Lewis meets Brouwer, or perhaps Heyting

Tadeusz Litak (FAU Erlangen-Nuremberg)
joint lecture with Albert Visser
course *Lewis meets Brouwer: Constructive strict implication*

ESSLLI 2021, part I

Partners in crime

- Albert Visser (Utrecht University)
- Jim de Groot and Dirk Pattinson (ANU)
- Igor Sedlar (CAS, Prague)
- Miriam Polzer (Google)
- ...

outline of the course

sound, smoke and cabbage

axiomatization for $\strictif$: $\mathsf{HLC}^{\flat}$ vs. $\mathsf{HLC}^{\sharp}$

saving C.I. Lewis?

profunctors, weakening relations and Haskell arrows

IELE: Intuitionistic Epistemic Logic of Entailments

Bonus derivation

Day 1. (Litak):

- Motivation, history.
- Axiomatization and algebraic semantics.
- The classical problems of Lewis.
- Curry–Howard correspondence, Hughes' arrows, monads and idioms in functional programming.
- Towards Intuitionistic Logic of Entailments (IELE).

Day 2. (Litak): Kripke semantics for $\mathsf{HLC}^\sharp$ (i.e., in the presence of axiom Di). Completeness and correspondence results via a Gödel-McKinsey-Tarski translation into a suitable bimodal classical logic: *explicit* or *extrinsic* perspective (van Benthem's terminology).

Day 3.

- Part I (Litak) What holds in the absence of Di? Algebraic semantics for $\mathsf{HLC}^\flat$. Countermodel search using tools such as Mace4:
  `https://www.cs.unm.edu/~mccune/mace4/`
- Part II (Visser): Introduction to arithmetical semantics of provability and preservativity logics

Day 4

- Part I (Visser): Introduction to arithmetical semantics of provability and preservativity logics (continuation)
- Part II (Litak): Extension stability. A glimpse at alternative semantics (generalizations of Veltman or Routley–Meyer frames)

Day 5 (Visser): Reverse mathematics of explicit fixpoints https://www8.cs.fau.de/people/dr-tadeusz-litak/esslli-2021-lectures/

A cabbage by any other name would swell as sweet.

Clyde Fitch, *Captain Jinks of the Horse Marines*, 1901
(with a little help from William Shakespeare)

Ich habe keinen Namen
Dafür! Gefühl ist alles;
Name ist Schall und Rauch,
Umnebelnd Himmelsglut.

JW von Goethe, *Faust I*, 1808
Translation: *I have no name*
For it! Feeling is everything;
(The) name is sound and smoke,
Enshrouding heaven's glow.

Schall und Rauch for our cabbage

- The implication/calculus of
  - Heyting-Lewis
  - Heyting-Lewis-Visser
  - Heyting-Lewis-Visser-Iemhoff
  - Heyting-Lewis-Visser-Iemhoff-Celani-Jansana
  - *any of the above options, but with Brouwer replacing Heyting and possibly permuting names*
  - Heyting-weak-Heyting (HwH)
- (The Calculus of) Constructive Strict Implication
- iP (Iemhoff and coauthors): because preservativity in HA
- iA (L. + Visser): because arrows in FP

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Virtual Special Issue - L.E.J. Brouwer after 50 years

Lewis meets Brouwer: Constructive strict implication

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- The history of $\dashv$ begins at UC Berkeley

  and then Harvard

- The history of constructive $\dashv$ begins at Utrecht U

  and then Amsterdam too, but only Utrecht has kept the flame alive

- Its variant first discovered by Visser in the context of preservativity in Heyting Arithmetic

  Evaluation, provably deductive equivalence in Heyting's Arithmetic of substitution instances of propositional formulas, 1985

- Iemhoff (PhD 2001), later also jointly with de Jongh and Zhou (Log. J. IGPL 2005), worked out the minimal theory and its main extensions

- Particularly natural Kripke semantics!

