Lewis meets Brouwer: Constructive strict implication

ESSLLI 2020 (2021) advanced course proposal
Area: Logic and Computation

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The text is mostly as submitted in 2019. Only the outline was edited, with corresponding references added

1 Abstract

C. I. Lewis invented modern modal logic as a theory of *strict implication* $\multimap$. Over the classical propositional calculus one can as well work with the unary box connective $\Box$. Intuitionistically, however, $\multimap$ has greater expressive power and allows distinctions invisible in the ordinary syntax. Thus, in this course we study constructive systems of strict implication. We discuss conditions to be imposed on Kripke semantics, axiomatization of the minimal system and some of its extensions, and some basic correspondence results. We illustrate

- when and how this logic collapses to that of unary box;
- how classical assumptions made the trivialization of Lewis original 1918 system inevitable.

Furthermore, we present two interpretations of this system. The first comes from provability logic, more specifically preservativity in extensions of Heyting's Arithmetic. The second, Curry-Howard one is provided by functional programming: the study of Haskell *arrows* as contrasted with *idioms* or *applicative functors*.

2 Motivation and Description

We are investigating a constructive variant of *strict implication* $\prec\!\!\!3$, the original connective of modal logic as proposed by C.I. Lewis [Lewis, 1912, 1918, Lewis and Langford, 1932]. Thus, our research merges Lewis' foundational project in logic with the intuitionistic reform of mathematics proposed by Brouwer and Heyting. In his later years, Lewis himself claimed that his attempt to turn the strict implication into a central primitive of intensional logic has largely failed, disintegrating into numerous isolated systems of "modal logic" based on the unary box connective $\Box$, which has since become the standard modal primitive. In our paper invited to the special issue of *Indagationes Mathematicae* commemorating 50th anniversary of Brouwer's death [Litak and Visser, 2018], we argue that the root of the problem is that Lewis did not attempt to base his system on the intuitionistic propositional calculus (IPC), despite his occasional favorable remarks about Brouwer's project. There is a mathematical motivation for our claim: while classically $\prec\!\!\!3$ is definable via $\Box$, constructively this is not the case. There is also a remarkable, yet hitherto under-investigated parallelism of the philosophical biographies of Brouwer and Lewis, and of their motivation to identify valid principles of reasoning based on subtler inference rules than those governing classical material implication. We do not know whether Lewis was familiar with the subsequent work of Kolmogorov, Heyting and Glivenko, turning Brouwer's philosophical ideas into a formal system.

We have been lead to the study of constructive strict implication via two different routes.

Albert Visser, Rosalie Iemhoff and the Utrecht group have encountered a version of it in their study of the metatheory of intuitionistic arithmetic, a subject fitting in the broader Utrecht panorama of intuitionistic mathematics (Dirk van Dalen, Ieke Moerdijk, Jaap van Oosten). In this setting, the Lewis arrow encodes *preservativity*, a relation between sentences over a given constructive theory. Classically, the study of preservativity collapses into the study of conservativity, but intuitionistically this is not the case. Investigation of the logic of preservativity presents a hopeful road to study principles of provability and fixpoint reasoning in intuitionistic arithmetic. This is despite the fact that this logic encompasses the seemingly simpler logic of ordinary intuitionistic provability (corresponding to the unary box), whose axiomatization remains an open problem after decades of research. It is a well-known mathematical paradox that a stronger result is sometimes easier to prove.

Tadeusz Litak's interest has been stimulated by theoretical computer science, more specifically by the famous *Curry-Howard correspondence* (sometimes called an isomorphism) between programs and proofs, inhabited types and provable formulas, functors and suitable intutionistic modalities. In fact, a restricted version of the distinction between $\prec\!\!\!3$ and $\Box$ has been (re)discovered in functional programming as the distinction between applicative functors and Haskell arrows as proposed by John Hughes. This connective also turns out to play a surprisingly natural role in the proof theory of guarded (co)recursion and productive (co)programming, which are currently hot subjects in TCS.

This course will provide a systematic introduction to this research area. At the very least, the participants can see it as a rather unorthodox introduction to intuitionistic modal logic and some of its fascinating and not always sufficiently well-known interpretations and applications.

We begin with historical and semantical motivation, illustrating how $\strictif$ naturally arises in the Kripke setting. The audience will also be able to see, for example, how the collapse of Lewis's original attempt at constructing a theory of strict implication [Lewis, 1918] resulted from his classical preferences.

