Guard Your Daggers and Traces: On The Equational Properties of Guarded (Co-)recursion

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Motivated by the recent interest in models of guarded (co-)recursion we study its equational properties. We formulate axioms for guarded fixpoint operators generalizing the axioms of iteration theories of Bloom and Ésik. Models of these axioms include both standard (e.g., cpo-based) models of iteration theories and models of guarded recursion such as complete metric spaces or the topos of trees studied by Birkedal et al. We show that the standard result on the satisfaction of all Conway axioms by a unique dagger operation generalizes to the guarded setting. We also introduce the notion of guarded trace operator on a category, and we prove that guarded trace and guarded fixpoint operators are in one-to-one correspondence. Our results are intended as first steps leading to the description of classifying theories for guarded recursion and hence completeness results involving our axioms of guarded fixpoint operators in future work.

1 Introduction

Our ability to describe concisely potentially infinite computations or infinite behaviour of systems relies on recursion, corecursion and iteration. Most programming languages and specification formalisms include a fixpoint operator. In order to give semantics to such operators one usually considers either

- models based on complete partial orders where fixpoint operators are interpreted by least fixpoints using the Kleene-Knaster-Tarski theorem or
- models based on complete metric spaces and unique fixpoints via Banach's theorem or
- term models where unique fixpoints arise by unfolding specifications syntactically.

In the last of these cases, one only considers *guarded* (co-)recursive definitions; see e.g. Milner's solution theorem for CCS [21] or Elgot's iterative theories [13]. Thus, the fixpoint operator becomes a partial operator defined only on a special class of maps. For a concrete example consider complete metric spaces which form a category with all non-expansive maps as morphisms, but unique fixpoints are taken only of contractive maps.

Recently, there has been a wave of interest in expressing guardedness by a new type constructor $\rhd$, a kind of "later" modality, which allows to make the fixpoint operator total, see, e.g., Nakano [23, 24], Appel et al. [4], Benton and Tabareau [7], Krishnaswami and Benton [19, 18], Birkedal et al. [9, 8] and Atkey and McBride [5]. For example, in the case of complete metric spaces $\rhd$ can be an endofunctor scaling the metric of any given space by a fixed factor $0 < r < 1$ so that non-expansive maps of type $\rhd X \to X$ are precisely contractive maps with a contraction factor of at most r. This allows to define a guarded (parametrized) fixpoint operator on *all* morphisms of type $\rhd X \times Y \to X$ of the model. So far various models allowing the interpretation of a typed language including a guarded fixpoint operator

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have been studied: complete metric spaces, the "topos of trees", i.e., presheaves on ω^{op} [9] or, more generally, sheaves on complete Heyting algebras with a well-founded basis [12, 9].

This paper initiates the study of the essential properties of guarded fixpoint operators. In the realm of ordinary fixpoint operators, it is well-known that iteration theories of Bloom and Ésik [10] completely axiomatize equalities of fixpoint terms in models based on complete partial orders (see also Simpson and Plotkin [25]). We make here the first steps towards similar completeness results in the guarded setting.

We begin with formalizing the notion of guarded fixpoint operator on a cartesian category. We discuss a number of models, including not only all those mentioned above, but also some not mentioned so far in the context of $\rhd$-guarded (co-)recursion. In fact, we consider the inclusion of examples such as the lifting functor on CPO (which also happens to be a paradigm example of a *fixpoint monad*, see Example 2.4.6 and the concluding remark of Section 2.7) or completely iterative monads (see Section 2.2) a pleasant by-product of our work and a potentially fruitful connection for future research. Then, we formulate generalizations of standard iteration theory axioms for guarded fixpoint operators and we establish these axioms are sound in all models under consideration. In particular, the central result of Section 2 is Theorem 2.16: models with *unique* guarded fixpoint operators satisfy all our axioms.

Hasegawa [16] proved that giving a parametrized fixpoint operator on a category satisfying the so-called *Conway axioms* (see, e.g., [10, 25] and Section 2.3 below) is equivalent to giving a *traced cartesian structure* [17] on that category.[1] Section 3 lifts this result to the guarded setting. We introduce a natural notion of a guarded trace operator on a category, and we prove in Theorem 3.5 that guarded traces and guarded fixpoint operators are in one-to-one correspondence. This extends to an isomorphism between the (2-)categories of guarded traced cartesian categories and guarded Conway categories.

Section 4 concludes and discusses further work.

Proofs of the major theorems will be made available in the full version.