  Indeed, once you see how it works, it seems clearly the intuitionistic generalization of C. I. Lewis's original modal connective

- Even in the Kripke setting, not reducible to unary $\Box$, quite unlike the classical situation

- As we all know (or do we?) the following is the original syntax of modern modal logic :

$$\mathcal{L}_{\multimap} \quad \varphi, \psi ::= \top \mid \bot \mid p \mid \varphi \to \psi \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \varphi \multimap \psi$$

- $\multimap$ is the strict implication of Clarence Irving Lewis

  who is not C.S. Lewis, David Lewis or Lewis Carroll

  His earliest series of papers on the subject: 1912–1915

  1918: *A Survey of Symbolic Logic*, University of California Press

  1932 (with C. H. Langford): *Symbolic Logic*, Dover
- $\Box\varphi$ is then definable as $\top \multimap \varphi$
- Over the Classical Propositional Calculus (CPC) and in the presence of Di, the converse holds too:
  classically $\varphi \multimap \psi$ is $\Box(\varphi \to \psi)$, i.e., $\top \multimap (\varphi \to \psi)$
- Informally: truth of strict implication at w = truth of material implication in all possible worlds seen from w
- The last item actually does hold intutionistically as well. But unlike the classical case, $\multimap$ does not collapse to $\Box$!

  We will say more about Kripke frames a bit later

axiomatization for $\multimap$: $\mathsf{HLC}^\flat$ vs. $\mathsf{HLC}^\sharp$

Who here knows …

- Intuitionistic Propositional Calculus (IPC)?
- Its Kripke semantics?
- Heyting algebras?
- The notion of the Lindenbaum-Tarski algebra of a logic?
  also known as the free algebra (on countably many generators)
- Some Abstract Algebraic Logic (AAL)?
- Some functional programming (FP)?
- The Curry-Howard Correspondence?
  also known as the Curry-Howard Isomorphism or or the proofs-as-programs and propositions- or formulae-as-types interpretation
- Some category theory?

Axioms and rules of IPC:

$$\varphi \to (\psi \to \varphi)$$
$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$$

$$\bot \to \varphi$$

$$(\varphi \wedge \psi) \to \varphi$$
$$(\varphi \wedge \psi) \to \psi$$
$$\varphi \to (\psi \to (\varphi \wedge \psi))$$

$$\varphi \to (\varphi \vee \psi)$$
$$\psi \to (\varphi \vee \psi)$$
$$(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi))$$

Axioms and rules of the minimal system $\mathsf{HLC}^\flat$(a.k.a. iA^- or iP^- or HL^-):

Those of IPC plus:

$$\mathsf{Tra} \quad (\varphi \dashv\!\!\!3\ \psi) \wedge (\psi \dashv\!\!\!3\ \chi) \to (\varphi \dashv\!\!\!3\ \chi)$$

transitivity of $\dashv\!\!\!3$

$$\mathsf{K_a} \quad (\varphi \dashv\!\!\!3\ \psi) \wedge (\varphi \dashv\!\!\!3\ \chi) \to (\varphi \dashv\!\!\!3\ (\psi \wedge \chi))$$

normality=normality in the second coördinate

$$\mathsf{N_a}\ \frac{\varphi \to \psi}{\varphi \dashv\!\!\!3\ \psi.}$$

binary generalization of necessitation

not only implies congruentiality, but also anti-monotonicity in the first coördinate

Axioms and rules of the full system $\mathsf{HLC}^\sharp$(a.k.a. iA or iP or HL):

All the axioms and rules of IPC and $\mathsf{HLC}^\flat$ and

$$\mathsf{Di} \quad ((\varphi \dashv\!\!\!3\ \chi) \wedge (\psi \dashv\!\!\!3\ \chi)) \to ((\varphi \vee \psi) \dashv\!\!\!3\ \chi).$$

should implication be anti-multiplicative in the first coördinate?

Running this axiom system via the AAL machinery yields:

Heyting algebras plus:

$$\mathsf{CTra} \quad (\varphi \multimapinv \psi) \wedge (\psi \multimapinv \chi) \leq \varphi \multimapinv \chi$$

$$\mathsf{CK_a} \quad (\varphi \multimapinv \psi) \wedge (\varphi \multimapinv \chi) = \varphi \multimapinv (\psi \wedge \chi)$$

$$\mathsf{CId} \quad \varphi \multimapinv \varphi = \top$$

The class of Lewis-Brouwer algebras:

all the equalities above plus

$$\mathsf{CDi} \quad (\varphi \multimapinv \chi) \wedge (\psi \multimapinv \chi) = (\varphi \vee \psi) \multimapinv \chi.$$

The $\multimapinv$-free reduct: Heyting algebras

The $\rightarrow$-free reduct:

weak Heyting algebras of Celani and Jansana

fusion, fibring or dovetailing along the shared bounded lattice reduct

Of course, we inherit as limiting cases:

- Normal modal logics over CPC
  = varieties of modal algebras (MAs, BAOs)
  (Boolean algebras with a single unary operator)
  Too many applications and references to discuss
- Normal modal logics over IPC (with $\Box$ only!)
  = varieties of HAMs or HA(d)Os
  (Heyting algebras with a modality =
  Heyting algebras with a single unary (dual) operator)
  Again, too many applications and references to discuss
- superintinuitionistic (intermediate) propositional logics
  = varieties of Heyting algebras

But there is much else ...

