APPENDIX TO: "ABSTRACT GSOS RULES AND A MODULAR TREATMENT OF RECURSIVE DEFINITIONS" [LOG. METHODS COMPUT. SCI XX (XX:XX), XX PP., 2013]

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ABSTRACT. This is an appendix that provides additional results and proof details that we omitted from our paper due to space constraints.

APPENDIX A. RESULTS FOR POINTED FUNCTORS

We mentioned in Remark 3.7 that Theorems 3.5 and 3.6 hold more generally for pointed endofunctors M in lieu of a free monad $M = \widehat{K}$. However, in this case we need our base category to be cocomplete. In this appendix we provide the details.

Assumption A.1. We assume that $\mathcal{A}$ is a cocomplete category, that $H : \mathcal{A} \to \mathcal{A}$ is a functor and that (M, η) is a *pointed functor* on $\mathcal{A}$, i. e., $M : \mathcal{A} \to \mathcal{A}$ is a functor and $\eta : \mathrm{Id} \to M$ is a natural transformation. As before $c : C \to HC$ is a terminal coalgebra for H.

Definition A.2. (1) An *algebra for* (M, η) is a pair (A, a) where A is an object of $\mathcal{A}$ and $a : MA \to A$ is a morphism satisfying the *unit law* $a \cdot \eta_A = \mathrm{id}_A$.

(2) A *distributive law of* M *over* H is a natural transformation $\lambda : MH \to HM$ such that the diagram

$$\begin{tikzcd} & H \arrow[dl, "\eta H"'] \arrow[dr, "H\eta"] & \\ MH \arrow[rr, "\lambda"] & & HM \end{tikzcd} \tag{A.1}$$

commutes.

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(3) Let (D, ε) be a copointed endofunctor on $\mathcal{A}$. A *distributive law of* (M, η) *over* (D, ε) is a distributive law $\lambda : MD \to DM$ of (M, η) over the functor D that makes, in addition to (A.1) with H replaced by D, the diagram

$$\begin{tikzcd} MD \arrow[rr, "\lambda"] \arrow[dr, "M\varepsilon"'] & & DM \arrow[dl, "\varepsilon M"] \\ & M & \end{tikzcd}$$

commute.

Remark A.3. (1) Every distributive law $\lambda : MH \to HM$ gives a distributive law of the cofree copointed functor $H \times \mathrm{Id}$ via

$$\begin{tikzcd} MH \arrow[rr, "\lambda"] & & HM \\ M(H \times \mathrm{Id}) \arrow[u, "M\pi_0"] \arrow[rr, dashed] \arrow[drr, "M\pi_1"'] & & (H \times \mathrm{Id})M \arrow[u, "\pi_0"'] \arrow[d, "\pi_1"] \\ & & M \end{tikzcd} \tag{A.2}$$

but not conversely.

(2) Analogously to Theorem 2.7, we have for any distributive law λ of M over the cofree copointed functor $H \times \mathrm{Id}$ a unique λ-*interpretation*, i. e., a unique morphism $b : MC \to C$ such that the diagram below commutes

$$\begin{tikzcd} MC \arrow[r, "M\langle c, \mathrm{id}_C\rangle"] \arrow[d, "b"'] & M(HC \times C) \arrow[r, "\lambda_C"] & HMC \times MC \arrow[r, "\pi_0"] & HMC \arrow[d, "Hb"] \\ C \arrow[rrr, "c"] & & & HC \end{tikzcd}$$

and (C, b) is an algebra for the pointed functor M, see [1]. Notice that b here corresponds to $\widehat{b} : \widehat{K}C \to C$ in Theorem 2.7. If we have a distributive law $\lambda : MH \to HM$, then we obtain one of M over the copointed functor $H \times \mathrm{Id}$ as in (A.2). We again call the resulting morphism $b : MC \to C$ the λ-interpretation in C. In this case, the diagram above simplifies to

$$\begin{tikzcd} MC \arrow[r, "Mc"] \arrow[d, "b"'] & MHC \arrow[r, "\lambda_C"] & HMC \arrow[d, "Hb"] \\ C \arrow[rr, "c"] & & HC\,. \end{tikzcd} \tag{A.3}$$

Next we shall need a version of Theorem 2.12 for a given distributive law λ of M over H (or over the cofree copointed functor $H \times \mathrm{Id}$). This is a variation of Theorem 4.2.2 of Bartels [1] (see also Lemma 4.3.2 in loc. cit.) using the cocompleteness of $\mathcal{A}$. Since one part (the uniqueness part) of the proof in [1] is only presented for Set we give a full proof here for the convenience of the reader.