1.1 Notational conventions

We will assume throughout that readers are familiar with basic notions from category theory. We denote the product of two objects by

$$A \xleftarrow{\pi_\ell} A \times B \xrightarrow{\pi_r} B,$$

and $\Delta : A \to A \times A$ denotes the diagonal. For every functor F we write $\mathsf{can} = \langle F\pi_\ell, F\pi_r\rangle : F(A \times B) \to FA \times FB$ for the canonical morphism.

We denote by CPO the category of complete partial orders (cpo's), i.e. partially ordered sets (not necessarily with a least element) having joins of ω-chains. The morphisms of CPO are Scott-continuous maps, i.e. maps preserving joins of ω-chains. And $\mathsf{CPO}_\perp$ is the full subcategory of CPO given by all cpo's with a least element $\perp$. We will also consider the category CMS of complete 1-bounded metric spaces and non-expansive maps.

2 Guarded Fixpoint Operators

In this section we define the notion of a guarded fixpoint operator on a cartesian category and present an extensive list of examples. Some of these examples like the lifting functor $(-)_\perp$ on CPO (see Example 2.4.6) or completely iterative monads (see Section 2.2) do not seem to have been considered as instances of the guarded setting before. We then introduce (equational) properties of guarded fixpoint operators.

[1] *Cartesian* here refers to the monoidal product being the ordinary categorical product.

These properties are motivated by and closely resemble properties of the fixpoint operator in iteration theories of Bloom and Ésik [10]. We conclude this section with Theorem 2.16 stating that unique fixpoint operators satisfy all the properties we study.

2.1 Definition and Examples of Guarded Fixpoint Operators

Assumption 2.1. We assume throughout the rest of the paper that $(\mathscr{C}, \rhd)$ is a pair consisting of a category $\mathscr{C}$ with finite products (also know as a *cartesian category*) and a pointed endofunctor $\rhd : \mathscr{C} \to \mathscr{C}$, i.e. we have a natural transformation $p : \mathsf{Id} \to \rhd$. The endofunctor $\rhd$ is called *delay*.

Remark 2.2. In references like [9, 8], much more is assumed about both the underlying category and the delay endofunctor. Whenever one wants to model simply-typed lambda calculus, one obviously imposes the condition of being cartesian closed. Furthermore, whenever one considers dependent types, one wants to postulate conditions like being a *type-theoretic fibration category* (see, e.g., [8, Definition IV.1]). In such a case, one also wants to impose some limit-preservation or at least finite-limit-preservation condition on the delay endofunctor, see [9, Definition 6.1]—e.g., to ensure the transfer of the guarded fixpoint operator to slices. We do not impose any of those restrictions because we do not need them in this paper. It is an interesting fact that all our derivations require no more than Assumption 2.1. For more on the connection with the setting of [9], see Proposition 2.6 below.

Definition 2.3. A *guarded fixpoint operator* on $(\mathscr{C}, \rhd)$ is a family of operations

$$\dagger_{X,Y} : \mathscr{C}(\rhd X \times Y, X) \to \mathscr{C}(Y, X)$$

such that for every $f : \rhd X \times Y \to X$ the following square commutes[2]:

$$\begin{tikzcd} Y \arrow[r, "f^\dagger"] \arrow[d, "\langle f^\dagger, Y\rangle"'] & X \\ X \times Y \arrow[r, "p_X \times Y"'] & \rhd X \times Y \arrow[u, "f"'] \end{tikzcd} \tag{2.1}$$

where (as usual) we drop the subscripts and write $f^\dagger : Y \to X$ in lieu of $\dagger_{X,Y}(f)$. We call the triple $(\mathscr{C}, \rhd, \dagger)$ a *guarded fixpoint category*.

Usually, one either assumes that $\dagger$ satisfies further properties or even that $f^\dagger$ is unique such that (2.1) commutes. We will come to the study of properties of guarded fixpoint operators in Section 2.3. Let us begin with a list of examples.

Examples 2.4. (1) Taking as $\rhd$ the identity functor on $\mathscr{C}$ and p_X the identity on X we arrive at the special case of categories with an ordinary fixpoint operator $\mathscr{C}(X \times Y, X) \to \mathscr{C}(Y, X)$ (see e.g. Hasegawa [16, 15] or Simpson and Plotkin [25]). Concrete examples are: the category $\mathsf{CPO}_\perp$ with its usual least fixpoint operator or (the dual of) any iteration theory of Bloom and Ésik [10].

