# Gödel-McKinsey-Tarski and Blok-Esakia for (sharp) Heyting-Lewis Calculus

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joint lecture with Albert Visser
course *Lewis meets Brouwer: Constructive strict implication*

ESSLLI 2021, part II

## What have we seen last time?

Axioms and rules of the minimal system $\mathsf{HLC}^\flat$:

Those of $\mathsf{IPC}$ plus:

$$\mathsf{Tra} \quad (\varphi \dashv\!3\, \psi) \wedge (\psi \dashv\!3\, \chi) \to (\varphi \dashv\!3\, \chi)$$

"syntactic transitivity" of $\dashv\!3$

$$\mathsf{K_a} \quad (\varphi \dashv\!3\, \psi) \wedge (\varphi \dashv\!3\, \chi) \to (\varphi \dashv\!3\, (\psi \wedge \chi))$$

normality=normality in the second coördinate

$$\mathsf{N_a} \ \frac{\varphi \to \psi}{\varphi \dashv\!3\, \psi.}$$

binary generalization of necessitation

not only implies congruentiality, but also anti-monotonicity in the first coördinate

Axioms and rules of the full system $\mathsf{HLC}^\sharp$:

All the axioms and rules of $\mathsf{IPC}$ and $\mathsf{HLC}^\flat$ and

$$\mathsf{Di} \quad ((\varphi \dashv\!3\, \chi) \wedge (\psi \dashv\!3\, \chi)) \to ((\varphi \vee \psi) \dashv\!3\, \chi).$$

should implication be anti-multiplicative in the first coördinate?

Running this axiom system via the AAL machinery yields:

Heyting algebras plus:

$$\mathsf{CTra} \quad (\varphi \mathrel{-\!\!3} \psi) \wedge (\psi \mathrel{-\!\!3} \chi) \leq \varphi \mathrel{-\!\!3} \chi$$

$$\mathsf{CK_a} \quad (\varphi \mathrel{-\!\!3} \psi) \wedge (\varphi \mathrel{-\!\!3} \chi) = \varphi \mathrel{-\!\!3} (\psi \wedge \chi)$$

$$\mathsf{CId} \qquad \varphi \mathrel{-\!\!3} \varphi = \top$$

The class of Heyting-Lewis algebras:

all the equalities above plus

$$\mathsf{CDi} \quad (\varphi \mathrel{-\!\!3} \chi) \wedge (\psi \mathrel{-\!\!3} \chi) = (\varphi \vee \psi) \mathrel{-\!\!3} \chi.$$

The $\mathrel{-\!\!3}$-free reduct: Heyting algebras
The $\to$-free reduct:

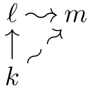
weak Heyting algebras of Celani and Jansana

fusion, fibring or dovetailing along the shared bounded lattice reduct

# Kripke semantics of HLC$^\sharp$

- Nonempty set of worlds/states $X$
- Two relations:
  - Intuitionistic poset relation $\preceq\ \subseteq X \times X$, drawn as $\rightarrow$;
  - Modal relation $\sqsubset\ \subseteq X \times X$, drawn as $\rightsquigarrow$.
  - These relations satisfy precomposition/prefixing:

$$\preceq \cdot \sqsubset \quad \subseteq \quad \sqsubset$$




$\longrightarrow$ intuitionistic $\preceq$

$\rightsquigarrow$ modal $\sqsubset$

- A valuation $V$ sends propositional atoms to $\preceq$-upward closed sets $up(X, \preceq)$

# Semantics of propositional connectives

- Semantics for $\wedge$, $\vee$, $\top$ and $\bot$: Tarskian/boolean clauses locally at a given state
- Semantics for $\rightarrow$:

$$X, V, w \Vdash \varphi \rightarrow \psi \text{ if for any } v \succeq w, v \Vdash \varphi \text{ implies } v \Vdash \psi$$