Theorem A.4. *Let $\lambda : MH \to HM$ be a distributive law of the pointed functor M over the functor H. Then for every λ-*equation *$e : X \to HMX$ there exists a unique solution, i. e., a unique*

morphism $e^\dagger : X \to C$ such that the diagram below commutes:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "e^\dagger"'] & HMX \arrow[d, "HMe^\dagger"] \\ C \arrow[r, "c"] & HC & HMC \arrow[l, "Hb"'] \end{tikzcd} \tag{A.4}$$

Before we proceed to the proof of the statement we need some auxiliary constructions and lemmas. We begin by defining an endofunctor S on our cocomplete category $\mathcal{A}$ as a colimit. We denote by M^n, $n \in \mathbb{N}$, the n-fold composition of M with itself. Now we consider the diagram D in the category of endofunctors on $\mathcal{A}$ given by the natural transformations in the picture below:

$$\begin{tikzcd} \mathrm{Id} \arrow[r, "\eta"] & M \arrow[r, bend left, "\eta M"] \arrow[r, bend right, "M\eta"'] & MM \arrow[r, bend left, "\eta MM"] \arrow[r, "M\eta M"] \arrow[r, bend right, "MM\eta"'] & \cdots \end{tikzcd}$$

More formally, the diagram D is formed by all natural transformations

$$M^{i+j} \xrightarrow{M^i \eta M^j} M^{i+1+j} \qquad i, j \in \mathbb{N}\,.$$

Let S be a colimit of this diagram D:

$$S = \operatorname{colim} D \qquad \text{with injections } \mathsf{inj}^i : M^i \to S.$$

Then S is a pointed endofunctor with the point $\mathsf{inj}^0 : \mathrm{Id} = M^0 \to S$.

Recall that colimits in the category of endofunctors of $\mathcal{A}$ are formed objectwise. So for any object X, SX is a colimit of the diagram D at that object X with colimit injections $\mathsf{inj}^n_X : M^n X \to SX$, $n \in \mathbb{N}$. This implies that for any endofunctor F of $\mathcal{A}$ the functor SF is a colimit with injections $\mathsf{inj}^n F : M^n F \to SF$.

The above definition of S appears in Bartels [1]. Next we define additional data using the universal property of the colimits SM and SH:

(1) a natural transformation $\chi : SM \to S$ uniquely determined by the commutativity of the triangles below:

$$\begin{tikzcd} M^{n+1} \arrow[d, "\mathsf{inj}^n M"'] \arrow[dr, "\mathsf{inj}^{n+1}"] & \\ SM \arrow[r, "\chi"'] & S \end{tikzcd} \qquad \text{for all } n \in \mathbb{N}.$$

(2) a natural transformation $\varepsilon : SM \to MS$ uniquely determined by the commutativity of the triangles below:

$$\begin{tikzcd} M^{n+1} \arrow[d, "\mathsf{inj}^n M"'] \arrow[dr, "M\mathsf{inj}^n"] & \\ SM \arrow[r, "\varepsilon"'] & MS \end{tikzcd} \qquad \text{for all } n \in \mathbb{N}.$$

(3) a natural transformation $\lambda^* : SH \to HS$; indeed, define first $\lambda^n : M^n H \to HM^n$ recursively as follows:

$$\begin{aligned} \lambda^0 &= \mathrm{id}_H : H \to H; \\ \lambda^{n+1} &= M^{n+1}H = MM^nH \xrightarrow{M\lambda^n} MHM^n \xrightarrow{\lambda M^n} HMM^n = HM^{n+1}\,. \end{aligned}$$

Then λ^* is uniquely determined by the commutativity of the squares below:

$$\begin{tikzcd} M^nH \arrow[r, "\lambda^n"] \arrow[d, "\mathsf{inj}^nH"'] & HM^n \arrow[d, "H\mathsf{inj}^n"] \\ SH \arrow[r, "\lambda^*"'] & HS \end{tikzcd} \qquad \text{for all } n \in \mathbb{N}.$$

Observe that λ^* is a distributive law of the pointed endofunctor S over H; the unit law is the above square for the case $n = 0$.

We now need to verify that the three natural transformations above are well-defined. More precisely, we need to prove that those natural transformations are induced by appropriate cocones. For $\chi : SM \to S$ and $\lambda^* : SH \to HS$, this follows from Lemma 4.3.2 in Bartels' thesis [1]. Hence, we make the explicit verification only for ε and leave the details for the other two natural transformations for the reader. To verify that the natural transformations $M\mathsf{inj}^n : M^{n+1} \to MS$ form a cocone for the appropriate diagram with colimit SM consider the triangles below:

$$\begin{tikzcd} M^{1+i+j} \arrow[rr, "MM^i\eta M^j"] \arrow[dr, "M\mathsf{inj}^n"'] & & M^{1+i+1+j} \arrow[dl, "M\mathsf{inj}^{n+1}"] \\ & MS & \end{tikzcd} \qquad \text{for all } n \in \mathbb{N},\ n = i + j.$$

These triangles commute since $\mathsf{inj}^n : M^n \to S$ form a cocone.