(2) Taking $\rhd$ to be the constant functor on the terminal object 1 and $p_X = ! : X \to 1$ the unique morphism, a trivial guarded fixpoint operator is given by the family of identity maps on the hom-sets $\mathscr{C}(Y, X)$.

(3) Take $\mathscr{C}$ to be the category CMS of complete 1-bounded metric spaces (see [19, 18] or [9, Section 5] and references therein), $\rhd_r$ $(0 < r < 1)$ to be an endofunctor which keeps the carrier of the space and multiplies all distances by r and $p_X : X \to \rhd_r X$ to be the obvious "contracted identity" mapping.

[2] Notice that we use the convention of simply writing objects to denote the identity morphisms on them.

Note that a non-expansive mapping $f : \rhd_r X \to X$ is the same as an *r-contractive* endomap, i.e. an endomap satisfying $d(fx, fy) \le r \cdot d(x,y)$. A guarded fixpoint operator is given by an application of Banach's unique fixpoint theorem: for every $f : \rhd_r X \times Y \to X$ we consider the map

$$\Phi_f : \mathsf{CMS}(Y,X) \to \mathsf{CMS}(Y,X), \qquad \Phi_f(m) = f \cdot (p_X \times Y) \cdot \langle m, Y \rangle;$$

notice that $\mathsf{CMS}(Y,X)$ is a complete metric space with the sup-metric $d_{Y,X}(m,n) = \sup_{y \in Y}\{d_X(my,ny)\}$; it is then easy to show that Φ_f is an r-contractive map, and so its unique fixpoint is a unique non-expansive map $f^\dagger : Y \to X$ such that (2.1) commutes.

(4) Let $\mathscr{A}$ be a category with finite products, and let $\mathscr{C}$ be the presheaf category $\mathsf{presh}(\omega, \mathscr{A}) := \mathscr{A}^{\omega^{\mathsf{op}}}$ of ω^{op}-chains in $\mathscr{A}$. The delay functor $\rhd$ takes a presheaf $X : \omega^{\mathsf{op}} \to \mathscr{A}$ to the presheaf $\rhd X$ with $\rhd X(0) = 1$ and $\rhd X(n+1) = X(n)$ for $n \ge 0$. And p_X is given by $(p_X)_0 : X(0) \to 1$ unique and $(p_X)_{n+1} = X(n+1 \ge n) : X(n+1) \to X(n)$. For every $f : \rhd X \times Y \to X$ there is a unique $f^\dagger : Y \to X$ making (2.1) commutative; it is defined as follows: given $f : \rhd X \times Y \to X$ (i.e. $f_0 : Y(0) \to X(0)$ and $f_{n+1} : X(n) \times Y(n+1) \to X(n+1)$) one defines $f^\dagger : Y \to X$ by $f_0^\dagger = f_0 : Y(0) \to X(0)$ and

$$f_{n+1}^\dagger = \big(Y(n+1) \xrightarrow{\langle f_n^\dagger \cdot Y(n+1 \ge n), Y(n+1) \rangle} X(n) \times Y(n+1) \xrightarrow{f_{n+1}} X(n+1)\big).$$

It is not difficult to prove that $f^\dagger$ is the unique morphism such that (2.1) commutes.

Notice that for $\mathscr{A} = \mathsf{Set}$, $\mathscr{C}$ is the "topos of trees" studied by Birkedal et al. [9]; they prove in Theorem 2.4 that $\mathsf{Set}^{\omega^{\mathsf{op}}}$ has a unique guarded fixpoint operator.

The next example generalizes this one.

(5) Assume $\mathfrak{W} := (W, <)$ is a well-founded poset, i.e, contains no infinite descending chains; for simplicity, we can assume $\mathfrak{W}$ has a root r. Furthermore, let $\mathscr{D}$ be a (small) complete category and $\mathscr{C} := \mathsf{presh}(\mathfrak{W}, \mathscr{D})$, i.e., $\mathscr{C} = \mathscr{D}^{(W,>)}$. Define $(\rhd X)(w)$ to be the limit of the diagram whose nodes are $X(u)$ for $u < w$ and whose arrows are restriction morphisms: $\rhd X(w) = \lim_{v<w} X(v)$. Then as $X(w)$ itself with restriction mappings forms a cone on that diagram, a natural $p_X : X \to \rhd X$ is given by the universal property of the limits. Note that for r, we have that $(\rhd X)(r)$ is the terminal object 1 of $\mathscr{D}$. The $\dagger$-operation is defined as follows: given $f : \rhd X \times Y \to X$ one defines $f^\dagger : Y \to X$ by induction on $(W,<)$; for the root r let $f_r^\dagger = f_r : Y(r) = 1 \times Y(r) \to X(r)$, and assuming that $f_v^\dagger$ is already defined for all $v < w$ let

