Characteristic Logics for Behavioural Metrics via Fuzzy Lax Extensions

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Fuzzy Relations

Definition

A *fuzzy relation* between A and B is a map $R: A \times B \to [0,1]$, also written $R: A \nrightarrow B$.

Let $R, R_1: A \nrightarrow B, R_2: B \nrightarrow C$ and $f: A \to B$.

- Converse relation $R^\circ: B \nrightarrow A$, $R^\circ(b,a) = R(a,b)$.
- Composition: $(R_1; R_2)(a,c) = \inf_{b \in B} R_1(a,b) \oplus R_2(b,c)$.
- ϵ-graph of f: $\mathsf{Gr}_{\epsilon,f}: A \nrightarrow B$,

$$\mathsf{Gr}_{\epsilon,f}(a,b) = \begin{cases} \epsilon, & \text{if } f(a) = b \\ 1, & \text{otherwise.} \end{cases}$$

We write $\mathsf{Gr}_f = \mathsf{Gr}_{0,f}$. If $f = \mathrm{id}_A$, we write $\mathsf{Gr}_{\epsilon,f} = \Delta_{\epsilon,A}$ and $\mathsf{Gr}_f = \Delta_A$.

Fuzzy Lax Extensions

Definition (Fuzzy lax extensions)

A *(fuzzy) lax extension* L of T maps each fuzzy relation $R\colon A \nrightarrow B$ to a fuzzy relation $LR\colon TA \nrightarrow TB$ such that

$$
\begin{aligned}
&\text{(L0)} \quad L(R^\circ) = (LR)^\circ \\
&\text{(L1)} \quad R_1 \le R_2 \Rightarrow LR_1 \le LR_2 \\
&\text{(L2)} \quad L(R;S) \le LR;LS \\
&\text{(L3)} \quad L\mathsf{Gr}_f \le \mathsf{Gr}_{Tf}
\end{aligned}
$$

for all sets A, B and for all $R, R_1, R_2\colon A \nrightarrow B$, $S\colon B \nrightarrow C$, $f\colon A \to B$. We say that L is *non-expansive*, if additionally

$$\text{(L4)} \quad L\Delta_{\epsilon,A} \le \Delta_{\epsilon,TA}$$

for all sets A and $\epsilon > 0$.

Nonexpansivity and Naturality

Lemma

Let L be a fuzzy lax extension of T. Then the following are equivalent.

1. *L satisfies Axiom (L4) (i.e. is non-expansive).*
2. *For all functions $f\colon A \to B$ and all $\epsilon > 0$,*
$$L\mathsf{Gr}_{\epsilon,f} \leq \mathsf{Gr}_{\epsilon,Tf}.$$
3. *For all sets A, B, the map $R \mapsto LR$ is non-expansive w.r.t. the supremum metric on $A \nrightarrow B$.*

Lemma (Naturality)

Let $R\colon A' \nrightarrow B'$, and let $f\colon A \to A', g\colon B \to B'$. Then
$$L(R \circ (f \times g)) = LR \circ (Tf \times Tg).$$

Functor Liftings

Lemma

A fuzzy relation $d\colon X \nrightarrow X$ *is a pseudometric iff*

$$d \le \Delta_X \quad \text{(reflexivity)}$$
$$d^\circ = d \quad \text{(symmetry)}$$
$$d \le d;d \quad \text{(triangle inequality).}$$

Lemma

1. *If* $d\colon X \nrightarrow X$ *is a pseudometric, then* Ld *is a pseudometric on* TX.
2. *If* f *is a nonexpansive map* $f\colon (X,d_1) \to (Y,d_2)$*, the map* $Tf\colon (TX, Ld_1) \to (TY, Ld_2)$ *is non-expansive.*

Thus, L *lifts* $T\colon \mathsf{Set} \to \mathsf{Set}$ to a functor $\overline{T}\colon \mathsf{PMet} \to \mathsf{PMet}$.