Next, notice that in the definition of λ^* above there are two possible canonical choices for λ^{n+1}. We now show that these two choices are equal:

Lemma A.5. *For all natural numbers n we have the commutative square below:*

$$\begin{tikzcd} M^{n+1}H \arrow[r, "M^n\lambda"] \arrow[d, "M\lambda^n"'] & M^nHM \arrow[d, "\lambda^nM"] \\ MHM^n \arrow[r, "\lambda M^n"'] & HM^{n+1}. \end{tikzcd}$$

Proof. We prove the result by induction on n. The base case $n = 0$ is clear: both composites in the desired square are simply $\lambda : MH \to HM$. For the induction step we need to verify that the diagram below commutes:

$$\begin{tikzcd} M^{n+1}MH \arrow[r, "M^{n+1}\lambda = MM^n\lambda"] \arrow[d, "MM\lambda^n"'] \arrow[dd, bend right=60, "M\lambda^{n+1}"'] & M^{n+1}HM \arrow[d, "M\lambda^nM"] \arrow[dd, bend left=60, "\lambda^{n+1}M"] \\ MMHM^n \arrow[r, "M\lambda M^n"] \arrow[d, "M\lambda M^n"'] & MHM^nM \arrow[d, "\lambda M^nM"] \\ MHM^{n+1} \arrow[r, "\lambda M^{n+1}"'] & HM^{n+1}M \end{tikzcd}$$

The left-hand and right-hand parts both commute due to the definition of λ^{n+1}. The lower square obviously commutes, and for the commutativity of the upper one apply the functor M to the induction hypothesis. Thus the desired outside square commutes. $\square$

Next we need to establish a couple of properties connecting the three natural transformations χ, ε and λ^*.

Lemma A.6. *The following diagram of natural transformations commutes:*

$$\begin{tikzcd} SMM \arrow[r, "\chi M"] \arrow[d, "\varepsilon M"'] & SM \arrow[d, "\varepsilon"] \\ MSM \arrow[r, "M\chi"'] & MS. \end{tikzcd}$$

Proof. To verify that the square in the statement commutes we extend that square by precomposing with the injections into the colimit SMM. This yields the following diagram:

$$\begin{tikzcd} M^nMM \arrow[rrr, equal] \arrow[dr, "\mathsf{inj}^n MM"] \arrow[ddd, equal] & & & M^{n+1}M \arrow[dl, "\mathsf{inj}^{n+1}M"'] \arrow[ddd, equal] \\ & SMM \arrow[r, "\chi M"] \arrow[d, "\varepsilon M"'] & SM \arrow[d, "\varepsilon"] & \\ & MSM \arrow[r, "M\chi"'] & MS & \\ MM^nM \arrow[rrr, equal] \arrow[ur, "M\mathsf{inj}^n M"] & & & MM^{n+1} \arrow[ul, "M\mathsf{inj}^{n+1}"'] \end{tikzcd}$$

The left-hand and right-hand inner squares commute by the definition of ε, and the upper and lower inner square commute by the definition of χ. Since the outside commutes obviously, so does the desired middle square when precomposed by any injection $\mathsf{inj}^n MM$ of the colimit SMM. Thus, the desired middle square commutes. $\square$

Lemma A.7. *The following square of natural transformations commutes:*

$$\begin{tikzcd} SMH \arrow[r, "S\lambda"] \arrow[d, "\chi H"'] & SHM \arrow[r, "\lambda^* M"] & HSM \arrow[d, "H\chi"] \\ SH \arrow[rr, "\lambda^*"'] & & HS. \end{tikzcd}$$

Proof. It suffices to verify that the desired square commutes when we extend it by precomposition with an arbitrary colimit injection $\mathsf{inj}^n MH$ of SMH. To this end we consider the diagram below:

$$\begin{tikzcd} M^nMH \arrow[rr, "M^n\lambda"] \arrow[dr, "\mathsf{inj}^n MH"] \arrow[ddd, equal] & & M^nHM \arrow[rr, "\lambda^n M"] \arrow[d, "\mathsf{inj}^n HM"] & & HM^nM \arrow[dl, "H\mathsf{inj}^n M"'] \arrow[ddd, equal] \\ & SMH \arrow[r, "S\lambda"] \arrow[d, "\chi H"'] & SHM \arrow[r, "\lambda^* M"] & HSM \arrow[d, "H\chi"] & \\ & SH \arrow[rr, "\lambda^*"'] & & HS & \\ M^{n+1}H \arrow[rrr, "\lambda^{n+1}"'] \arrow[ur, "\mathsf{inj}^{n+1}H"] & & & HM^{n+1} \arrow[ul, "H\mathsf{inj}^{n+1}"'] \end{tikzcd}$$

