, \top, \bot)$
- The $\strictif$-free reduct $(A, \to, \wedge, \vee, \top, \bot)$ is a Heyting algebra
  Bounded distributive lattice with $\to$ adjoint to/residual of $\wedge$
- The laws for $\strictif$ (flat and sharp) already discussed on three different occasions
- A Heyting-Lewis-Kripke frame can be always turned into a Heyting-Lewis algebra
- $\mathfrak{F} = (X, \preceq, \sqsubset)$, $\mathfrak{F}^{+}$ has $\mathit{up}(X, \preceq)$ as its carrier
- The lattice part is interpreted set-theoretically
- The two implications: by algebraizing the forcing relation

$$a \underline{\to} b = \{x \in X \mid \text{if } x \preceq y \text{ and } y \in a \text{ then } y \in b\}$$

$$a \underline{\strictif} b = \{x \in X \mid \text{if } x \sqsubset y \text{ and } y \in a \text{ then } y \in b\}$$

Mace4 for counterexamples

- Developed by William Walker McCune (December 17, 1953 – May 2, 2011)

Mace4 for counterexamples

- Developed by William Walker McCune (December 17, 1953 – May 2, 2011)
- Wikipedia: *an American computer scientist and logician working in the fields of automated reasoning, algebra, logic, and formal methods. He was best known for the development of the Otter, Prover9, and Mace4 automated reasoning systems, and the automated proof of the Robbins conjecture using the EQP theorem prover*

Mace4 for counterexamples

- Developed by William Walker McCune (December 17, 1953 – May 2, 2011)
- Wikipedia: *an American computer scientist and logician working in the fields of automated reasoning, algebra, logic, and formal methods. He was best known for the development of the Otter, Prover9, and Mace4 automated reasoning systems, and the automated proof of the Robbins conjecture using the EQP theorem prover*
- MACE stands for *Models And Counter-Examples*

Mace4 for counterexamples

- Developed by William Walker McCune (December 17, 1953 – May 2, 2011)
- Wikipedia: *an American computer scientist and logician working in the fields of automated reasoning, algebra, logic, and formal methods. He was best known for the development of the Otter, Prover9, and Mace4 automated reasoning systems, and the automated proof of the Robbins conjecture using the EQP theorem prover*
- MACE stands for *Models And Counter-Examples*
- Available online (together with Prover9) and still perfectly usable

Mace4 for counterexamples

- Developed by William Walker McCune (December 17, 1953 – May 2, 2011)
- Wikipedia: *an American computer scientist and logician working in the fields of automated reasoning, algebra, logic, and formal methods. He was best known for the development of the Otter, Prover9, and Mace4 automated reasoning systems, and the automated proof of the Robbins conjecture using the EQP theorem prover*
- MACE stands for *Models And Counter-Examples*
- Available online (together with Prover9) and still perfectly usable
- `https://www.cs.unm.edu/~mccune/mace4/`

  See also `https://github.com/theoremprover-museum`

Mace4 for counterexamples

- Developed by William Walker McCune (December 17, 1953 – May 2, 2011)
- Wikipedia: *an American computer scientist and logician working in the fields of automated reasoning, algebra, logic, and formal methods. He was best known for the development of the Otter, Prover9, and Mace4 automated reasoning systems, and the automated proof of the Robbins conjecture using the EQP theorem prover*
- MACE stands for *Models And Counter-Examples*
- Available online (together with Prover9) and still perfectly usable
- `https://www.cs.unm.edu/~mccune/mace4/`
  See also `https://github.com/theoremprover-museum`
- Let's unrelate siblings of transitivity with Mace4: $4^{\circ}_{\mathsf{a}} \nvdash_{\flat} 44^{\circ}_{\mathsf{a}}$