- Semantics for $⥽$:

$$X, V, w \Vdash \varphi ⥽ \psi \text{ if for any } v \sqsupset w, v \Vdash \varphi \text{ implies } v \Vdash \psi$$

- Global satisfaction and validity defined as usual
- Exercise Show semantics for $\Box$:
  $X, V, w \Vdash \Box\varphi$ if for any $v \sqsupset w$, $v \Vdash \varphi$
- Exercise: show that denotations of all connectives are upward closed

  For $⥽$ this is equivalent to the prefixing condition from the previous slide!
- Exercise: show that all axioms of $\mathsf{HLC}^\sharp$, in particular $\mathsf{Di}$, are valid

$$\mathsf{Tra} \quad (\varphi \strictif \psi) \wedge (\psi \strictif \chi) \to (\varphi \strictif \chi)$$

Assume that (a) $w \Vdash \varphi \strictif \psi$, (b) $w \Vdash \psi \strictif \chi$, (c) $v \sqsupset w$, and (d) $v \Vdash \varphi$. We have that (a) , (c) and (d) yield (e) $v \Vdash \psi$ and then (b) , (c) and (e) yield $v \Vdash \chi$

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Assume that (a) $w \Vdash \varphi \strictif \psi$, (b) $w \Vdash \varphi \strictif \chi$, (c) $v \sqsupset w$, and (d) $v \Vdash \varphi$. Then (a) , (c) and (d) yield (e) $v \Vdash \psi$ and (b) , (c) and (d) yield (f) $v \Vdash \chi$. From (e) and (f) , we infer $v \Vdash \psi \wedge \chi$

$$\mathsf{N_a} \ \frac{\varphi \to \psi}{\varphi \strictif \psi}$$

Assume that (a) $\varphi \to \psi$ is globally forced (b) $v \sqsupset w$, and (c) $v \Vdash \varphi$. But then just (a) and (c) yield that $v \Vdash \psi$

$$\mathsf{Di} \quad ((\varphi \strictif \chi) \wedge (\psi \strictif \chi)) \to ((\varphi \vee \psi) \strictif \chi).$$

Assume that (a) $w \Vdash \varphi \strictif \chi$, (b) $w \Vdash \psi \strictif \chi$, (c) $v \sqsupset w$, and (d) $v \Vdash \varphi \vee \psi$. By the satisfaction clause for $\vee$, this means that either $v \Vdash \varphi$ or $v \Vdash \psi$. Split cases and use either (a) or (b) .

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- If so, can we always make countermodel finite, i.e., do we have the finite model property?

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  Note that for a finitely axiomatizable logic, the finite model property implies decidability
- And do we have strong completeness, i.e., completeness for theories?

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- An example: a countermodel for $p \vee \neg p$ and $\neg\neg p \to p$:

$$\begin{array}{c} \ell \Vdash p \\ \uparrow \\ k \end{array}$$

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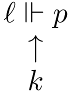
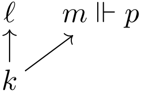



- A countermodel for $\neg p \vee \neg\neg p$:



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- Is there a systematic way of deriving such completeness and correspondence results for suitably large classes of axioms?

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- Several important classes of formulas for which the fmp holds (for Sahlqvist, does not hold automatically!)
- One such important syntactic class: uniform formulas (Fine)

  every occurrence of every atom within the scope of the same number of boxes

# Subframeness for FMP

- An even more important class is defined semantically: transitive subframe logics
- A modal logic $\Lambda$ is subframe if whenever
  - $(X, \sqsubset) \Vdash \Lambda$ and
  - $S \subseteq X$

  then $(S, \sqsubset|_{S \times S}) \Vdash \Lambda$
- If the class of frames for $\Lambda$ is defined by a FO formula $\varphi$, its subframeness is equivalent to $\varphi$
- Fine: transitive subframe logics have the fmp