The left-hand and right-hand parts commute by the definition of χ, and the lower and the upper right-hand parts commute by the definition of λ^*. The upper left-hand part commutes by the naturality of inj^n. Finally, the outside commutes by the definition of λ^{n+1} together with Lemma A.5. Thus, the desired middle square commutes when extended by any colimit injection $\mathsf{inj}^n MH$ of the colimit SMH. □

Lemma A.8. *The following diagram of natural transformations commutes:*

$$\begin{tikzcd}
SMH \arrow[r, "S\lambda"] \arrow[d, "\varepsilon H"'] & SHM \arrow[r, "\lambda^* M"] & HSM \arrow[d, "H\varepsilon"] \\
MSH \arrow[r, "M\lambda^*"'] & MHS \arrow[r, "\lambda S"'] & HMS
\end{tikzcd}$$

Proof. Once more it is sufficient to verify that the desired square commutes when extended by any injection of the colimit SMH. So consider the diagram below:

$$\begin{tikzcd}
M^n MH \arrow[rr, "M^n\lambda"] \arrow[dr, "\mathsf{inj}^n MH"] \arrow[ddd, equal] & & M^n HM \arrow[rr, "\lambda^n M"] \arrow[d, "\mathsf{inj}^n HM"] & & HM^n M \arrow[dl, "H\mathsf{inj}^n M"'] \arrow[ddd, equal] \\
& SMH \arrow[r, "S\lambda"] \arrow[d, "\varepsilon H"'] & SHM \arrow[r, "\lambda^* M"] & HSM \arrow[d, "H\varepsilon"] & \\
& MSH \arrow[r, "M\lambda^*"'] & MHS \arrow[r, "\lambda S"'] & HMS & \\
MM^n H \arrow[rr, "M\lambda^n"'] \arrow[ur, "M\mathsf{inj}^n H"] & & MHM^n \arrow[rr, "\lambda M^n"'] \arrow[u, "MH\mathsf{inj}^n"] & & HMM^n \arrow[ul, "HM\mathsf{inj}^n"']
\end{tikzcd}$$

The left-hand and right-hand parts commute by the definition of ε, and the lower left-hand and upper right-hand parts commute by the definition of λ^*. The upper left-hand and the lower right-hand parts both commute due to the naturality of inj^n and λ, respectively. Finally, the outside commutes by Lemma A.5. Thus, the desired inner square commutes when extended by any colimit injection $\mathsf{inj}^n MH : M^n MH \to SMH$. □

We are now prepared to prove the statement of Theorem A.4.

Proof of Theorem A.4. Let $e : X \to HMX$ be any λ-equation. We form the following H-coalgebra:

$$\overline{e} \equiv SX \xrightarrow{Se} SHMX \xrightarrow{\lambda^*_{MX}} HSMX \xrightarrow{H\chi_X} HSX\,. \tag{A.5}$$

Since $c : C \to HC$ is a terminal H-coalgebra there exists a unique H-coalgebra homomorphism h from $(SX, \overline{e})$ to (C, c). We shall prove that the morphism

$$e^\dagger \equiv X \xrightarrow{\mathsf{inj}^0_X} SX \xrightarrow{h} C \tag{A.6}$$

is the desired unique solution of the λ-equation e.

(1) $e^\dagger$ is a solution of e. It is our task to establish that the outside of the diagram below commutes (cf. Diagram A.4):

$$\begin{tikzcd}
X \arrow[r, "\mathsf{inj}^0_X"] \arrow[rr, bend left, "e^\dagger"] \arrow[dd, "e"'] & SX \arrow[r, "h"] \arrow[d, "Se"'] & C \arrow[dd, "c"] \\
& SHMX \arrow[d, "\lambda^*_{MX}"] & \\
& HSMX \arrow[d, "H\chi_X"] & \\
& HSX \arrow[r, "Hh"'] & HC \\
HMX \arrow[uuur, "\mathsf{inj}^0_{HMX}"] \arrow[uur, "H\mathsf{inj}^0_{MX}"] \arrow[ur, "H\mathsf{inj}^1_X"'] \arrow[rr, "HMe^\dagger"'] & & HMC \arrow[u, "Hb"']
\end{tikzcd}$$

The upper part commutes by the definition of $e^\dagger$, and the upper right-hand square commutes since h is a coalgebra homomorphism. The upper left-hand part commutes due to the naturality of inj^0, the triangle below that commutes by the definition of λ^*, and the lowest triangle commutes by the definition of χ. It remains to verify that the lowest part commutes. To this end we will now establish the following equation

$$b \cdot Me^\dagger = h \cdot \mathsf{inj}^1_X \,. \tag{A.7}$$

Consider the diagram below:

$$\begin{tikzcd}
MX \arrow[rr, "Me^\dagger"] \arrow[dr, "M\mathsf{inj}^0_X"] \arrow[ddr, "\mathsf{inj}^0_{MX}"] \arrow[ddd, "\mathsf{inj}^1_X"'] & & MC \arrow[ddd, "b"] \\
& MSX \arrow[ur, "Mh"'] & \\
& SMX \arrow[u, "\varepsilon_X"'] \arrow[dl, "\chi_X"] & \\
SX \arrow[rr, "h"'] & & C
\end{tikzcd}$$