Mace4 input

```
formulas(assumptions).
x ^ (y ^ z) = (x ^ y) ^ z.
x ^ x = x.
x ^ y = y ^ x.
x ^ 1 = x.
x ^ 0 = 0.
x * x = 1.
x ^ (x * y) = x ^ y.
y ^ (x * y) = y.
x * (y ^ z) = (x * y) ^ (x * z).
(x + y) ^ (x + z) = x + (y ^ z).
((x + y) ^ (y + z)) ^ (x + z) = (x + y) ^ (y + z).
x + x = 1.
(x + y) ^ (x + (x + y)) = x + y.
end_of_list.

formulas(goals).
(x + (y + z)) ^ (x + (y + (x + (y + z)))) = x + (y + z).
end_of_list.
```

Mace4 output 1

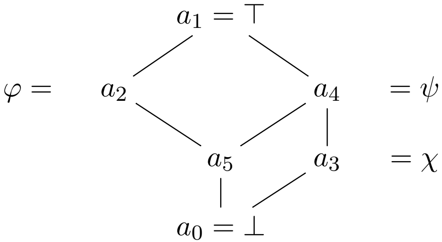
```
formulas(mace4_clauses).
x ^ (y ^ z) = (x ^ y) ^ z.
x ^ x = x.
x ^ y = y ^ x.
x ^ 1 = x.
x ^ 0 = 0.
x * x = 1.
x ^ (x * y) = x ^ y.
x ^ (y * x) = x.
x * (y ^ z) = (x * y) ^ (x * z).
(x + y) ^ (x + z) = x + (y ^ z).
((x + y) ^ (y + z)) ^ (x + z) = (x + y) ^ (y + z).
x + x = 1.
(x + y) ^ (x + (x + y)) = x + y.
(c1 + (c2 + c3)) ^ (c1 + (c2 + (c1 + (c2 + c3)))) != c1 + (c2 + c3).
end_of_list.
```

Mace4 output 2

```
interpretation( 6, [number=1, seconds=8], [

        function(c1, [ 2 ]),

        function(c2, [ 4 ]),

        function(c3, [ 3 ]),

        function(*(_,_), [
   1, 1, 1, 1, 1, 1,
   0, 1, 2, 3, 4, 5,
   3, 1, 1, 3, 4, 4,
   2, 1, 2, 1, 1, 2,
   0, 1, 2, 3, 1, 2,
   3, 1, 1, 3, 1, 1 ]),
```

Mace4 output 3

```
        function(+(_,_), [
   1, 1, 1, 1, 1, 1,
   0, 1, 0, 0, 0, 0,
   5, 1, 1, 5, 2, 2,
   0, 1, 0, 1, 1, 0,
   0, 1, 0, 4, 1, 0,
   4, 1, 1, 4, 1, 1 ]),

        function(^(_,_), [
   0, 0, 0, 0, 0, 0,
   0, 1, 2, 3, 4, 5,
   0, 2, 2, 0, 5, 5,
   0, 3, 0, 3, 3, 0,
   0, 4, 5, 3, 4, 5,
   0, 5, 5, 0, 5, 5 ])
]).
```



$\ni$	$\bot$	$\top$	a_2	a_3	a_4	a_5
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$	$\bot$
a_2	a_5	$\top$	$\top$	a_5	a_2	a_2
a_3	$\bot$	$\top$	$\bot$	$\top$	$\top$	$\bot$
a_4	$\bot$	$\top$	$\bot$	a_4	$\top$	$\bot$
a_5	a_4	$\top$	$\top$	a_4	$\top$	$\top$

Advanced Exercise: Use Mace4 to find counterexamples for other derivations that fail in the flat setting!