Intuitionistic $\dashv$

- Metatheory of arithmetic
  Σ^0_1-preservativity for a theory T extending HA
  For more, cf. the Brouwer paper or ***Lewisian Fixpoints I: two incomporable constructions***? https://arxiv.org/abs/1905.09450
- Functional programming: (Haskell) arrows of John Hughes (versus *applicative functors/idioms* of McBride/Patterson)
  A series of papers (Lindley & Wadler & Yallop, Atkey, Jacobs & Heunen & Hasuo ...) and numerous Haskell libraries
- Proof theory of guarded (co)recursion
  Nakano and more recently Clouston&Goré
- Analysis of intuitionistic Kripke semantics
  generalizing defining conditions of profunctors/weakening relations
- Intuitionistic Epistemic Logic of Entailments (IELE)
  Generalizing Artemov& Protopopescu's IEL: recently with Jim de Groot
- Saving Lewis' original systems?
  You're going to hear more about it now: blame Paolo Aglianò for this!

saving C.I. Lewis?

- Lewis indeed wanted to have classical (involutive) negation
- In fact, he introduced $-\!\!3$ as defined using $\Diamond$

  somehow did not explicitly work with $\Box$ in the signature
- Especially earlier variants of his systems freely used negation in the axiomatization
- This lead to two problems: one immediate, one in the long-term perspective

The immediate problem

- The early system Lewis developed in *A Survey of Symbolic Logic* (1918) turn out to collapse $\strictif$ to $\rightarrow$

  As found out by Post; see Lewis's *Strict implication–An emendation*, J. Philos. Psychol. Sci. Methods (1920)

- Lewis' later summary:

  In developing the system, I had worked for a month to avoid this principle, which later turned out to be false. Then, finding no reason to think it false, I sacrificed economy and put it in.

- In Appendix II to *Symbolic Logic* (1932), Lewis was more cautious, creating several *lines of retreat* (Parry): S3, S2 and S1
- You might wonder now about S4 and S5?
- Lewis did not like them at all;
  they were forced on him from the outside

  Becker, then Gödel

Those interested in the merely mathematical properties of such systems of symbolic logic tend to prefer more comprehensive and less strict systems such as S5 *and material implication. The interests of logical study would probably be best served by an exactly opposite tendency.*

The final words of *Symbolic Logic* (Appendix II)

APPENDIX II

THE STRUCTURE OF THE SYSTEM OF STRICT IMPLICATION[1]

The System of Strict Implication, as presented in Chapter V of *A Survey of Symbolic Logic* (University of California Press, 1918), contained an error with respect to one postulate. This was pointed out by Dr. E. L. Post, and was corrected by me in the *Journal of Philosophy, Psychology, and Scientific Method* (XVII [1920], 300). The amended postulates (set A below) compare with those of Chapter VI of this book (set B below) as follows:

[1] This appendix is written by Mr. Lewis, but the points demonstrated are, most of them, due to other persons.

Groups II and III, below, were transmitted to Mr. Lewis by Dr. M. Wajsberg, of the University of Warsaw, in 1927. Dr. Wajsberg's letter also contained the first proof ever given that the System of Strict Implication is not reducible to Material Implication, as well as the outline of a system which is equivalent to that deducible from the postulates of Strict Implication with the addition of the postulate later suggested in Becker's paper and cited below as C11. It is to be hoped that this and other important work of Dr. Wajsberg will be published shortly.

Groups I, IV, and V are due to Dr. William T. Parry, who also discovered independently Groups II and III. Groups I, II, and III are contained in his doctoral dissertation, on file in the Harvard University Library. Most of the proofs in this appendix have been given or suggested by Dr. Parry.