The upper triangle commutes by the definition of $e^\dagger$, the left-hand triangle commutes by the definition of χ and the inner triangle commutes by the definition of ε. In order to establish that the right-hand part commutes we will use that C is a terminal H-coalgebra. Thus, we shall exhibit H-coalgebra structures on the five objects and then show that all edges of the right-hand part of the diagram are H-coalgebra homomorphisms. Then by the uniqueness of coalgebra homomorphisms into the terminal coalgebra (C, c), we conclude that the desired part of the above diagram commutes.

For C, we use $c : C \to HC$, and for MC we use $\lambda_C \cdot Mc$. We already know that $b : MC \to C$ is a coalgebra homomorphism (see (A.3)). For SX, we use $\overline{e}$ from (A.5); again, we already know that $h : SX \to C$ is a coalgebra homomorphism. For MSX we use $\lambda_{SX} \cdot M\overline{e}$. The verification

that Mh is a coalgebra morphism comes from the diagram below:

$$\begin{tikzcd}
MSX \arrow[r, "Mh"] \arrow[d, "M\overline{e}"'] & MC \arrow[d, "Mc"] \\
MHSX \arrow[r, "MHh"] \arrow[d, "\lambda_{SX}"'] & MHC \arrow[d, "\lambda_C"] \\
HMSX \arrow[r, "HMh"'] & HMC
\end{tikzcd}$$

To see that the upper square commutes, remove M and recall that h is a coalgebra homomorphism from $(SX, \overline{e})$ to (C, c). The lower square commutes by the naturality of λ.

Now we show that $\varepsilon_X : SMX \to MSX$ is a coalgebra homomorphism, where the structure on SMX is the composite on the left below:

$$\begin{tikzcd}
SMX \arrow[rr, "\varepsilon_X"] \arrow[d, "SMe"'] & & MSX \arrow[d, "MSe"] \arrow[ddd, bend left, "M\overline{e}"] \\
SMHMX \arrow[rr, "\varepsilon_{HMX}"] \arrow[d, "S\lambda_{MX}"'] & & MSHMX \arrow[d, "M\lambda^*_{MX}"] \\
SHMMX \arrow[d, "\lambda^*_{MMX}"'] & & MHSMX \arrow[dl, "\lambda_{SMX}"'] \arrow[d, "MH\chi_X"] \\
HSMMX \arrow[r, "H\varepsilon_{MX}"] \arrow[d, "H\chi_{MX}"'] & HMSMX \arrow[dr, "HM\chi_X"'] & MHSX \arrow[d, "\lambda_{SX}"] \\
HSMX \arrow[rr, "H\varepsilon_X"'] & & HMSX
\end{tikzcd}$$

The upper square commutes by the naturality of ε, and the inner triangle commutes by the naturality of λ. To see that the right-hand part commutes, remove M and consider the definition of $\overline{e}$. The lowest part commutes due to Lemma A.6, and the middle part commutes by Lemma A.8.

Finally, we show that $\chi_X : SMX \to SX$ is a coalgebra homomorphism. To do this we consider the following diagram:

$$\begin{tikzcd}
SMX \arrow[r, "\chi_X"] \arrow[d, "SMe"'] & SX \arrow[d, "Se"] \\
SMHMX \arrow[r, "\chi_{HMX}"] \arrow[d, "S\lambda_{MX}"'] & SHMX \arrow[dd, "\lambda^*_{MX}"] \\
SHMMX \arrow[d, "\lambda^*_{MMX}"'] & \\
HSMMX \arrow[r, "H\chi_{MX}"] \arrow[d, "H\chi_{MX}"'] & HSMX \arrow[d, "H\chi_X"] \\
HSMX \arrow[r, "H\chi_X"'] & HSX
\end{tikzcd}$$

The upper square commutes by the naturality of χ, the middle square commutes by Lemma A.7, and the lower square commutes obviously. This concludes the proof that $e^\dagger$ is a solution of e.