# An overview of GMT, Blok-Esakia, and Wolter-Zakharyaschev in the unary case

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$$\mathcal{L}_{\mathsf{i,m}} \quad \varphi, \psi ::= \top \mid \bot \mid p \mid \varphi \to \psi \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \Box_{\mathsf{i}}\varphi \mid \Box_{\mathsf{m}}\varphi$$

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- The brutal Gödel-(McKinsey-Tarski) translation for $\mathcal{L}_\Box$:

$$t_\Box^{\mathsf{bru}}(\Box\varphi) := \Box_{\mathsf{i}}\Box_{\mathsf{m}}(t_\Box^{\mathsf{bru}}\varphi)$$

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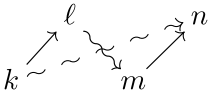
- Each such interval has a maximal element, obtained with the help of the Grzegorczyk axiom for $\Box$ and

$$\mathsf{mix} \qquad \Box_{\mathsf{m}}\varphi \to \Box_{\mathsf{i}}\Box_{\mathsf{m}}\Box_{\mathsf{i}}\varphi$$

Recall mix/brilliancy :

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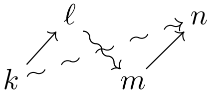


Denote as $\mathsf{S4Mix}$ the extension of $\mathsf{S4K}$ with $\mathsf{mix}$

- We can now refine the translation:

$$t_\Box^{\mathsf{mix}}(\Box\varphi) := \Box_{\mathsf{m}}(t_\Box^{\mathsf{mix}}\varphi)$$

We can also optimize modulo $\mathsf{S4}$:
dropping $\Box_{\mathsf{i}}$ in front of $\wedge$, $\vee$, $\top$ and $\bot$.

# The Grzegorczyk axiom

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- This version implies reflexivity and transitivity, i.e., $\mathsf{S4}$
- But it also implies (weak) Noetherianity: the lack of (strictly) infinite ascending chains and proper clusters

- The translation reflects decidability, completeness, fmp. Above mix, it also reflects canonicity

  enough to find one S4Mix-counterpart with the desired property!

- To establish such results for extensions of S4Mix, one can use classical modal metatheory

  e.g., the Sahlqvist/SQEMA algorithm for canonicity and completeness

- W & Z showed this using a suitable notion of "descriptive frames" (equivalent to an Esakia-style duality)
- As a by-product, they obtained a variant of the Blok-Esakia theorem:

*the lattice of those extensions of* S4Mix
*that include the Grzegorczyk axiom*
*is isomorphic to*
*the lattice of all intuitionistic unimodal logics with a normal box*

# Our job today

- Extending the Gödel-(McKinsey-Tarski) translation to $\mathcal{L}_{\multimap}$

$$t^{\mathsf{bru}}_{\multimap}(\varphi \multimap \psi) := \Box_{\mathsf{i}}\Box_{\mathsf{m}}(t^{\mathsf{bru}}_{\multimap}\varphi \to t^{\mathsf{bru}}_{\multimap}\psi)$$

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- Extending the Gödel-(McKinsey-Tarski) translation to $\mathcal{L}_{\prec}$

$$t_{\prec}^{\mathsf{bru}}(\varphi \prec \psi) := \Box_{\mathsf{i}}\Box_{\mathsf{m}}(t_{\prec}^{\mathsf{bru}}\varphi \rightarrow t_{\prec}^{\mathsf{bru}}\psi)$$

- Obviously, one needs to replace $\mathsf{mix}$ with

$$\mathsf{HL} \quad \Box_{\mathsf{m}}\varphi \rightarrow \Box_{\mathsf{i}}\Box_{\mathsf{m}}\varphi$$

# Our job today

- Extending the Gödel-(McKinsey-Tarski) translation to $\mathcal{L}_{\multimapinv}$

$$t^{\mathsf{bru}}_{\multimapinv}(\varphi \multimapinv \psi) := \Box_{\mathsf{i}}\Box_{\mathsf{m}}(t^{\mathsf{bru}}_{\multimapinv}\varphi \to t^{\mathsf{bru}}_{\multimapinv}\psi)$$