After that, we provide a more systematic overview of syntactic and semantic results on basic systems. The study of their relationships illustrates how intuitionism makes precise classically invisible distinctions. Armed with this background, we can study the arithmetical and computational interpretations discussed above. We also overview the highlights of our recent technical work regarding the definability of fixpoints and regarding uniform interpolation —some of it is still being prepared for publication as of the moment of writing.

If there is sufficient interest, we can also devote some time to experimenting with computer software and formalizations. This, however, would depend on audience preferences and the degree of comfort with proof assistants or programming languages. The constraints of the lecture would not permit an independent introduction to such subjects.

3 Outline (edited 2021)

Day 1. (Litak): Motivation, history, basics of syntax and semantics. The classical problems of Lewis. Curry–Howard correspondence, arrows and idioms in functional programming. Towards Intuitionistic Logic of Entailments?

Main reference: Litak and Visser [2018]

Day 2. (Litak): Axiomatization, algebraic and Kripke semantics (in the presence of axiom Di). Completeness and correspondence results via translation into a Gödel-McKinsey-Tarski translation into a suitable bimodal classical logic: explicit or extrinsic perspective (van Benthem terminology).

Main reference: de Groot, Litak, Pattinson, Gödel–McKinsey–Tarski and Blok–Esakia for Heyting–Lewis Implication, LiCS 2021. Full version (technical report): `https://arxiv.org/abs/2105.01873`

Day 3.
- Part I (Litak) What holds in the absence of Di? Algebra as an alternative semantics. Countermodel search using tools such as Mace4: `https://www.cs.unm.edu/~mccune/mace4/`
- Part II (Visser): Introduction to arithmetical semantics of provability and preservativity logics

Day 4.
- Part I (Visser): Introduction to arithmetical semantics of provability and preservativity logics (continuation)
- Part II (Litak): Extension stability. A glimpse at alternative semantics (generalizations of Veltman or Routley–Meyer frames)

Day 5. (Visser): Reverse mathematics of explicit fixpoints

Main reference for the second half of the course: Litak and Visser [2019]

4 Expected Level and Prerequisites

The course is designed as an advanced one. The audience is supposed to be familiar with those areas in the reading list (Section 5) which include at least one MD in the corresponding subsection, and strongly recommended to have some familiarity with the remaining ones. In more detail:

- We assume reasonable familiarity with intuitionistic propositional logic and modal logic, including their Kripke semantics, morphisms and bisimulations, soundness and completeness of basic systems, and basics of definability and correspondence.
- For the lecture dealing with preservativity and provability interpretation, it is necessary to have some familiarity with classical results and techniques for Peano Arithmetic, as giving a full introduction in the limited timeframe is not possible.
- Some familiarity with the Curry-Howard correspondence or functional programming, intuitionistic arithmetic or intuitionistic modal logic can be helpful, although we do not want to assume detailed knowledge here.

5 Reading Material

5.1 Legenda

MD part of a mandatory disjunction.

B possible non-mandatory background

5.2 Intuitionistic Propositional Logic

- MD: An excellent textbook is van Dalen [2013].
- MD: A treatment focusing on semantics and connections with modal logic is Chagrov and Zakharyaschev [1997]. This is also a good reference for modal logic itself, cf. Subsection 5.3
- MD: Another treatment can be found in [Sørensen and Urzyczyn, 2006, Ch.2]. Further chapters of this reference are B in Subsection 5.7 below.

5.3 Modal Logic

- MD: Chagrov and Zakharyaschev [1997].
- MD: Blackburn et al. [2001].
- MD: Opening chapters of Boolos [1993], cf. also Subsection 5.6 below.

5.4 Intuitionistic Modal Logic and Constructive Strict Implication

- B: Obviously the most natural one in this course is Litak and Visser [2018].
- B: Earlier reasonably self-contained references include Iemhoff [2001], Iemhoff et al. [2005].
- B: For "ordinary" intuitionistic modal logic with $\Box$ and/or $\Diamond$, there are numerous references. Simpson [1994] is a standard example, in particular the material on Kripke semantics. Litak [2014] contains some information on "provability smack" in intuitionistic propositional logic. Participants familiar with the Gödel-McKinsey-Tarski translation of intuitionistic propositional logic into modal $\mathsf{S4}$ can also find Wolter and Zakharyaschev [1997, 1998] instructive.