$$f_w^\dagger = \big(Y(w) \xrightarrow{\langle k, Y(w) \rangle} \rhd X(w) \times Y(w) \xrightarrow{f_w} X(w)\big),$$

where $k : Y(w) \to \rhd X(w)$ is the morphism uniquely induced by the cone $f_v^\dagger \cdot Y(w > v) : Y(w) \to Y(v) \to X(v)$ for every $v < w$. One can prove that $f^\dagger$ is a morphism of presheaves and that it is the unique one such that (2.1) commutes. Details will be given in the full version. Regarding the examples given in [9], see also Proposition 2.6 below.

(6) Let $\rhd$ be the lifting functor $(-)_\perp$ on CPO, i.e. for any cpo X, $X_\perp$ is the cpo with a newly added least element. The natural transformation $p_X : X \to X_\perp$ is the embedding of X into $X_\perp$. Then CPO has a guarded fixpoint operator given by taking least fixpoints. To see this notice that the hom-sets $\mathsf{CPO}(X,Y)$ are cpos with the pointwise order: $f \le g$ iff $f(x) \le g(x)$ for all $x \in X$. Now any continuous $f : X_\perp \times Y \to X$ gives rise to a continuous map Φ_f on $\mathsf{CPO}(Y, X_\perp)$:

$$\Phi_f : \mathsf{CPO}(Y, X_\perp) \to \mathsf{CPO}(Y, X_\perp), \qquad \Phi_f(m) = p_X \cdot f \cdot \langle m, Y \rangle.$$

Using the least fixpoint s of Φ_f one then defines:

$$f^\dagger = (Y \xrightarrow{\langle s,Y\rangle} X_\perp \times Y \xrightarrow{f} X);$$

using that $s = \Phi_f(s)$ it is not difficult to prove that $f^\dagger$ makes (2.1) commutative.

Birkedal et al. [9] provide a general setting for topos-theoretic examples like (4) and (5) (the latter restricted to the case of Set-presheaves) by defining a notion of *a model of guarded recursive terms* and showing that *sheaves over complete Heyting algebras with a well-founded basis* proposed by [12] are instances of this notion. The difference between Definition 6.1 in [9] and our Definition 2.3 is that in the former a) the delay endofunctor $\rhd$ is also assumed to preserve finite limits. On other hand b) our equality (2.1) is only postulated in the case when Y is the terminal object, i.e., only non-parametrized fixpoint identity is assumed but c) the dagger in this less general version of (2.1) is assumed to be unique. Now, one can show that assumptions a) and c) imply our parametrized identity (2.1) *whenever the underlying category is cartesian closed*, in particular whenever $\mathscr{C}$ is a topos. Let us state both the definition and the result formally:

Definition 2.5 ([9])**.** A *model of guarded fixpoint terms* is a triple $(\mathscr{C}, \rhd, \ddagger)$, where

- $(\mathscr{C}, \rhd)$ satisfy our general Assumption 2.1, i.e., $\rhd : \mathscr{C} \to \mathscr{C}$ is a pointed endofunctor (with point $p : \mathsf{Id} \to \rhd$) and $\mathscr{C}$ has finite limits
- $\rhd$ preserves finite limits and
- $\ddagger$ is a family of operations $\ddagger_X : \mathscr{C}(\rhd X, X) \to \mathscr{C}(1, X)$ such that for every $f : \rhd X \to X$, $f^\ddagger$ is a unique morphism making the following square commute:

$$\begin{tikzcd} 1 \arrow[r, "f^\ddagger"] \arrow[d, "f^\ddagger"'] & X \\ X \arrow[r, "p_X"'] & \rhd X \arrow[u, "f"'] \end{tikzcd} \tag{2.2}$$

We write $\mathsf{can}^{-1}_{X,Y} : \rhd X \times \rhd Y \to \rhd(X \times Y)$ for the isomorphism provided by the assumption of limit preservation for the special case of product[3] of X and Y.