It follows from Dr. Wajsberg's work that there is an unlimited number of groups, or systems, of different cardinality, which satisfy the postulates of Strict Implication. Mr. Paul Henle, of Harvard University, later discovered another proof of this same fact. Mr. Henle's proof, which can be more easily indicated in brief space, proceeds by demonstrating that any group which satisfies the Boole-Schröder Algebra will also satisfy the postulates of Strict Implication if ◊p be determined as follows:

$$\Diamond p = 1 \text{ when and only when } p \neq 0;$$
$$\Diamond p = 0 \text{ when and only when } p = 0.$$

This establishes the fact that there are as many distinct groups satisfying the postulates as there are powers of 2, since it has been shown by Huntington that there is a group satisfying the postulates of the Boole-Schröder Algebra for every power of 2 (“Sets of I

A1. $p\,q \,.\!\prec\!.\, q\,p$	B1. $p\,q \,.\!\prec\!.\, q\,p$
A2. $q\,p \,.\!\prec\!.\, p$	B2. $p\,q \,.\!\prec\!.\, p$
A3. $p \,.\!\prec\!.\, p\,p$	B3. $p \,.\!\prec\!.\, p\,p$
A4. $p(q\,r) \,.\!\prec\!.\, q(p\,r)$	B4. $(p\,q)r \,.\!\prec\!.\, p(q\,r)$
A5. $p \prec \sim(\sim p)$	B5. $p \prec \sim(\sim p)$
A6. $p \prec q \,.\, q \prec r \,:\!\prec\!.\, p \prec r$	B6. $p \prec q \,.\, q \prec r \,:\!\prec\!.\, p \prec r$
A7. $\sim\Diamond p \prec \sim p$	B7. $p \,.\, p \prec q \,:\!\prec\!.\, q$
A8. $p \prec q \,.\!\prec\!.\, \sim\Diamond q \prec \sim\Diamond p$	B8. $\Diamond(p\,q) \prec \Diamond p$
	B9. $(\exists\, p, q) : \sim(p \prec q) \,.\, \sim(p \prec \sim q)$

The primitive ideas and definitions are not identical in the two cases; but they form equivalent sets, in connection with the postulates.

Comparison of these two sets of postulates, as well as many other points concerning the structure of Strict Implication, will be facilitated by consideration of the following groups. Each of these is based upon the same matrix for the relation $p\,q$ and the function of negation $\sim p$. (This is a four-valued matrix which satisfies the postulates of the Boole-Schröder Algebra.) The groups differ by their different specification of the function $\Diamond p$. We give the fundamental matrix for $p\,q$ and $\sim p$ in the first case only. The matrix for $p \prec q$, resulting from this and the particular determination of $\Diamond p$, is given for each group:

GROUP I

$p\,q$	q: 1	2	3	4	$\sim p$
p: 1	1	2	3	4	4
2	2	2	4	4	3
3	3	4	3	4	2
4	4	4	4	4	1

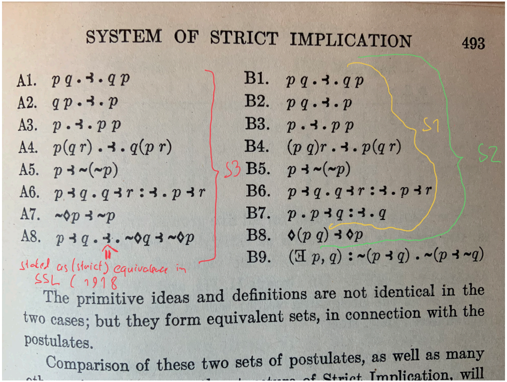
◊	⥽	1	2	3	4
1	1	2	4	4	4
1	2	2	2	4	4
1	3	2	4	2	4
3	4	2	2	2	2

GROUP II

◊	⥽	1	2	3	4
1	1	1	4	3	4
2	2	1	1	3	3

GROUP III

◊	⥽	1	2	3	4
1	1	1	4	4	4
1	2	1	1	4	4
1	3	1	4	1	4



SYSTEM OF STRICT IMPLICATION 493

A1.	$p\,q \,.\strictif.\, q\,p$	B1.	$p\,q \,.\strictif.\, q\,p$
A2.	$q\,p \,.\strictif.\, p$	B2.	$p\,q \,.\strictif.\, p$
A3.	$p \,.\strictif.\, p\,p$	B3.	$p \,.\strictif.\, p\,p$
A4.	$p(q\,r) \,.\strictif.\, q(p\,r)$	B4.	$(p\,q)r \,.\strictif.\, p(q\,r)$
A5.	$p \strictif \sim(\sim p)$	B5.	$p \strictif \sim(\sim p)$
A6.	$p \strictif q \,.\, q \strictif r :\strictif.\, p \strictif r$	B6.	$p \strictif q \,.\, q \strictif r :\strictif.\, p \strictif r$
A7.	$\sim\Diamond p \strictif \sim p$	B7.	$p \,.\, p \strictif q :\strictif.\, q$
A8.	$p \strictif q \,.\strictif.\, \sim\Diamond q \strictif \sim\Diamond p$	B8.	$\Diamond(p\,q) \strictif \Diamond p$
		B9.	$(\exists\, p, q) : \sim(p \strictif q) \,.\, \sim(p \strictif \sim q)$

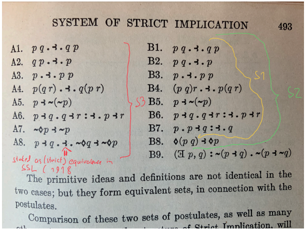
The primitive ideas and definitions are not identical in the two cases; but they form equivalent sets, in connection with the postulates.