Hausdorff lifting

H is a nonexpansive fuzzy lax extension of the powerset functor $\mathcal{P}$, given by

$$HR(U,V) = \max(\sup_{a\in U}\inf_{b\in V} R(a,b),\ \sup_{b\in V}\inf_{a\in U} R(a,b)).$$

for $U \subseteq A, V \subseteq B$.

Note the parallels to the Egli-Milner extension, which gives rise to the usual notion of bisimulation on Kripke frames:

$$(V,W) \in \overline{\mathcal{P}}(R) \iff (\forall x \in V.\, \exists y \in W.\, (x,y) \in R) \wedge (\forall y \in W.\, \exists x \in V.\, (x,y) \in R).$$

Quantitative Bisimulations

Definition

Let L be a lax extension of T, and let $\alpha\colon A \to TA$ and $\beta\colon B \to TB$ be coalgebras.

1. $R\colon A \nrightarrow B$ is an *L-bisimulation* if $LR \circ (\alpha \times \beta) \leq R$.
2. *L-behavioural distance* $d^L_{\alpha,\beta}\colon A \nrightarrow B$ is given by:

$$d^L_{\alpha,\beta} = \inf\{R\colon A \nrightarrow B \mid R \text{ is an } L\text{-bisimulation}\}.$$

If $\alpha = \beta$, we write $d^L_\alpha = d^L_{\alpha,\beta}$ instead.

Note that by Knaster-Tarski $d^L_{\alpha,\beta}$ is itself an L-bisimulation and also

$$Ld^L_{\alpha,\beta} \circ (\alpha \times \beta) = d^L_{\alpha,\beta}.$$

Behavioural Equivalence

Every fuzzy lax extension L gives rise to a *crisp* lax extension L_c, where for $R\colon A \times B \to \{0,1\}$, $L_c R(t_1, t_2) = 1 \iff LR(t_1, t_2) > 0$.

Definition (Behavioural equivalence)

Two states a and b in coalgebras (A, α) and (B, β) are *behaviourally equivalent* if there is a coalgebra (C, γ) and morphisms $f\colon (A, \alpha) \to (C, \gamma), g\colon (B, \beta) \to (C, \gamma)$ with $f(a) = g(b)$.

In [MV15] it is shown that L_c characterizes behavioural equivalence if it *preserves diagonals*. In our setting this becomes:

Lemma

Let L be a fuzzy lax extension of T such that $L\Delta_A$ is a metric for every set A. Then a, b are behaviourally equiv. iff $d^L_{\alpha,\beta}(a, b) = 0$.

Bisimulation Example

Let $TX = [0,1] \times \mathcal{P}X$ with nonexpansive lax extension L, given by

$$LR((p,U),(q,V)) = \frac{1}{2}(|p-q| + HR(U,V)),$$

where $R\colon A \nrightarrow B$, $p, q \in [0,1], U \subseteq A, V \subseteq B$.
Now consider the T-coalgebras (A,α) and (B,β) below:

Let $R\colon A \nrightarrow B$, with $R(a_1,b_1) = 0.2, R(a_2,b_3) = 0.1, R(a_3,b_2) = 0.05$ and $R(a_i,b_j) = 1$ otherwise.

Then $d^L_{\alpha,\beta}(a_1,b_1) \leq 0.2$, as witnessed by the L-bisimulation R.

Fuzzy Predicate Liftings

An n*-ary (fuzzy) predicate lifting* is a natural transformation

$$\lambda\colon \mathcal{Q}^n \Rightarrow \mathcal{Q} \circ T.$$

The *dual* $\bar{\lambda}$ of λ is given by $\bar{\lambda}(f_1, \dots, f_n) = 1 - \lambda(1 - f_1, \dots, 1 - f_n)$.

We call λ

- *monotone* if
  $f_1 \leq g_1, \dots, f_n \leq g_n \implies \lambda(f_1, \dots, f_n) \leq \lambda(g_1, \dots, g_n)$.
- *nonexpansive* if

$$\|\lambda_X(f_1, \dots, f_n) - \lambda_X(g_1, \dots, g_n)\|_\infty \leq \max(\|f_1 - g_1\|_\infty, \dots, \|f_n - g_n\|_\infty).$$

Nonexpansive Pairs

Definition

Let $R: A \nrightarrow B$. A pair (f, g) of functions $f: A \to [0,1]$ and $g: B \to [0,1]$ is *R-nonexpansive* if $f(a) - g(b) \leq R(a,b)$ for all $a \in A, b \in B$.