(2) $e^\dagger$ in (A.6) is the unique solution of e. Suppose now that $e^\dagger$ is any solution of the λ-equation e. Recall that the object SX is a colimit of the diagram D at object X with the colimit injections $\mathsf{inj}^n_X : M^nX \to SX$, $n \in \mathbb{N}$. We will use the universal property of that colimit to define a morphism $h : SX \to C$. To this end we need to give a cocone $h_n : M^nX \to C$, $n \in \mathbb{N}$, for the appropriate diagram. We define this cocone inductively as follows:

$$\begin{aligned} h_0 &= e^\dagger : M^0X = X \to C; \\ h_{n+1} &= M^{n+1}X = MM^nX \xrightarrow{Mh_n} MC \xrightarrow{b} C, \qquad n \in \mathbb{N}. \end{aligned}$$

We now verify by induction on n that the morphisms h_n, $n \in \mathbb{N}$ do indeed form a cocone. For the base case consider the diagram below:

$$\begin{tikzcd} M^0X = X \arrow[r, "\eta_X"] \arrow[dr, "h_0"] \arrow[ddr, bend right, "h_0"'] & MX = M^1X \arrow[d, "Mh_0"] \\ & C \arrow[r, "\eta_C"] \arrow[dr, equal] & MC \arrow[d, "b"] \\ & & C \end{tikzcd}$$

The upper part commutes by the naturality of η, the lower triangle commutes since $b : MC \to C$ is an algebra for the pointed endofunctor M, and the left-hand part is trivial. For the induction step consider for any natural number $n = i + j$ the following diagram:

$$\begin{tikzcd} M^{n+1}X = MM^{i+j}X \arrow[rr, "MM^i\eta_{M^jX}"] \arrow[dr, "Mh_n"] \arrow[ddr, bend right, "h_{n+1}"'] & & MM^iMM^j = M^{n+2}X \arrow[dl, "Mh_{n+1}"'] \arrow[ddl, bend left, "h_{n+2}"] \\ & MC \arrow[d, "b"] & \\ & C & \end{tikzcd}$$

This diagram commutes: for the upper triangle remove M and use the induction hypothesis, and the remaining two inner parts commute by the definition of h_{n+1} and h_{n+2}, respectively.

Now we obtain a unique morphism $h : SX \to C$ such that for any natural number n the triangle below commutes:

$$\begin{tikzcd} M^nX \arrow[d, "\mathsf{inj}^nX"'] \arrow[dr, "h_n"] & \\ SX \arrow[r, "h"'] & C\,. \end{tikzcd} \tag{A.8}$$

Next we show that $h : SX \to C$ is a coalgebra homomorphism from $(SX, \overline{e})$ to the terminal coalgebra (C, c). To this end we will now verify that the lower part in the diagram below commutes:

$$\begin{tikzcd}
M^nX \arrow[r, "M^ne"] \arrow[d, "\mathsf{inj}^n_X"'] & M^nHMX \arrow[r, "\lambda^n_{MX}"] \arrow[d, "\mathsf{inj}^n_{HMX}"] & HM^nMX \arrow[r, equal] \arrow[d, "H\mathsf{inj}^n_{MX}"] & HM^{n+1}X \arrow[d, "H\mathsf{inj}^{n+1}_X"] \\
SX \arrow[r, "Se"] \arrow[d, "h"'] & SHMX \arrow[r, "\lambda^*_{MX}"] & HSMX \arrow[r, "H\chi_X"] & HSX \arrow[d, "Hh"] \\
C \arrow[rrr, "c"] & & & HC
\end{tikzcd}$$

$$h_n \qquad Hh_{n+1}$$

It suffices to show that the desired lower part commutes when extended by any colimit injection inj^n_X. Indeed, the left-hand part of the diagram above commutes by Diagram (A.8), and for the commutativity of the right-hand part, remove H and use Diagram (A.8) again. The upper left-hand square commutes by the naturality of inj^n, the upper middle square commutes by the definition of λ^*, and for the commutativity of the upper right-hand part remove H and use the definition of χ. It remains to verify that the outside of the diagram commutes. We will now prove this by induction on n. For the base case $n = 0$ we obtain the following diagram

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[dd, "h_0"'] & HMX \arrow[d, "HMh_0"'] \\
& HMC \arrow[d, "Hb"'] \\
C \arrow[r, "c"'] & HC
\end{tikzcd}$$

$$Hh_1$$

This diagram commutes: for the commutativity of the right-hand part remove H and use the definition of h_1, and the left-hand part commutes since $h_0 = e^\dagger$ is a solution of the λ-equation e.

Finally, for the induction step we consider the diagram below:

$$\begin{tikzcd}
M^{n+1}X \arrow[r, "M^{n+1}e"] \arrow[d, "Mh_n"'] & M^{n+1}HMX \arrow[r, "M\lambda^n_{MX}"] \arrow[rr, bend left, "\lambda^{n+1}_{MX}"] & MHM^nMX \arrow[r, "\lambda_{M^{n+1}X}"] \arrow[d, "MHh_{n+1}"] & HM^{n+2}X \arrow[d, "HMh_{n+1}"'] \\
MC \arrow[rr, "Mc"] \arrow[d, "b"'] & & MHC \arrow[r, "\lambda_C"] & HMC \arrow[d, "Hb"] \\
C \arrow[rrr, "c"'] & & & HC
\end{tikzcd}$$

$$h_{n+1} \qquad Hh_{n+2}$$

We see that this diagram commutes as follows: the lower part commutes by the definition of b (see (A.3)), the left-hand part commutes by the definition of h_{n+1}, and for the commutativity of the right-hand part remove H and use the definition of h_{n+2}. The small upper part commutes by the definition of λ^{n+1}, the upper right-hand square commutes by the naturality of λ, and finally, to see the commutativity of the upper left-hand square remove M and use the induction hypothesis.