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- Apart from this, everything works, yielding even a suitable variant of the Blok-Esakia Theorem

# Our job today

- Extending the Gödel-(McKinsey-Tarski) translation to $\mathcal{L}_{\multimap}$

$$t_{\multimap}^{\mathsf{bru}}(\varphi \multimap \psi) := \Box_{\mathsf{i}}\Box_{\mathsf{m}}(t_{\multimap}^{\mathsf{bru}}\varphi \to t_{\multimap}^{\mathsf{bru}}\psi)$$

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- Apart from this, everything works, yielding even a suitable variant of the Blok-Esakia Theorem
- Particularly nice in the presence of

$$\mathsf{P} \qquad (\varphi \multimap \psi) \to \Box(\varphi \multimap \psi)$$

ensuring transitivity of the modal relation $\sqsubset$ $(\rightsquigarrow)$

# Our job today

- Extending the Gödel-(McKinsey-Tarski) translation to $\mathcal{L}_{-\!3}$

$$t^{\mathsf{bru}}_{-\!3}(\varphi \mathrel{-\!3} \psi) := \Box_{\mathsf{i}}\Box_{\mathsf{m}}(t^{\mathsf{bru}}_{-\!3}\varphi \to t^{\mathsf{bru}}_{-\!3}\psi)$$

- Obviously, one needs to replace mix with

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- Apart from this, everything works, yielding even a suitable variant of the Blok-Esakia Theorem
- Particularly nice in the presence of

$$\mathsf{P} \qquad (\varphi \mathrel{-\!3} \psi) \to \Box(\varphi \mathrel{-\!3} \psi)$$

  ensuring transitivity of the modal relation $\sqsubset$ $(\leadsto)$
- Still better in the presence of strength

- $t_{\strictif}^{\mathsf{bru}}$ embeds faithfully every extension of $\mathsf{HLC}^\flat$ into an interval of extensions of $\mathsf{S4K}$
- Each such interval has a maximal element, obtained with the help of the Grzegorczyk axiom for $\Box_{\mathsf{i}}$ and

$$\mathsf{HL} \quad \Box_{\mathsf{m}}\varphi \to \Box_{\mathsf{i}}\Box_{\mathsf{m}}\varphi$$

Recall prefixing (persistence for $\strictif$):

$$\begin{array}{ccc} \ell & \rightsquigarrow & m \\ \uparrow & \nearrow\!\!\!\sim & \\ k & & \end{array}$$

Denote as $\mathsf{S4HL}$ the extension of $\mathsf{S4K}$ with $\mathsf{mix}$

- We can now refine the translation:

$$t_{\strictif}^{\mathsf{HL}}(\Box\varphi) := \Box_{\mathsf{m}}(t_{\strictif}^{\mathsf{HL}}\varphi)$$

We can also optimize modulo $\mathsf{S4}$:
dropping $\Box_{\mathsf{i}}$ in front of $\wedge$, $\vee$, $\top$ and $\bot$.

# Three maps between classes of logics

- For every $\mathsf{S4K}$-logic $\mathsf{M}$, define

$$\rho\mathsf{M} := \{\varphi \in \mathcal{L}_{\neg\!3} \mid t^{\mathsf{bru}}_{\neg\!3}(\varphi) \in \mathsf{M}\}$$

# Three maps between classes of logics

- For every $\mathsf{S4K}$-logic $\mathsf{M}$, define

$$\rho\mathsf{M} := \{\varphi \in \mathcal{L}_{-\!<} \mid t^{\mathsf{bru}}_{-\!<}(\varphi) \in \mathsf{M}\}$$