Comparison of these two sets of postulates, as well as many ... of Strict Implication, will

- Lewis rules are

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi} \qquad\qquad \frac{\varphi \quad \varphi \strictif \psi}{\psi} \qquad\qquad \frac{\varphi \strictiff \psi}{\chi(\varphi) \strictiff \chi(\psi)}$$

plus uniform substitution, of course



SYSTEM OF STRICT IMPLICATION 493

A1.	$p\,q \,.\!\strictif\!.\, q\,p$	B1.	$p\,q \,.\!\strictif\!.\, q\,p$
A2.	$q\,p \,.\!\strictif\!.\, p$	B2.	$p\,q \,.\!\strictif\!.\, p$
A3.	$p \,.\!\strictif\!.\, p\,p$	B3.	$p \,.\!\strictif\!.\, p\,p$
A4.	$p(q\,r) \,.\!\strictif\!.\, q(p\,r)$	B4.	$(p\,q)r \,.\!\strictif\!.\, p(q\,r)$
A5.	$p \strictif \sim(\sim p)$	B5.	$p \strictif \sim(\sim p)$
A6.	$p \strictif q \,.\, q \strictif r :\strictif.\, p \strictif r$	B6.	$p \strictif q \,.\, q \strictif r :\strictif.\, p \strictif r$
A7.	$\sim\Diamond p \strictif \sim p$	B7.	$p \,.\, p \strictif q :\strictif.\, q$
A8.	$p \strictif q \,.\!\strictif\!.\, \sim\Diamond q \strictif \sim\Diamond p$	B8.	$\Diamond(p\,q) \strictif \Diamond p$
		B9.	$(\exists\, p, q) : \sim(p \strictif q) \,.\, \sim(p \strictif \sim q)$

The primitive ideas and definitions are not identical in the two cases; but they form equivalent sets, in connection with the postulates.

Comparison of these two sets of postulates, as well as many

- The 1932 axioms rely much less on involutive negation!
- Perhaps there would have been even less of it were it not for Lewis' determination to make

$$\frac{\vdash_{\mathsf{CPC}} \ \varphi \leftrightarrow \psi}{\varphi \mathrel{\varepsilon\!\!\exists} \psi}$$

  an admissible rule
- The only axiom among B1 – B8 underivable in $\mathsf{HLC}^\sharp$ is B7
- We will revisit it in the functional programming context

The long-term problem

- Definability in terms of $\Box$ might be another reason why $\prec$ slid into irrelevance ...
- ... which did not seem to make Lewis happy
- He didn't even like the name "modal logic"

There is a logic restricted to indicatives; the truth-value logic most impressively developed in "Principia Mathematica". But those who adhere to it usually have thought of it—so far as they understood what they were doing—as being the universal logic of propositions which is independent of mode. And when that universal logic was first formulated in exact terms, they failed to recognize it as the only logic which is independent of the mode in which propositions are entertained and dubbed it "modal logic".

"Alternative Systems of Logic", *The Monist*, 1932

- Curiously, Lewis seemed sympathetic towards non-classical systems (mostly the Łukasiewicz logic)
    - A detailed discussion in *Symbolic Logic*, 1932
    - A paper on "Alternative Systems of Logic", *The Monist*, same year
    - Both papers analyze possible implication connectives in finite valued logics
- I found just one reference where he mentions (quite favourably) Brouwer and intuitionism

[T]he mathematical logician Brouwer has maintained that the law of the Excluded Middle is not a valid principle at all. The issues of so difficult a question could not be discussed here; but let us suggest a point of view at least something like his. . . . The law of the Excluded Middle is not writ in the heavens: it but reflects our rather stubborn adherence to the simplest of all possible modes of division, and our predominant interest in concrete objects as opposed to abstract concepts. The reasons for the choice of our logical categories are not themselves reasons of logic any more than the reasons for choosing Cartesian, as against polar or Gaussian coördinates, are themselves principles of mathematics, or the reason for the radix 10 is of the essence of number.