5.5 Intuitionistic Arithmetic

- B: Troelstra [1973] is still a very good source. We also recommend its part: Smoryński [1973].
- B: Dragalin [1988] is another good and not sufficiently appreciated reference.

- B: One can also find some information in the intuitionistically oriented Section 9 of Artemov and Beklemishev [2004].
- B: Some basic information, perhaps sufficient for the purpose of the course, can be found in several references quoted in Subsection 5.4 below, especially Litak and Visser [2018] or Iemhoff [2001].

5.6 Provability and Interpretability Logic

- MD: Švejdar [2000].
- MD: Lindström [1996].
- MD: Boolos [1993], Chapter 1–9. Of course, the whole book is B.
- MD: Smoryński [1985], Chapter 1–4. Of course, the whole book is B.
- MD: Japaridze and de Jongh [1998], except Section 10, 15, 16. The rest is B.
- MD: Artemov and Beklemishev [2004], Part I, except Sections 6, 7, 8 and 10. As mentioned above, Section 9 is intuitionistically oriented. The rest is B.
- B: Visser [1998].
- B: Boolos and Sambin [1991].
- B: de Jongh and Visser [1991].
- B: Visser and Zoethout [2019].
- B: Visser [2008]
- B: Iemhoff [2001]

5.7 Curry-Howard Correspondence and Arrows in FP

- B: Sørensen and Urzyczyn [2006]
- B: Hughes [2000], Lindley et al. [2011], McBride and Paterson [2008]
- B: Benton et al. [1998], Fairtlough and Mendler [1997], Kobayashi [1997], Moggi [1991]
- B: Litak and Visser [2018]

5.8 Fixpoints Calculations

- B: Litak and Visser [2019]
- B: [Smoryński, 1985, Ch. 4]
- B: de Jongh and Visser [1991]

References

S. Artemov and L. Beklemishev. Provability logic. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic, 2nd ed.*, volume 13, pages 229–403. Springer, Dordrecht, 2004.

P. N. Benton, G. M. Bierman, and V. de Paiva. Computational types from a logical perspective. *J. Funct. Programming*, 8(2):177–193, 1998.

P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic.* Cambridge University Press, 2001.

G. Boolos. *The logic of provability.* Cambridge University Press, Cambridge, 1993.

G. Boolos and G. Sambin. Provability: the emergence of a mathematical modality. *Studia Logica*, 50:1–23, 1991.

A. Chagrov and M. Zakharyaschev. *Modal Logic.* Number 35 in Oxford Logic Guides. Clarendon Press, Oxford, 1997.

D. van Dalen. *Logic and Structure.* Berlijn, Springer, 2013. Fifth edition.

D. de Jongh and A. Visser. Explicit fixed points in interpretability logic. *Studia Logica*, 50:39–50, 1991.

A. G. Dragalin. *Mathematical Intuitionism: Introduction to Proof Theory*, volume 67 of *Translations of Mathematical Monographs.* American Mathematical Society, 1988. Translated by E. Mendelson, ed. by B. Silver.

M. Fairtlough and M. Mendler. Propositional lax logic. *Inform. and Comput.*, 137(1): 1–33, 1997.

J. Hughes. Generalising monads to arrows. *Sci. Comput. Program.*, 37(1-3):67–111, 2000. doi: 10.1016/S0167-6423(99)00023-4. URL `http://dx.doi.org/10.1016/S0167-6423(99)00023-4`.

R. Iemhoff. *Provability Logic and Admissible Rules.* PhD thesis, University of Amsterdam, 2001.

R. Iemhoff, D. De Jongh, and C. Zhou. Properties of intuitionistic provability and preservativity logics. *Logic Journal of IGPL*, 13(6):615–636, 2005.

G. Japaridze and D. de Jongh. The logic of provability. In S. Buss, editor, *Handbook of proof theory*, pages 475–546. North-Holland Publishing Co., Amsterdam, 1998.

S. Kobayashi. Monad as modality. *Theoret. Comput. Sci.*, 175(1):29 – 74, 1997. ISSN 0304-3975. doi: DOI:10.1016/S0304-3975(96)00169-7. URL `http://www.sciencedirect.com/science/article/pii/S0304397596001697`.