We have finished the proof that $h : SX \to C$ is a coalgebra homomorphism from $(SX, \overline{e})$ to the terminal coalgebra (C, c). Since h is uniquely determined, it follows that the solution $e^\dagger = h \cdot \mathsf{inj}^0_X$ is uniquely determined, too. This completes our proof. $\square$

Remark A.9. As explained by Bartels in [1], Theorem A.4 extends to the case where a distributive law λ of M over the cofree copointed functor $H \times \mathrm{Id}$ is given. We briefly explain the ideas.

Let $D = H \times \mathrm{Id}$ and $\varepsilon = \pi_1 : D \to \mathrm{Id}$.

(1) A coalgebra for the copointed functor (D, ε) is a pair (X, x) where $x : X \to DX$ is such that $\varepsilon_X \cdot x = \mathrm{id}_X$. Homomorphisms of coalgebras for (D, ε) are the usual D-coalgebra homomorphisms. It is trivial to prove that

$$C \xrightarrow{\langle c, \mathrm{id}_C \rangle} HC \times C$$

is a terminal coalgebra for (D, ε).

(2) One verifies that λ-equations $e : X \to HMX$ are in bijective correspondence with morphisms $f : X \to DMX$ such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\eta_X"'] & DMX \arrow[d, "\varepsilon_{MX}"] \\ & MX \end{tikzcd}$$

commutes, and also that solutions of e correspond bijectively to solutions of f, i. e., morphisms $f^\dagger : X \to C$ such that Diagram (A.4) commutes with H replaced by D and c replaced by $\langle c, \mathrm{id}_C \rangle$:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[d, "f^\dagger"'] & & DMX \arrow[d, "DMf^\dagger"] \\ C \arrow[r, "\langle c, \mathrm{id}_C \rangle"] & DC & DMC \arrow[l, "Db"'] \end{tikzcd}$$

See [1], Lemma 4.3.9.

(3) The same proof as the one for Theorem A.4 shows that for every $f : X \to DMX$ as in (2) above there exists a unique solution $f^\dagger$. One only replaces H by D, c by $\langle c, \mathrm{id}_C \rangle$, and one has to verify that the coalgebra $\overline{e} : SX \to DSX$ from (A.5) is a coalgebra for the copointed endofunctor, see [1], Lemma 4.3.7.

To sum up, we obtain the following

Corollary A.10. *Let λ be a distributive law of the pointed functor M over the copointed one $H \times \mathrm{Id}$. Then for every $e : X \to HMX$ there exists a unique solution, i. e., a unique $e^\dagger : X \to C$ such that (A.4) commutes.*

Theorem A.11. *Let λ be a distributive law of the pointed functor M over the copointed one $H \times \mathrm{Id}$, and let $b : MC \to C$ be its λ-interpretation. Consider the algebra*

$$k = (\ HMC \xrightarrow{Hb} HC \xrightarrow{c^{-1}} C\ ).$$

Then (C, k) is a cia for HM.

Indeed, to prove this result copy the proof of Theorem 3.5 replacing $\widehat{b} : \widehat{K}C \to C$ by $b : MC \to C$.

However, for our version of Theorem 3.6 in the current setting we need a different proof. We start with an auxiliary lemma.

Lemma A.12. *Let $\lambda : MH \to HM$ be a distributive law of the pointed functor M over the functor H, and let $b : MC \to C$ be its interpretation. Then the natural transformation $\lambda' = \lambda M \cdot M\lambda : MMH \to HMM$ is a distributive law of the pointed functor MM over H, and $Mb \cdot b$ is the λ'-interpretation in C.*

Proof. Clearly $(MM, \eta M \cdot \eta = M\eta \cdot \eta : \mathrm{Id} \to MM)$ is a pointed endofunctor. The following commutative diagram

$$\begin{tikzcd}
 & & H \arrow[dl, "\eta H"'] \arrow[dr, "H\eta"] & & \\
 & MH \arrow[dl, "M\eta H"'] \arrow[dr, "MH\eta"] & & HM \arrow[dl, "\eta HM"'] \arrow[dr, "H\eta M"] & \\
MMH \arrow[rr, "M\lambda"'] & & MHM \arrow[rr, "\lambda M"'] & & HMM
\end{tikzcd}$$

shows that $\lambda' = \lambda M \cdot M\lambda$ is a distributive law of the pointed functor MM over H. In fact, the triangles commute by the assumption on λ, and the remaining upper square commutes by naturality of η; thus the outside triangle commutes.