Proposition 2.6. *If* $(\mathscr{C}, \rhd, \ddagger)$ *is a model of guarded recursive terms and* $\mathscr{C}$ *is* cartesian closed *with*

$$\begin{array}{lll} \mathsf{curry}^X_{Y,Z} & : \mathscr{C}(X \times Y, Z) & \to \mathscr{C}(X, Z^Y), \\ \mathsf{uncurry}^X_{Y,Z} & : \mathscr{C}(X, Z^Y) & \to \mathscr{C}(X \times Y, Z), \\ \mathsf{eval}_{Y,Z} & : Y \times Z^Y & \to Z, \end{array}$$

then the operator $\dagger_{X,Y} : \mathscr{C}(\rhd X \times Y, X) \to \mathscr{C}(Y, X)$ *defined as*

$$\mathsf{uncurry}^1_{Y,X}([\mathsf{curry}^{\rhd(X^Y)}_{Y,X}(f \cdot \langle(\rhd \mathsf{eval}_{Y,X}) \cdot \mathsf{can}^{-1}_{Y,X^Y} \cdot (p_Y \times \rhd(X^Y)), \pi_\ell\rangle)]^\ddagger)$$

is a guarded fixpoint operator on $(\mathscr{C}, \rhd)$.

[3] One can note here that for the purpose of stating and proving Proposition 2.6, the assumption of finite limit preservation in Definition 2.5 can be weakened to finite product preservation. We only keep the stronger assumption for full consistency with [9, Definition 6.1].

Obviously, we implicitly identified Y and $1 \times Y$ above. Note that the converse implication does not hold. Example 2.4.6 is a a guarded fixpoint category, but $(-)_\perp$ clearly fails to preserve even finite products and hence it does not yield a model of guarded recursive terms.

Also, while we do not have a counterexample at the moment, Proposition 2.6 is not likely to hold when the assumption that $\mathscr{C}$ is cartesian closed is removed: we believe there are examples of models of guarded recursive terms which are not guarded fixpoint categories. However, to apply Proposition 2.6, it is enough that $(\mathscr{C}, \rhd, \ddagger)$ is a *full subcategory* of a cartesian closed model of guarded recursive terms such that, moreover, the inclusion functor preserves products and $\rhd$.

Remark 2.7. Monads provide perhaps the most natural and well-known examples of pointed endofunctors. The reader may ask whether delay endofunctors in Example 2.4 happen to be monads. Clearly, the delay functors in (1), (2) and (6) are. In fact, while the first two ones are rather trivial monads, 6 is a paradigm example of a *fixpoint monad* of Crole and Pitts [11]. In (3), i.e. the CMS example, the type $\rhd\rhd A \to \rhd A$ is still inhabited (by any constant mapping), but one can easily show that monad laws cannot hold whatever candidate for monad multiplication is postulated. In the remaining (i.e., topos-theoretic) examples, monad laws fail more dramatically: $\rhd\rhd A \to \rhd A$ is not even always inhabited. The following section discusses perhaps the most interesting subclass of monads which happen to be delay endofunctors with unique dagger.

2.2 Completely Iterative Theories

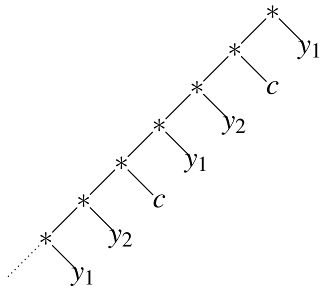
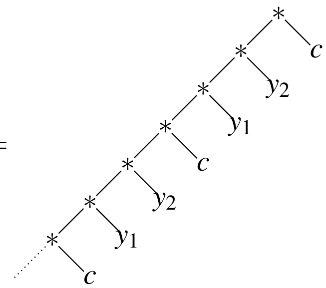
In this subsection we will explain how categories with guarded fixpoint operator capture a classical setting in which guarded recursive definitions are studied—Elgot's (completely) iterative theories [13, 14]. The connection to guarded fixpoint operators is most easily seen if we consider monads in lieu of Lawvere theories, and so we follow the presentation of *(completely) iterative monads* in [20]. The motivating example for completely iterative monads are infinite trees on a signature, and we recall this now. Let Σ be a signature, i.e. a sequence $(\Sigma_n)_{n<\omega}$ of sets of operation symbols with prescribed arity n. A Σ-tree t on a set X of generators is a rooted and ordered (finite or infinite) tree whose nodes with $n > 0$ children are labelled by n-ary operation symbols from Σ and a leaf is labelled by a constant symbol from Σ_0 or by a generator from X. One considers systems of mutually recursive equations of the form