Do get in touch with me if you're interested, but not sure if you're doing it right

Still More Advanced: Use a more recent tool of your choice

Do tell me how it went

The use of SQEMA via GMT

- Here is the (refined, not brutal) GMT translation of P

$$\Box_{\mathsf{m}}(\Box_{\mathsf{i}}p \to \Box_{\mathsf{i}}q) \to \Box_{\mathsf{m}}\Box_{\mathsf{m}}(\Box_{\mathsf{i}}p \to \Box_{\mathsf{i}}q).$$

The use of SQEMA via GMT

- Here is the (refined, not brutal) GMT translation of P

$$\Box_{\mathsf{m}}(\Box_{\mathsf{i}} p \to \Box_{\mathsf{i}} q) \to \Box_{\mathsf{m}}\Box_{\mathsf{m}}(\Box_{\mathsf{i}} p \to \Box_{\mathsf{i}} q).$$

- Try to run it through SQEMA:
  `http://dimiter.slavi.biz/sqema/`

The use of SQEMA via GMT

- Here is the (refined, not brutal) GMT translation of P

$$\Box_{\mathsf{m}}(\Box_{\mathsf{i}} p \to \Box_{\mathsf{i}} q) \to \Box_{\mathsf{m}}\Box_{\mathsf{m}}(\Box_{\mathsf{i}} p \to \Box_{\mathsf{i}} q).$$

- Try to run it through SQEMA:
  `http://dimiter.slavi.biz/sqema/`

- It does yield a FO counterpart, but not equivalent to transitivity over arbitrary (quasi-ordered) frames for S4HL

The use of SQEMA via GMT

- Here is the (refined, not brutal) GMT translation of P

$$\Box_{\mathsf{m}}(\Box_{\mathsf{i}} p \to \Box_{\mathsf{i}} q) \to \Box_{\mathsf{m}}\Box_{\mathsf{m}}(\Box_{\mathsf{i}} p \to \Box_{\mathsf{i}} q).$$

- Try to run it through SQEMA:
`http://dimiter.slavi.biz/sqema/`

- It does yield a FO counterpart, but not equivalent to transitivity over arbitrary (quasi-ordered) frames for S4HL
- One needs to transform manually the FO formula in question using the assumption of antisymmetry

The use of SQEMA via GMT

- Here is the (refined, not brutal) GMT translation of P

$$\Box_{\mathsf{m}}(\Box_{\mathsf{i}} p \to \Box_{\mathsf{i}} q) \to \Box_{\mathsf{m}}\Box_{\mathsf{m}}(\Box_{\mathsf{i}} p \to \Box_{\mathsf{i}} q).$$

- Try to run it through SQEMA:
`http://dimiter.slavi.biz/sqema/`

- It does yield a FO counterpart, but not equivalent to transitivity over arbitrary (quasi-ordered) frames for S4HL
- One needs to transform manually the FO formula in question using the assumption of antisymmetry
- Alternatively, over partial orders, the following rule is admissible:

$$\text{from } \varphi(\Box_{\mathsf{i}} p \to \Box_{\mathsf{i}} q), \text{ derive } \varphi(r),$$
$$\text{where } p \text{ and } q \text{ are fresh for } \varphi(r)$$

The use of SQEMA via GMT

- Here is the (refined, not brutal) GMT translation of P

$$\Box_{\mathsf{m}}(\Box_{\mathsf{i}} p \to \Box_{\mathsf{i}} q) \to \Box_{\mathsf{m}}\Box_{\mathsf{m}}(\Box_{\mathsf{i}} p \to \Box_{\mathsf{i}} q).$$

- Try to run it through SQEMA:
`http://dimiter.slavi.biz/sqema/`

- It does yield a FO counterpart, but not equivalent to transitivity over arbitrary (quasi-ordered) frames for S4HL
- One needs to transform manually the FO formula in question using the assumption of antisymmetry
- Alternatively, over partial orders, the following rule is admissible:

$$\text{from } \varphi(\Box_{\mathsf{i}} p \to \Box_{\mathsf{i}} q), \text{ derive } \varphi(r),$$
$$\text{where } p \text{ and } q \text{ are fresh for } \varphi(r)$$