- For every $-\!<$-logic $\mathsf{L} = \mathsf{HLC}^{\sharp} \oplus \Gamma$, define

$$\tau\mathsf{L} := (\mathsf{S4} \otimes \mathsf{K}) \oplus t^{\mathsf{bru}}_{-\!<}(\Gamma) \oplus \mathsf{HL}$$

$$\sigma\mathsf{L} := (\mathsf{Grz_i} \otimes \mathsf{K}) \oplus t^{\mathsf{HL}}_{-\!<}(\Gamma) \oplus \mathsf{HL}$$

# Three maps between classes of logics

- For every $\mathsf{S4K}$-logic $\mathsf{M}$, define

$$\rho\mathsf{M} := \{\varphi \in \mathcal{L}_{-\!3} \mid t_{-\!3}^{\mathsf{bru}}(\varphi) \in \mathsf{M}\}$$

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- Each $-\!3$-logic is embeddable by $t$ in any logic $\mathsf{M}$ in the interval

$$[(\mathsf{S4} \otimes \mathsf{K}) \oplus t_{-\!3}^{\mathsf{bru}}(\Gamma), \sigma\mathsf{L}].$$

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- The map $\rho$ preserves decidability, Kripke completeness and the finite model property.

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- The map $\rho$ preserves decidability, Kripke completeness and the finite model property.
- The map $\rho$ preserves canonicity of $\mathsf{S4HL}$-logics.

# Three maps between classes of logics

- For every S4K-logic M, define

$$\rho \mathsf{M} := \{\varphi \in \mathcal{L}_{-\!<} \mid t_{-\!<}^{\mathsf{bru}}(\varphi) \in \mathsf{M}\}$$

- For every $-\!<$-logic $\mathsf{L} = \mathsf{HLC}^\sharp \oplus \Gamma$, define

$$\tau \mathsf{L} := (\mathsf{S4} \otimes \mathsf{K}) \oplus t_{-\!<}^{\mathsf{bru}}(\Gamma) \oplus \mathsf{HL}$$
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# Three maps between classes of logics

- For every $\mathsf{S4K}$-logic $\mathsf{M}$, define

$$\rho\mathsf{M} := \{\varphi \in \mathcal{L}_{\multimap} \mid t^{\mathsf{bru}}_{\multimap}(\varphi) \in \mathsf{M}\}$$

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# Transfer of the fmp

- In order to use such transfer results not just for completeness and canonicity, but also for the fmp, we need such criteria for $\mathcal{L}_{\mathsf{i,m}}$-logics

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- In order to use such transfer results not just for completeness and canonicity, but also for the fmp, we need such criteria for $\mathcal{L}_{\mathsf{i},\mathsf{m}}$-logics
- W & Z provide some results based on the notion of (cofinal) subframe logic when $R_{\mathsf{m}}$ is transitive
- In the absence of $R_{\mathsf{i}}$-clusters, this is ensured by

$$\mathsf{P} \qquad (\varphi \multimap \psi) \to \Box(\varphi \multimap \psi)$$

### Theorem

*Suppose* $\mathsf{M}$ *is a canonical extension of* $\mathsf{S4} \otimes \mathsf{K4}$ *containing* $\mathsf{HL}$ *that is closed under forming (*$R_{\mathsf{m}}$*-cofinal) subframes. Then:*

1. $\mathsf{M}$ *has the finite model property.*
2. *If moreover* $\mathsf{M}$ *contains the classical strength axiom*

$$\mathsf{S_c} \qquad \Box_{\mathsf{i}} p \to \Box_{\mathsf{m}} p.$$

   *then for any (*$R_{\mathsf{m}}$*-cofinal) subframe logic* $\Gamma \subseteq \mathcal{L}_{\mathsf{m}}$*, the logic* $\mathsf{M} \oplus \Gamma$ *has the finite model property.*

## Corollary

*Let $L$ be a $\multimap\!\exists$-logic extending* $\mathsf{P}$.