$$x_i \approx t_i(\vec{x}, \vec{y}) \qquad i \in I,$$

where $X = \{x_i \mid i \in I\}$ is a set of recursion variables and each t_i is a Σ-tree on $X + Y$ with Y a set of parameters (i.e. generators that do not occur on the left-hand side of a recursive equation). A system of recursive equations is *guarded* if none of the trees t_i is only a recursion variable $x \in X$. Every guarded system has a unique *solution*, which assigns to every recursion variable $x_i \in X$ a Σ-tree $t_i^\dagger(\vec{y})$ on Y such that $t_i^\dagger(\vec{y}) = t_i[\vec{t}^\dagger(\vec{y})/\vec{x}]$, i.e. t_i with each x_j replaced by $t_j^\dagger(\vec{y})$. For a concrete example, let Σ consist of a binary operation symbol $*$ and a constant symbol c, i.e. $\Sigma_0 = \{c\}$, $\Sigma_2 = \{*\}$ and $\Sigma_n = \emptyset$ else. Then the following system

$$x_1 \approx x_2 * y_1 \qquad x_2 \approx (x_1 * y_2) * c,$$

where y_1 and y_2 are parameters, has the following unique solution:

$t_1^\dagger =$  and $t_2^\dagger =$ 

For any set X, let $T_\Sigma(X)$ be the set of Σ-trees on X. It has been realized by Badouel [6] that T_Σ is the object part of a monad. A system of equations is then nothing but a map

$$f : X \to T_\Sigma(X+Y)$$

and a solution is a map $f^\dagger : X \to T_\Sigma Y$ such that the following square commutes:

$$\begin{tikzcd}
X \arrow[r, "f^\dagger"] \arrow[d, "f"'] & T_\Sigma Y \\
T_\Sigma(X+Y) \arrow[r, "{[f^\dagger, \eta_Y]}"'] & T_\Sigma T_\Sigma Y \arrow[u, "\mu_Y"']
\end{tikzcd}$$

where η and μ are the unit and multiplication of the monad T_Σ, respectively.

It is clear that the notion of equation and solution can be formulated for every monad S. However, the notion of guardedness requires one to speak about *non-variables* in S. This is enabled by Elgot's notion of *ideal theory* [13], which for a finitary monad on Set is equivalent to the notion recalled in the following definition. We assume for the rest of this subsection that $\mathscr{A}$ is a category with finite coproducts such that coproduct injections are monomorphic.

Definition 2.8 ([2])**.** By an *ideal monad* on $\mathscr{A}$ is understood a six-tuple

$$(S, \eta, \mu, S', \sigma, \mu')$$

consisting of a monad (S, η, μ) on $\mathscr{A}$, a subfunctor $\sigma : S' \hookrightarrow S$ and a natural transformation $\mu' : S'S \to S'$ such that

(1) $S = S' + \mathsf{Id}$ with coproduct injections σ and η, and

(2) μ restricts to μ' along σ, i.e., the square below commutes:

$$\begin{tikzcd}
S'S \arrow[r, "\mu'"] \arrow[d, "\sigma S"'] & S' \arrow[d, "\sigma"] \\
SS \arrow[r, "\mu"'] & S
\end{tikzcd}$$

The subfunctor S' of an ideal monad S allows us to formulate the notion of a guarded equation system abstractly; this leads to the notion of completely iterative theory of Elgot et al. [14] for which we here present the formulation with monads from [20]:

Definition 2.9. Let $(S,\eta,\mu,S',\sigma,\mu')$ be an ideal monad on $\mathscr{A}$.

1. By an *equation morphism* is meant a morphism
$$f : X \to S(X+Y)$$
in $\mathscr{A}$, where X is an object ("of variables") and Y is an object ("of parameters").

2. By a *solution* of f is meant a morphism $f^\dagger : X \to SY$ for which the following square commutes:
$$\begin{tikzcd} X \arrow[r, "f^\dagger"] \arrow[d, "f"'] & SY \\ S(X+Y) \arrow[r, "{S[f^\dagger,\eta_Y]}"'] & SSY \arrow[u, "\mu_Y"'] \end{tikzcd} \tag{2.3}$$

3. The equation morphism f is called *guarded* if it factors through the summand $S'(X+Y)+Y$ of $S(X+Y) = S'(X+Y)+X+Y$:
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, dotted] & S(X+Y) \\ & S'(X+Y)+Y \arrow[u, "{[\sigma_{X+Y},\eta_{X+Y}\cdot\mathsf{inr}]}"'] \end{tikzcd}$$

4. The given ideal monad is called *completely iterative* if every guarded equation morphism has a unique solution.

Examples 2.10. We only briefly mention two examples of completely iterative monads. More can be found in [2, 20, 3].