- This yiels

$$\Box_{\mathsf{m}} r \to \Box_{\mathsf{m}}\Box_{\mathsf{m}} r$$

The use of SQEMA via GMT

- Here is the (refined, not brutal) GMT translation of P

$$\Box_{\mathsf{m}}(\Box_{\mathsf{i}}p \to \Box_{\mathsf{i}}q) \to \Box_{\mathsf{m}}\Box_{\mathsf{m}}(\Box_{\mathsf{i}}p \to \Box_{\mathsf{i}}q).$$

- Try to run it through SQEMA:
`http://dimiter.slavi.biz/sqema/`

- It does yield a FO counterpart, but not equivalent to transitivity over arbitrary (quasi-ordered) frames for S4HL
- One needs to transform manually the FO formula in question using the assumption of antisymmetry
- Alternatively, over partial orders, the following rule is admissible:

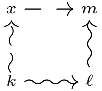
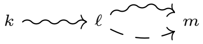
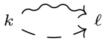
$$\text{from } \varphi(\Box_{\mathsf{i}}p \to \Box_{\mathsf{i}}q), \text{ derive } \varphi(r),$$
$$\text{where } p \text{ and } q \text{ are fresh for } \varphi(r)$$

- This yiels

$$\Box_{\mathsf{m}}r \to \Box_{\mathsf{m}}\Box_{\mathsf{m}}r$$

- Just transitivity of R_{m}!

Some FO counterparts

4	$\Box\,\varphi \to \Box\,\Box\,\varphi$ semi-transitivity	$k \sqsubset \ell \sqsubset m \Rightarrow$ $\exists x. k \sqsubset x \preceq m$	
4_{a}	$\varphi \strictif \Box\,\varphi$ gathering	$k \sqsubset \ell \sqsubset m \Rightarrow \ell \preceq m$	
4°_{a}	$(\varphi \strictif \psi) \to (\varphi \strictif (\varphi \strictif \psi))$ gather-transitivity	$x \sqsubset y \sqsubset z \Rightarrow$ $x \sqsubset z$ or $y \preceq z$	
S	$\varphi \to \Box\,\varphi$ strength	$k \sqsubset \ell \Rightarrow k \preceq \ell$	

Exercise: Compute them using the online implementation + transformations from the previous slide!
Exercise: Try a chosen axioms for monads (arrows with apply)
Advanced Exercise: Do these computations fully manually, if you know Sahlqvist/SQEMA

Coda: a few words about extension stability

- You learned from Albert yesterday that provability/interpretability/preservativity principles are not stable under expansions with additional axioms

Coda: a few words about extension stability

- You learned from Albert yesterday that provability/interpretability/preservativity principles are not stable under expansions with additional axioms
- Those that are, are called extension stable

  Interestingly, fixpoint principles discussed tomorrow do have this property

Coda: a few words about extension stability

- You learned from Albert yesterday that provability/interpretability/preservativity principles are not stable under expansions with additional axioms
- Those that are, are called extension stable

  Interestingly, fixpoint principles discussed tomorrow do have this property
- An algebraic perspective shows that extension stability is a variant of subframe property which we needed on day 2

Coda: a few words about extension stability

- You learned from Albert yesterday that provability/interpretability/preservativity principles are not stable under expansions with additional axioms
- Those that are, are called extension stable

  Interestingly, fixpoint principles discussed tomorrow do have this property
- An algebraic perspective shows that extension stability is a variant of subframe property which we needed on day 2
- Thus, Di (Kripkeanity) is not subframe!

Coda: a few words about extension stability

- You learned from Albert yesterday that provability/interpretability/preservativity principles are not stable under expansions with additional axioms
- Those that are, are called extension stable

  Interestingly, fixpoint principles discussed tomorrow do have this property
- An algebraic perspective shows that extension stability is a variant of subframe property which we needed on day 2
- Thus, Di (Kripkeanity) is not subframe!
- We need one of alternative state-based semantics to get a deeper insight into this claim . . .