1. *If its* $\mathsf{S4HL}$*-counterparts include a canonical logic preserved by forming (cofinal) subframes, $L$ has the fmp.*
2. *Furthermore, if $L$ extends* $\mathsf{S_a}$ *and its* $\mathsf{S4HL}$*-counterparts include a logic obtained by extending a canonical (cofinal) subframe logic with a collection of $\mathcal{L}_\mathsf{m}$-axioms preserved by $R_m$-subframes, $L$ has the fmp.*

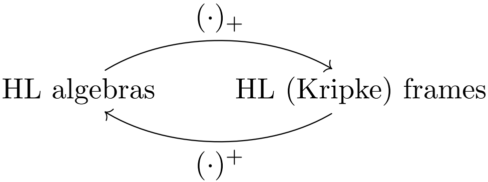
*In either case, $L$ is decidable whenever finitely axiomatizable.*

This covers the $\mathsf{P}$ axiom itself, the strength axiom, a strong variant of the Löb axiom, the axiom of monads $\mathsf{App_a}$ ...

However, some creativity is needed ...

# Descriptive frames or Esakia/Priestley dualities

# Begin with frames w/o topology/no admissible sets




- For the Heyting reduct, proceed as usual
- For $\mathfrak{F} = (X, \preceq, \sqsubset)$, $\mathfrak{F}^+$ has $\mathit{up}(X, \preceq)$ as its carrier

$$a \mathrel{\underline{\strictif}} b = \{x \in X \mid \text{if } x \sqsubset y \text{ and } y \in a \text{ then } y \in b\}$$

- The carrier of $\mathcal{A}_+$ is just $\mathit{pf}\,\mathcal{A}$: its prime filters

$$\mathfrak{p} \sqsubset \mathfrak{q} \quad \text{iff} \quad \forall a, b \in A (a \strictif b \in \mathfrak{p} \text{ and } a \in \mathfrak{q} \text{ implies } b \in \mathfrak{q}).$$

- Showing that $((\cdot)_+)^+$ is a HL-embedding yields Kripke completeness

  For the base system, and a few other ones, has been proved previously in a finitary manner (Iemhoff).

# Sketch of the $\multimap$-clause

- Suppose $a \multimap b \notin \mathfrak{p}$.

# Sketch of the $\strictif$-clause

- Suppose $a \strictif b \notin \mathfrak{p}$.
- We need to construct $\mathfrak{q} \in \mathit{pf}\,A$ s.t.

# Sketch of the $\multimap$-clause

- Suppose $a \multimap b \notin \mathfrak{p}$.
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  - $a \in \mathfrak{q}$, $b \notin \mathfrak{q}$ and

# Sketch of the $\dashv\!\!\!\!\!\supset$-clause

- Suppose $a \dashv\!\!\!\!\supset b \notin \mathfrak{p}$.
- We need to construct $\mathfrak{q} \in \mathit{pf}\, A$ s.t.
  - $a \in \mathfrak{q}$, $b \notin \mathfrak{q}$ and
  - $\mathfrak{p} \sqsubset \mathfrak{q}$, i.e., $\forall a', b' \in A(a' \dashv\!\!\!\!\supset b' \in \mathfrak{p}$ and $a' \in \mathfrak{q}$ implies $b' \in \mathfrak{q})$.

# Sketch of the $\dashv\!\!\!\!3$-clause

- Suppose $a \multimap b \notin \mathfrak{p}$.
- We need to construct $\mathfrak{q} \in \mathit{pf}\, A$ s.t.
  - $a \in \mathfrak{q}$, $b \notin \mathfrak{q}$ and
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- Let $[a) = \{c \in A \mid a \leq c\}$ and $I := \{d \in A \mid d \multimap b \in \mathfrak{p}\}$.