(1) The monad T_Σ of Σ-trees is a completely iterative monad.

(2) A more general example is given by parametrized final coalgebras. Let $H : \mathscr{A} \to \mathscr{A}$ be an endofunctor such that for every object X of $\mathscr{A}$ a final coalgebra TX for $H(-)+X$ exists. Then T is the object assignment of a completely iterative monad; in fact, T is the free completely iterative monad on H (see [20]).

We will now explain how completely iterative monads are subsumed by the notion of categories with a guarded fixpoint operator. To this end we fix a completely iterative monad S. We will show that the dual of its Kleisli category $\mathscr{C} = (\mathscr{A}_S)^{op}$ is equipped with a guarded fixpoint operator. First notice, that since $\mathscr{A}_S$ has coproducts given by the coproducts in $\mathscr{A}$ we see that $\mathscr{C}$ has products. Next we need to obtain the endofunctor $\rhd$ on $\mathscr{C}$. This will be given as the dual of an extension of the subfunctor $S' : \mathscr{A} \to \mathscr{A}$ of S to the Kleisli category $\mathscr{A}_S$. Indeed, it is well-known that to have an extension of S' to $\mathscr{A}_S$ is equivalent to having a distributive law of the functor S' over the monad S (see Mulry [22]).

But it is easy to verify that the natural transformation
$$S'S \xrightarrow{\mu'} S' \xrightarrow{\eta S'} SS'$$
satisfies the two required laws and thus yields a distributive law. Moreover, the ensuing endofunctor $\rhd^{op} = S'$ on $\mathscr{A}_S$ is copointed, i.e. we have a natural transformation p from S' to $\mathsf{Id} : \mathscr{A}_S \to \mathscr{A}_S$; indeed,

its components at X are given by the coproduct injections $\sigma_X : S'X \to SX$, and it is not difficult to verify that this is a natural transformation; thus, $\rhd$ is a pointed endofunctor on $\mathscr{C}$.

Now observe that a morphism $f : \rhd X \times Y \to X$ is equivalently a morphism

$$f : X \to S(S'X + Y)$$

in $\mathscr{A}$. We are ready to describe the guarded fixpoint operator on $\mathscr{C}$.

Construction 2.11. For any morphism $f : X \to S(S'X + Y)$ form the following morphism

$$\overline{f} = (X \xrightarrow{f} S(S'X + Y) \xrightarrow{S(\sigma_X + \eta_Y)} S(SX + SY) \xrightarrow{S\mathsf{can}} SS(X + Y) \xrightarrow{\mu_{X+Y}} S(X + Y)),$$

where $\mathsf{can} = [S\mathsf{inl}, S\mathsf{inr}] : SX + SY \to S(X + Y)$. It is not difficult to verify that $\overline{f}$ is a guarded equation morphism for S, and we define $f^\dagger : X \to SY$ to be the unique solution of $\overline{f}$.

Proposition 2.12. *For every f, $f^\dagger$ from Construction 2.11 is a unique morphism $Y \to X$ in $\mathscr{C}$ such that (2.1) commutes.*

In fact, to prove this proposition one shows that solutions of $\overline{f} : X \to S(X + Y)$ (i.e. morphisms $s : X \to SY$ such that (2.3) commutes) are in one-to-one correspondence with morphisms $Y \to X$ is $\mathscr{C}$ such that (2.1) commutes.

2.3 Properties of Guarded Fixpoint Operators

In this section we study properties of guarded fixpoint operators. Except for uniformity these properties are purely equational. They are generalizing analogous properties of iteration theories; more precisely, they would collapse to the original, unguarded counterparts when $\rhd$ is instantiated to the identity endofunctor (see Example 2.4(1)).

Definition 2.13. Let $(\mathscr{C}, \rhd, \dagger)$ be a guarded fixpoint category. We define the following properties of $\dagger$:

(1) Fixpoint Identity. For every $f : \rhd X \times Y \to X$ the diagram (2.1) commutes. This is built into the definition of guarded fixpoint categories and only mentioned here again for the sake of completeness.