"Alternative Systems of Logic", *The Monist*, 1932

- No indication he was aware of Kolmogorov, Heyting, Glivenko …
- Maybe he should've followed up on that …
- …especially given all analogies between him and Brouwer
  - almost perfectly parallel life dates
  - wrote his 1910 PhD on *The Place of Intuition in Knowledge*
  - a solid background in/influence of idealism and Kant …
  - …see our paper for more on Lewis and Brouwer as parallel lives and parallel fruitful failures

The error of philosophers: *The philosopher believes that the value of his philosophy lies in the whole, in the structure. Posterity finds it in the stone with which he built and with which, from that time forth, men will build oftener and better—in other words, in the fact that the structure may be destroyed and yet have value as material.*

Nietzsche, *Human, All-Too-Human, Part II*, translated by P.V. Cohn.

profunctors, weakening relations and Haskell arrows

Kripke semantics for intuitionistic $\Box$:

- Nonempty set of worlds
- Two relations:
  - Intuitionistic partial order relation $\preceq$, drawn as $\to$;
  - Modal relation $\sqsubset$, drawn as $\rightsquigarrow$.
- Semantics for $\Box$: $w \Vdash \Box\varphi$ if for any $v \sqsupset w$, $v \Vdash \varphi$
- Semantics for $\strictif$:

$$w \Vdash \varphi \strictif \psi \text{ if for any } v \sqsupset w, v \Vdash \varphi \text{ implies } v \Vdash \psi$$

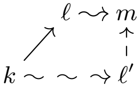
- Nice and important exercise: this semantics makes Di valid

- What is the minimal condition to guarantee persistence?
- That is, given A, B upward closed, is

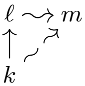
$$A \multimap B = \{w \mid \text{ for any } v \sqsupset w, v \in A \text{ implies } v \in B\}$$

  upward closed?
- Is it it stronger than the one ensuring persistence for $\Box A$?

Four frame conditions (known since 1980's)

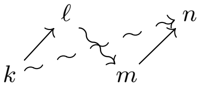


$\Box$-p
(persistence for $\Box$)

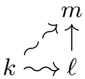


prefixing
(persistence for ⥽)

$\longrightarrow$ intuitionistic $\preceq$
$\rightsquigarrow$ modal $\sqsubset$



mix / brilliancy
profunctors/weakening rels.



postfixing

$\Longleftarrow$ both equivalent in presence of $\Box$-p, collapsing ⥽ to $\Box$

- brilliancy obtains naturally in, e.g., Stone-Jónsson-Tarski for $\Box$
- ... but ⥽ can feel it! $\Longrightarrow$ collapse of ⥽ to $\Box$
- Over prefixing (or ⥽-frames) $\Box(\varphi \to \psi)$ implies φ ⥽ ψ, but not the other way around

Idioms are oblivious, arrows are meticulous, monads are promiscuous

Sam Lindley, Philip Wadler and Jeremy Yallop

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The University of Edinburgh

Abstract

We revisit the connection between three notions of computation: Moggi's *monads*, Hughes's *arrows* and McBride and Paterson's *idioms* (also called *applicative functors*). We show that idioms are equivalent to arrows that satisfy the type isomorphism $A \rightsquigarrow B \simeq 1 \rightsquigarrow (A \to B)$ and that monads are equivalent to arrows that satisfy the type isomorphism $A \rightsquigarrow B \simeq A \to (1 \rightsquigarrow B)$. Further, idioms embed into arrows and arrows embed into monads.

Keywords: applicative functors, idioms, arrows, monads

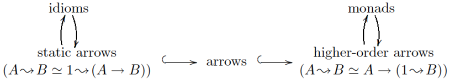

Fig. 1. Idioms, arrows and monads

$\rightsquigarrow$ here is our ⥽

ENTCS 2011, proceedings of MSFP 2008

Recap of the Curry-Howard correspondence (isomorphism?)