Definition

Let $R: A \nrightarrow B$ and $f: A \to [0,1]$. Then we define $R[f]: B \to [0,1]$ as follows:

$$R[f](b) = \sup_{a \in A} f(a) \ominus R(a,b),$$

where for $x, y \in [0,1]$, $x \ominus y = \max(x - y, 0)$.

Lemma

(f, g) is R-nonexpansive $\iff R[f] \leq g$.

The Kantorovich Lifting

Definition (Kantorovich Lifting)

Let Λ be a set of monotone predicate liftings that is closed under duals. The *Kantorovich lifting* K_Λ is defined as follows: for $R: A \nrightarrow B$, $K_\Lambda R: TA \nrightarrow TB$ is given by

$$K_\Lambda R(t_1, t_2) = \sup\{\lambda_A(f_1, \dots, f_n)(t_1) - \lambda_B(g_1, \dots, g_n)(t_2) \mid \lambda \in \Lambda \ n\text{-ary}, (f_1, g_1), \dots (f_n, g_n) \ R\text{-nonexpansive}\}.$$

Theorem

Let Λ be a set of monotone predicate liftings that is closed under duals. The Kantorovich lifting K_Λ is a lax extension. If all $\lambda \in \Lambda$ are nonexpansive, then K_Λ is nonexpansive as well.

Kantorovich Examples

Example

The standard Kantorovich lifting K of the discrete distribution functor $\mathcal{D}$ is an instance of the generic one, for the single predicate lifting $\Diamond(f)(\mu) = \mathbb{E}_\mu(f)$.

Example

Consider the *fuzzy neighbourhood functor* $\mathcal{N} = \mathcal{Q} \circ \mathcal{Q}$; it has a subfunctor $\mathcal{M}$ given by

$$\mathcal{M}X = \{A \in \mathcal{N}X \mid A\colon \mathcal{Q}X \to [0,1] \text{ is monotone and nonexp.}\}$$

We put $LR(A,B) =$

$$\max(\sup_{f \in \mathcal{Q}X} A(f) \ominus B(R[f]), \sup_{g \in \mathcal{Q}X} B(g) \ominus A(R^\circ[g]))$$

for $R\colon A \nrightarrow B$, $A \in \mathcal{M}X$, $B \in \mathcal{M}Y$. Then $L = K_{\{\lambda\}}$ where λ is the predicate lifting given by $\lambda_X(f)(A) = A(f)$.

Couplings

Definition (Couplings)

Let $t_1 \in TA, t_2 \in TB$ for sets A, B. The set of *couplings* of t_1 and t_2 is $\mathsf{Cpl}(t_1, t_2) = \{t \in T(A \times B) \mid T\pi_1(t) = t_1, T\pi_2(t) = t_2\}$.

If T preserves weak pullbacks, we have:

Lemma (Gluing Lemma)

Let A, B and C be sets and let $t_1 \in TA, t_2 \in TB, t_3 \in TC$. Let $t_{12} \in \mathsf{Cpl}(t_1, t_2)$ and $t_{23} \in \mathsf{Cpl}(t_2, t_3)$. Then there exists $t_{123} \in \mathsf{Cpl}(t_1, t_2, t_3)$ such that

$$T\langle \pi_1, \pi_2 \rangle (t_{123}) = t_{12} \quad \textit{and} \quad T\langle \pi_2, \pi_3 \rangle (t_{123}) = t_{23},$$

where the π_i are the projections of the product $A \times B \times C$. Moreover, $t_{13} := T\langle \pi_1, \pi_3 \rangle (t_{123}) \in \mathsf{Cpl}(t_1, t_3)$.