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- Let $[a) = \{c \in A \mid a \leq c\}$ and $I := \{d \in A \mid d \multimap b \in \mathfrak{p}\}$.
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# Sketch of the $-\!\!3$-clause

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  - $a \in \mathfrak{q}$, $b \notin \mathfrak{q}$ and
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- Let $[a) = \{c \in A \mid a \leq c\}$ and $I := \{d \in A \mid d \multimap b \in \mathfrak{p}\}$.
- $I$ is an ideal (thanks to $\mathsf{CDi}$!) s.t.
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# Sketch of the $\rightarrow\!\!\!\!\!\!\,\mathfrak{3}$-clause

- Suppose $a \rightarrow\!\!\!\!\,3\, b \notin \mathfrak{p}$.
- We need to construct $\mathfrak{q} \in \mathit{pf}\, A$ s.t.
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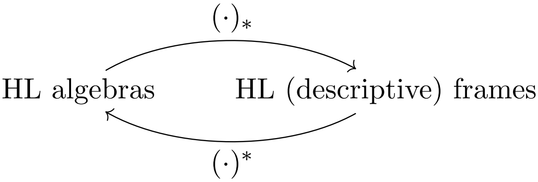
- The Prime Filter Lemma yields a suitable $\mathfrak{q}$:
  - One needs to use the fact that $\mathfrak{q}$ is a maximal element ...
  - ...and then one also needs to use all non-$\mathsf{CDi}$ axioms.

# Upgrading to dual equivalence (descriptive-style)




- Limit valuations to admissible upsets: a general frame is $(X, \preceq, \sqsubset, P)$ with $P \subseteq up(X, \preceq)$ closed under $\cap, \cup, \underline{\rightarrow}, \underline{-\!\!3}$.
- It is called descriptive if additionally it is
  - compact: For every $A \subseteq P$ and $B \subseteq \{X \setminus a \mid a \in P\}$, if $A \cup B$ has the f.i.p. then $\bigcap(A \cup B) \neq \emptyset$;
  - $\preceq$-refined: For all $x, y \in X$, if $x \npreceq y$ then there exists $a \in P$ such that $x \in a$ and $y \notin a$;
  - $-\!\!3$-refined: For all $x, y \in X$, if $x \not\sqsubset y$ then there exist $a, b \in P$ such that $x \in a \,\underline{-\!\!3}\, b$ and $y \in a$ and $y \notin b$.

  Morphisms: bounded wrt $\sqsubset$ and $\preceq$ + inverse images of admissibles admissible

# Upgrading to dual equivalence (Esakia- or Priestley-style)

- A strict implication space is a tuple $(X, \preceq, \sqsubset, \tau)$ s.t.
  - $(X, \preceq, \tau)$ is an Esakia space;
  - $x \preceq y \sqsubset z$ implies $x \sqsubset z$ for all $x, y, z \in X$;
  - $\downarrow_{\sqsubset} a = \{x \in X \mid x \sqsubset y \text{ for some } y \in a\}$ is clopen for every clopen $a \subseteq X$;
  - $\uparrow_{\sqsubset} x = \{y \in X \mid x \sqsubset y\}$ is closed in $(X, \tau)$ for all $x \in X$.
- Morphisms are continuous functions bounded wrt $\sqsubset$ and $\preceq$.
- Extending the Heyting case: SIS is an isomorphic category to descriptive ⥽-frames.
- But also limiting Celani and Jansana:

  “Bounded distributive lattices with strict implication”, *Mathematical Logic Quarterly*, vol. 51, pp. 219–246, 2005.
- SIS are (isomorphic to) a subcategory of their WH-spaces
- Priestley-style rather than Esakia-style

# Whence that Grzegorczyk axiom for $\Box_{\mathsf{i}}$?

- When turning a general/descriptive $\multimap$-frame into one for $\mathsf{S4K}$ (in fact, $\mathsf{S4HL}$) the natural solution is to close the admissibles under booleans

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- If the underlying $\Box_i$-relation is a partial order, validity of $\mathcal{L}_{i,m}$-formulas is unaffected after going there and back again
- And every extension of S4K is complete wrt such frames

  More demanding proofs than in the unimodal case