> *The Brouwer - Heyting - Kolmogorov - Schönfinkel - Curry - Meredith - Kleene - Feys - Gödel - Läuchli - Kreisel - Tait - Lawvere - Howard - de Bruijn - Scott - Martin-Löf - Girard - Reynolds - Stenlund - Constable - Coquand - Huet - ...- isomorphism might be a more appropriate name, still not including all the contributors.*
>
> Sørensen and Urzyczyn, *Lectures on the Curry-Howard Isomorphism*

- The Curry-Howard isomorphism = the most commonly accepted specification of the Brouwer-Heyting-Kolmogorov (BHK) interpretation of intuitionistic connectives

  Lewis does indeed meet Brouwer here!

logic	lambda-calculus
formula	type
propositional variable	type variable
connective	type constructor
implication	function space
conjunction	product
disjunction	disjoint sum
absurdity	empty type
proof	term
assumption	object variable
introduction	constructor
elimination	destructor
proof detour	redex
normalization	reduction
normal proof	normal form
provability	inhabitation

Many programs and libraries involve components that are "function-like", in that they take inputs and produce outputs, but are not simple functions from inputs to outputs ... [S]uch "notions of computation" defin[e] a common interface, called "arrows". ... Monads ... serve a similar purpose, but arrows are more general. In particular, they include notions of computation with static components, independent of the input, as well as computations that consume multiple inputs.

Ross Paterson

- I'd suggest calling FP arrows strong arrows
- They satisfy in addition the axiom

$$\mathsf{S_a}(\varphi \to \psi) \to (\varphi \multimap \psi)$$

- ... or, equivalently, $\mathsf{S}_\Box \quad \varphi \to \Box\varphi$
- Why "equivalently"?

  After all, many $\Box$-principles not equivalent to $\multimap$-counterparts

$$\begin{aligned} \varphi \to \psi &\leq \Box(\varphi \to \psi) \\ &\leq \varphi \multimap \psi \end{aligned}$$

- $\mathsf{S}_\Box$ forces $\sqsubset$ to be contained in $\preceq$:
  - rather degenerate in the classical case

    only three consistent logics of (disjoint unions of) singleton(s)
  - and yet intuitionistically you have a whole CS zoo

    type inhabitation of idioms, arrows, strong monads/PLL ...

    plus superintuitionistic logics as a degenerate case

    also a recent proposal for Intuitionistic Epistemic Logic (IEL) by Artemov and Protopopescu, which we generalize to IELE

Haskell arrows as proposed by John Hughes

```
class Arrow a where
  arr   :: (b -> c) -> a b c
  (>>>) :: a b c ->  a c d -> a b d
  first :: a b c ->  a (b, d) (c, d)
```

$$\mathsf{S_a} \qquad (\beta \to \gamma) \to (\beta \multimap \gamma)$$

$$\mathsf{Tra} \quad (\beta \multimap \gamma) \wedge (\gamma \multimap \delta) \to (\beta \multimap \delta)$$

$$\mathsf{K'_a} \qquad (\beta \multimap \gamma) \to ((\beta \wedge \delta) \multimap (\gamma \wedge \delta))$$

Where's Di ?

```
class Arrow arr => ArrowChoice arr where
  (|||) :: arr a c -> arr b c ->
           arr (Either a b) c
```

Di $\quad ((\alpha \rightsquigarrow \gamma) \wedge (\beta \rightsquigarrow \gamma)) \to ((\alpha \vee \beta) \rightsquigarrow \gamma)$

Arrows with choice?

- + function spaces
- + applicative functors/ arrows with delay
- + monads (see the next slide)
- + Kleisli arrows
- + co-Kleisli arrows if monad distributes over coproducts
- + list processors the interleaving pattern in the output is modelled on the interleaving of the input

- \- automata transforming elements of type a to elements of type b that satisfy the isomorphism $\mathsf{A}\,a\,b \cong a \to b \times (\mathsf{A}\,a\,b)$
- \- functions on infinite streams

Monads

(recall they allow decomposing $\beta \strictif \gamma$ as $\beta \to \Box\gamma$)

```
class Arrow  => ArrowApply a where
app :: a (a b c, b) c
```

$$\mathsf{App_a} \qquad ((\beta \strictif \gamma) \land \beta) \strictif \gamma$$

Recall it's precisely Lewis's B7!
The only S2 axiom underivable in $\mathsf{HLC}^\sharp$...