To see that $b \cdot Mb$ is the λ'-interpretation in C, consider the following diagram:

$$\begin{tikzcd}
MMC \arrow[r, "MMc"] \arrow[d, "Mb"'] & MMHC \arrow[r, "M\lambda_C"] & MHMC \arrow[r, "\lambda_{MC}"] \arrow[d, "MHb"'] & HMMC \arrow[d, "HMb"] \\
MC \arrow[rr, "Mc"] \arrow[d, "b"'] & & MHC \arrow[r, "\lambda_C"] & HMC \arrow[d, "Hb"] \\
C \arrow[rrr, "c"'] & & & HC
\end{tikzcd}$$

It commutes since b is the λ-interpretation in C and by the naturality of λ. In addition, $b \cdot Mb$ is easily seen to be an algebra for the pointed functor $(MM, \eta M \cdot \eta)$ since b is one for (M, η) and η is a natural transformation:

$$\begin{tikzcd}
C \arrow[d, "\eta_C"'] \arrow[dr, equal] & & \\
MC \arrow[r, "b"'] \arrow[d, "\eta_{MC}"'] & C \arrow[d, "\eta_C"'] \arrow[dr, equal] & \\
MMC \arrow[r, "Mb"'] & MC \arrow[r, "b"'] & C
\end{tikzcd}$$

This concludes the proof. □

Theorem A.13. *Let $\lambda : MH \to HM$ be a distributive law of the pointed functor M over the functor H, and let $b : MC \to C$ be its λ-interpretation. Consider the algebra*

$$k' = (MHMC \xrightarrow{Mk} MC \xrightarrow{b} C),$$

where $k = c^{-1} \cdot Hb$ as in Theorem A.11. Then (C, k') is a cia for MHM.

Proof. We have to prove that for every flat equation morphism $e : X \to MHMX + C$ for MHM there is a unique solution $e^\dagger : X \to C$ in $k' = b \cdot Mc^{-1} \cdot MHb : MHMC \to C$, i. e., a unique

morphism $e^\dagger$ such that the outside of the diagram

$$\begin{tikzcd}
X \arrow[rrr, "e^\dagger"] \arrow[dr, "\overline{e}"] \arrow[dd, "e"'] & & & C \\
& HMMX+C \arrow[dr, "HMMe^\dagger+C"'] & HC+C \arrow[ur, "{[c^{-1},C]}"] & MC+C \arrow[u, "{[b,C]}"'] \arrow[d, shift right, "Mc+C"'] \\
& & HMC+C \arrow[u, "Hb+C"] & MHC+C \arrow[l, "\lambda_C+C"'] \arrow[u, shift right, "Mc^{-1}+C"'] \arrow[uu, bend right=60, "{[k',C]}"'] \\
& & HMMC+C \arrow[u, "HMb+C"] & \\
MHMX+C \arrow[uur, "\lambda_{MX}+C"] \arrow[rrr, "MHMe^\dagger+C"'] & & & MHMC+C \arrow[uu, "MHb+C"'] \arrow[ul, "\lambda_{MC}+C"]
\end{tikzcd}$$

commutes. To this end, we define the flat equation morphism

$$\overline{e} = (X \xrightarrow{e} MHMX+C \xrightarrow{\lambda_{MX}+C} HMMX+C)$$

for HMM (this is the left-hand triangle). According to Lemma A.12 and Theorem A.11,

$$HMMC \xrightarrow{HMb} HMC \xrightarrow{Hb} HC \xrightarrow{c^{-1}} C$$

is a cia for HMM. So $\overline{e}$ has a unique solution $e^\dagger$ in this cia, i. e., the big inner part of the diagram commutes. In the upper right-hand part, b is the λ-interpretation in C. Since that part and the two remaining squares also commute (due to naturality of λ), the desired outside commutes. Thus, $e^\dagger$ also is a solution of e in the algebra $k' : MHMC \to C$.

This solution is unique, since any other solution $e^\ddagger$ of e in k' (i. e., the outside of the above diagram with $e^\ddagger$ in lieu of $e^\dagger$ commutes) is a solution of $\overline{e}$ in the cia $c^{-1} \cdot Hb \cdot HMb : HMMC \to C$ (i. e., the inner part commutes with $e^\ddagger$ in lieu of $e^\dagger$), thus $e^\ddagger = e^\dagger$. $\square$

REFERENCES

[1] F. Bartels. *On generalized coinduction and probabilistic specification formats*. PhD thesis, Vrije Universiteit Amsterdam, 2004.