C. Lewis. *A Survey of Symbolic Logic*. University of California Press, 1918.

C. Lewis and C. Langford. *Symbolic Logic*. Dover, 1932.

C. I. Lewis. Implication and the algebra of logic. *Mind*, 21(84):522–531, 1912. ISSN 00264423, 14602113. URL http://www.jstor.org/stable/2249157.

S. Lindley, P. Wadler, and J. Yallop. Idioms are oblivious, arrows are meticulous, monads are promiscuous. *ENTCS*, 229(5):97 – 117, 2011. ISSN 1571-0661. doi: http://dx.doi.org/10.1016/j.entcs.2011.02.018. URL http://www.sciencedirect.com/science/article/pii/S1571066111000557. Proceedings of MSFP.

P. Lindström. Provability logic – a short introduction. *Theoria*, 62(1-2):19–61, 1996.

T. Litak. Constructive modalities with provability smack. In G. Bezhanishvili, editor, *Leo Esakia on duality in modal and intuitionistic logics*, volume 4 of *Outstanding Contributions to Logic*, pages 179–208. Springer, 2014. doi: 10.1007/978-94-017-8860-1_8. URL https://arxiv.org/abs/1708.05607.

T. Litak and A. Visser. Lewis meets Brouwer: constructive strict implication. *Indagationes Mathematicae*, 29:36–90, February 2018. ISSN 0019-3577. doi: 10.1016/j.indag.2017.10.003. URL https://arxiv.org/abs/1708.02143. A special issue "L.E.J. Brouwer, fifty years later".

T. Litak and A. Visser. Lewisian fixed points I: two incomparable constructions. *CoRR*, abs/1905.09450, 2019. URL http://arxiv.org/abs/1905.09450.

C. McBride and R. Paterson. Applicative programming with effects. *J. Funct. Programming*, 18(1):1–13, 2008.

E. Moggi. Notions of computation and monads. *Inform. and Comput.*, 93:55–92, July 1991. ISSN 0890-5401. doi: 10.1016/0890-5401(91)90052-4. URL http://portal.acm.org/citation.cfm?id=116981.116984.

A. K. Simpson. *The Proof Theory and Semantics of Intuitionistic Modal Logic*. PhD thesis, University of Edinburgh, 1994. URL http://homepages.inf.ed.ac.uk/als/Research/thesis.ps.gz.

C. Smoryński. Applications of Kripke Models. In A. Troelstra, editor, *Metamathematical Investigations of Intuitionistic Arithmetic and Analysis*, Springer Lecture Notes 344, pages 324–391. Springer, Berlin, 1973.

C. Smoryński. *Self-Reference and Modal Logic*. Universitext. Springer, New York, 1985.

M. H. Sørensen and P. Urzyczyn. *Lectures on the Curry-Howard Isomorphism*, volume 149 of *Stud. Logic Found. Math.* Elsevier Science Inc., New York, NY, USA, 2006. ISBN 0444520775.

V. Švejdar. On provability logic. *Nordic Journal of Philosophical Logic*, 4(2):95–116, 2000.

A. Troelstra. *Metamathematical investigations of intuitionistic arithmetic and analysis.* Springer Lecture Notes 344. Springer Verlag, Berlin, 1973.

A. Visser. An Overview of Interpretability Logic. In M. Kracht, M. de Rijke, H. Wansing, and M. Zakharyaschev, editors, *Advances in Modal Logic*, volume 1, 87 of *CSLI Lecture Notes*, pages 307–359. Center for the Study of Language and Information, Stanford, 1998.

A. Visser. Closed fragments of provability logics of constructive theories. *Journal of Symbolic Logic*, 73(3):1081–1096, 2008.

A. Visser and J. Zoethout. Provability logic and the completeness principle. *Annals of Pure and Applied Logic*, 170(6):718–753, 2019. doi: 10.1016/j.apal.2019.02.001.

F. Wolter and M. Zakharyaschev. On the relation between intuitionistic and classical modal logics. *Algebra and Logic*, 36:121–125, 1997.

F. Wolter and M. Zakharyaschev. Intuitionistic modal logics as fragments of classical bimodal logics. In E. Orlowska, editor, *Logic at Work, Essays in honour of Helena Rasiowa*, pages 168–186. Springer–Verlag, 1998.