A1.	$p\,q \,.\, ⥽ \,.\, q\,p$	B1.	$p\,q \,.\, ⥽ \,.\, q\,p$
A2.	$q\,p \,.\, ⥽ \,.\, p$	B2.	$p\,q \,.\, ⥽ \,.\, p$
A3.	$p \,.\, ⥽ \,.\, p\,p$	B3.	$p \,.\, ⥽ \,.\, p\,p$
A4.	$p(q\,r) \,.\, ⥽ \,.\, q(p\,r)$	B4.	$(p\,q)r \,.\, ⥽ \,.\, p(q\,r)$
A5.	$p ⥽ \sim(\sim p)$	B5.	$p ⥽ \sim(\sim p)$
A6.	$p ⥽ q \,.\, q ⥽ r :⥽.\, p ⥽ r$	B6.	$p ⥽ q \,.\, q ⥽ r :⥽.\, p ⥽ r$
A7.	$\sim\Diamond p ⥽ \sim p$	B7.	$p \,.\, p ⥽ q :⥽.\, q$
A8.	$p ⥽ q \,.\, ⥽ \,.\, \sim\Diamond q ⥽ \sim\Diamond p$	B8.	$\Diamond(p\,q) ⥽ \Diamond p$
		B9.	$(\exists\, p, q) : \sim(p ⥽ q) \,.\, \sim(p ⥽ \sim q)$

S3 (A1–A8); S1 (B1–B7); S2 (B1–B8)

stated as (strict) equivalence in SSL (1918

The primitive ideas and definitions are not identical in the two cases; but they form equivalent sets, in connection with the postulates.

Comparison of these two sets of postulates, as well as many

Intuitionistic coreflection

- Artemov and Protopopescu use the BHK interpretation and the Curry-Howard correspondence to argue for coreflection in their IEL (Intuitionistic Epistemic Logic)
- That is, if $\Box$ is read as "knowledge", $\mathsf{S}_\Box$ should be valid
- With Jim de Groot (and Dirk Pattinson), we propose IELE (Intuitionistic Epistemic Logic of Entailments)

  $\varphi \dashv\!\!\!\!3\, \psi \equiv$ "the (Brouwerian) agent knows that φ entails ψ".

  which includes $\mathsf{S_a}$, as

  Intuitionistic implication $\Rightarrow$ knowledge of implication
- Recall $\mathsf{S}_\Box$ and $\mathsf{S_a}$ are equivalent

Intuitionistic reflection

- Conversely, known implications cannot be false
- Therefore one cannot intuitionistically falsify any implication that is known
- This gives rise to the following generalisation of intuitionistic reflection ($\Box\varphi \to \neg\neg\varphi$)

$$\mathsf{IR}\ (\varphi \dashv\!\!3\ \psi) \to \neg\neg(\varphi \to \psi)$$

- Intuitionistically equivalent to

$$\mathsf{IR}'\ (\varphi \dashv\!\!3\ \psi) \to (\varphi \to \neg\neg\psi)$$

- Lemma: In IELE, we can derive $\neg(\varphi \dashv\!\!3\ \psi) \leftrightarrow \neg(\varphi \to \psi)$
- But otherwise, $\varphi \dashv\!\!3\ \psi$ does not collapse to either $\varphi \to \psi$, $\Box(\varphi \to \psi)$ or $\varphi \to \Box\psi$

Questions

- What are other valid principles?
- In particular should Di be valid under this interpretation?
- Is there an "epistemic arrow calculus"' comparable to those developed in FP?
- How does our proposal compare to other intuitionistic epistemic logics?

Bonus derivation

Collapsing $⥽$ to $\Box$ with (C)Di and CPC

On the one hand,

$$\varphi ⥽ \psi \leq \varphi ⥽ (\varphi \to \psi)$$

On the other hand,

$$\begin{aligned} \top &= (\neg\varphi) \to (\varphi \to \psi) \\ &= (\neg\varphi) ⥽ (\varphi \to \psi) \end{aligned}$$

Putting both hands together

$$\varphi ⥽ \psi \leq (\varphi ⥽ (\varphi \to \psi)) \wedge ((\neg\varphi) ⥽ (\varphi \to \psi))$$

Now using (C)Di

$$\varphi ⥽ \psi \leq (\varphi \vee \neg\varphi) ⥽ (\varphi \to \psi)$$

Collapsing $\prec$ to $\Box$ with (C)Di and CPC II

In the opposite direction,

$$\alpha \prec (\beta \to \gamma) \leq (\alpha \wedge \beta) \to \gamma$$

is another variant of $\mathsf{K_a}$. Thus

$$(\varphi \vee \neg\varphi) \prec (\varphi \to \psi) \leq ((\varphi \vee \neg\varphi) \wedge \varphi) \prec \psi \qquad = \varphi \prec \psi$$

Putting both slides together

$$\varphi \prec \psi = (\varphi \vee \neg\varphi) \prec (\varphi \to \psi)$$

Note no other classical tautology would work as the strict antecedent!