Wasserstein Lifting (1)

Definition (Wasserstein lifting)

Let Λ be a set of unary predicate liftings. The *Wasserstein lifting* W_Λ of T is defined for $R\colon A \nrightarrow B$ by

$$W_\Lambda R(t_1, t_2) = \sup_{\lambda \in \Lambda} \inf\{\lambda_{A \times B}(R)(t) \mid t \in \mathsf{Cpl}(t_1, t_2)\}.$$

Wasserstein Lifting (2)

Definition

Let λ be a unary predicate lifting.

1. λ is *subadditive* if for all sets X and all $f, g \in \mathcal{Q}X$, $\lambda_X(f \oplus g) \leq \lambda_X(f) \oplus \lambda_X(g)$.
2. λ *preserves the zero function* if for all sets X, $\lambda_X(0_X) = 0_{TX}$, where $0_X : x \mapsto 0$.
3. λ is *standard* if it is monotone, subadditive, and preserves the zero function.

Theorem

If T preserves weak pullbacks and Λ is a set of standard predicate liftings, then the Wasserstein lifting W_Λ is a lax extension. If additionally all $\lambda \in \Lambda$ are nonexpansive, then W_Λ is nonexpansive as well.

Wasserstein Examples

Example

The Hausdorff lifting H is the Wasserstein lifting $W_{\{\lambda\}}$ for $\mathcal{P}$, where $\lambda_X(f)(A) = \sup f[A]$ for $A \subseteq X$.

Example

The convex powerset functor $\mathcal{C}$ [BSS17] maps a set X to the set of nonempty convex subsets of $\mathcal{D}X$. The Wasserstein lifting $W_{\{\lambda\}}$, where $\lambda_X(f)(A) = \sup_{\mu \in A} \mathbb{E}_\mu(f)$ for $A \in \mathcal{C}X$, is a non-expansive lax extension of $\mathcal{C}$.

Adequacy and Separation

Let L be a lax extension of T and Λ be a set of predicate liftings.

Definition

An n-ary predicate lifting λ *preserves nonexpansivity* if for all fuzzy relations R and all R-nonexpansive pairs $(f_1, g_1), \ldots, (f_n, g_n)$, the pair $(\lambda_A(f_1, \ldots, f_n), \lambda_B(g_1, \ldots, g_n))$ is LR-nonexpansive. A set Λ of predicate liftings *preserves nonexpansivity* if all $\lambda \in \Lambda$ do.

Lemma (Adequacy)

We have $K_\Lambda R \leq LR$ if and only if Λ preserves nonexpansivity.

Definition (Separation)

A set Λ of predicate liftings is *separating* for L if $K_\Lambda R \geq LR$ for all fuzzy relations R.

Finitary Separability

Definition (Finitary part)

The *finitary part* of T is the functor T_ω given by

$$T_\omega X = \bigcup\{Ti[TY] \mid Y \subseteq X \text{ finite}, i\colon Y \to X \text{ inclusion}\}$$

Definition (Finitary Separability)

A fuzzy lax extension L for the functor T is *finitarily separable* if for every set X, $T_\omega X$ is a dense subset of TX wrt. to the pseudometric $L\Delta_X$.

Example

1. The Kantorovich lifting K of $\mathcal{D}$ is finitarily separable.
2. The Hausdorff lifting H of $\mathcal{P}$ is not finitarily separable.

Lax Extensions as Kantorovich Liftings

Theorem

If L is a finitarily separable lax extension of T, then there exists a set Λ of monotone predicate liftings that preserves nonexpansivity and is separating for L, i.e. $L = K_\Lambda$. Moreover, L is nonexpansive iff Λ can be chosen such that all $\lambda \in \Lambda$ are nonexpansive.

We work with a *finitary presentation* of T_ω, consisting of:

- a *signature* Σ of operations with given finite arities
- for each $\sigma \in \Sigma$ of arity n a natural transformation $\sigma\colon (-)^n \Rightarrow T_\omega$

such that every element of $T_\omega X$ has the form $\sigma_X(x_1, \dots, x_n)$ for some $\sigma \in \Sigma$.

Moss Liftings

Definition

Let $\sigma \in \Sigma$ be n-ary. The *Moss lifting* $\mu^\sigma \colon \mathcal{Q}^n \Rightarrow \mathcal{Q} \circ T$ is defined as follows:

$$\mu^\sigma_X(f_1, \dots, f_n)(t) = L\mathrm{ev}_X(\sigma_{\mathcal{Q}X}(f_1, \dots, f_n), t),$$

where $\mathrm{ev}_X \colon \mathcal{Q}X \nrightarrow X$ is given by $\mathrm{ev}_X(f, x) = f(x)$.

We take Λ to be the set of all Moss liftings and their duals:

$$\Lambda = \{\mu^\sigma \mid \sigma \in \Sigma\} \cup \{\overline{\mu^\sigma} \mid \sigma \in \Sigma\}$$

Lemma

Λ is a set of monotone predicate liftings that preserves nonexpansivity and is separating for L. If L is nonexpansive, then so are all predicate liftings in Λ.

Real-valued Coalgebraic Modal Logic

Let Λ be a set of predicate liftings.

Syntax:

$$\varphi, \psi ::= c \mid \varphi \ominus c \mid \neg\varphi \mid \varphi \wedge \psi \mid \lambda(\varphi_1, \ldots, \varphi_n)$$

Semantics:

$$\begin{aligned}
[\![c]\!](a) &= c \\
[\![\varphi \ominus c]\!](a) &= \max([\![\varphi]\!](a) - c, 0) \\
[\![\neg\varphi]\!](a) &= 1 - [\![\varphi]\!](a) \\
[\![\varphi \wedge \psi]\!](a) &= \min([\![\varphi]\!](a), [\![\psi]\!](a)) \\
[\![\lambda(\varphi_1, \ldots, \varphi_n)]\!](a) &= \lambda_A([\![\varphi_1]\!], \ldots, [\![\varphi_n]\!])(\alpha(a))
\end{aligned}$$

Nonexpansivity of Modal Logic

Definition

The Λ-*logical distance* between states $a \in A$, $b \in B$ in T-coalgebras (A, α), (B, β) is $d^\Lambda(a, b) = \sup\{|[\![\varphi]\!](a) - [\![\varphi]\!](b)| \mid \varphi \in \mathcal{L}_\Lambda\}$.

Because the logical connectives are nonexpansive, we have:

Lemma

If Λ preserves nonexpansivity with respect to a lax extension L, then $d^\Lambda \leq d^L$.

Fixpoint Iteration

Theorem

Let L be a non-expansive finitarily separable lax extension of T. Given T-coalgebras (A, α), (B, β), define a sequence $(d_n \colon A \nrightarrow B)_{n<\omega}$ and $d_\omega \colon A \nrightarrow B$ by

$$d_0 = 0, \qquad d_{n+1} = Ld_n \circ (\alpha \times \beta), \qquad d_\omega = \sup_{n<\omega} d_n.$$

Then $Ld_\omega \circ (\alpha \times \beta) = d_\omega$ and $d^L_{\alpha,\beta} = d_\omega$.

Hennessy-Milner Theorem

Theorem ([KMM18])

Let Λ be a set of predicate liftings such that iterative approximation of the fixpoint d^{K_Λ} as on the previous slide stabilizes in ω steps. Then $d^\Lambda \geq d^{K_\Lambda}$.

Corollary (Coalgebraic quantitative Hennessy-Milner theorem)

Let L be a non-expansive finitarily separable fuzzy lax extension, and let Λ be a separating set of monotone non-expansive predicate liftings for L. Then $d^\Lambda \geq d^L$.

Future work

- Generalize to *quantale*-valued relations.
- Consider different flavours of semantics for fuzzy logic, such as Łukasiewicz semantics
- As a special case of our main result, we show that all Wasserstein extensions are also Kantorovich extensions. How does this tie in with the *Kantorovich-Rubinstein Duality* for the discrete distribution functor $\mathcal{D}$?

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