(2) Parameter Identity. For every $f : \rhd X \times Y \to X$ and every $h : Z \to Y$ we have

$$Z \xrightarrow{h} Y \xrightarrow{f^\dagger} X = (\rhd X \times Z \xrightarrow{\rhd X \times h} \rhd X \times Y \xrightarrow{f} X)^\dagger.$$

(3) (Simplified) Composition Identity. Given $f : \rhd X \times Y \to Z$ and $g : Z \to X$ we have

$$(\rhd X \times Y \xrightarrow{f} Z \xrightarrow{g} X)^\dagger = (Y \xrightarrow{(f\cdot(\rhd g \times Y))^\dagger} Z \xrightarrow{g} X).$$

(4) Double Dagger Identity. For every $f : \rhd X \times \rhd X \times Y \to X$ we have

$$(Y \xrightarrow{f^{\dagger\dagger}} X) = (\rhd X \times Y \xrightarrow{\Delta \times Y} \rhd X \times \rhd X \times Y \xrightarrow{f} X)^\dagger.$$

(5) Uniformity. Given $f : \rhd X \times Y \to X$, $g : \rhd X' \times Y \to X'$ and $h : X \to X'$ we have

$$\begin{tikzcd} \rhd X \times Y \arrow[r, "f"] \arrow[d, "\rhd h \times Y"'] & X \arrow[d, "h"] \\ \rhd X' \times Y \arrow[r, "g"'] & X' \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} & X \arrow[d, "h"] \\ Y \arrow[ur, "f^\dagger"] \arrow[r, "g^\dagger"'] & X' \end{tikzcd}$$

We call the first four properties (1)–(4) the *Conway* axioms.

Notice that the Conway axioms are equational properties while (5) is quasiequational (i.e. an implication between equations).

Next we shall show that in the presence of certain of the above properties the natural transformation $p : \mathsf{Id} \to \rhd$ is a derived structure. Let $(\mathscr{C}, \rhd)$ be equipped with an operator $\dagger$ *not necessarily satisfying (2.1)*. For every object X of $\mathscr{C}$ define $q_X : X \to \rhd X$ as follows: consider

$$f_X = (\,\rhd(\rhd X \times X) \times X \xrightarrow{\rhd\pi_r \times X} \rhd X \times X\,)$$

and form

$$q_X = (\,X \xrightarrow{f_X^\dagger} \rhd X \times X \xrightarrow{\pi_\ell} \rhd X\,).$$

Lemma 2.14. *Let $(\mathscr{C}, \rhd)$ be equipped with the operator $\dagger$. Then:*

1. *If $\dagger$ satisfies the parameter identity and uniformity, then $q : \mathsf{Id} \to \rhd$ is a natural transformation.*
2. *If $\dagger$ satisfies the fixpoint identity, then $q_X = p_X$ for all X.*

Definition 2.15. A guarded fixpoint category $(\mathscr{C}, \rhd, \dagger)$ satisfying the Conway axioms (i.e. fixpoint, parameter, composition and double dagger identities) is called a *guarded Conway category*.

If in addition uniformity is satisfied, we call $(\mathscr{C}, \rhd, \dagger)$ a *uniform guarded Conway category*.

And $(\mathscr{C}, \rhd, \dagger)$ is called a *unique guarded fixpoint category* if for every $f : \rhd X \times Y \to X$, $f^\dagger : Y \to X$ is the unique morphism such that (2.1) commutes. In this case, we can just write a pair $(\mathscr{C}, \rhd)$ rather than a triple $(\mathscr{C}, \rhd, \dagger)$.

The next theorem states that such a unique $\dagger$ satisfies all the properties in Definition 2.13.

Theorem 2.16. *If $(\mathscr{C}, \rhd)$ is a unique guarded fixpoint category, then it is a uniform guarded Conway category.*

Examples 2.17. (1) Several of our examples in 2.4 are unique guarded fixpoint categories and hence their unique $\dagger$ satisfies all the properties in Definition 2.13. This holds for Examples 2.4(2)–(6), and also for the example of completely iterative monads in Section 2.2.

(2) One can prove that Example 2.4(7), i.e., $\mathscr{C} = \mathsf{CPO}$ with the lifting functor $\rhd = (-)_\perp$ satisfies all the properties of Definition 2.13, i.e. $(\mathsf{CPO}, (-)_\perp)$ is a uniform guarded Conway category. But it is not a unique guarded fixpoint category: for let $X = \{0,1\}$ be the two-chain, $Y = 1$ the one element cpo and $f : X_\perp = X_\perp \times Y \to X$ be the map with $f(0) = f(\perp) = 0$ and $f(1) = 1$. Then both $0 : 1 \to X$ and $1 : 1 \to X$ make (2.1) commutative.

3 Guarded Trace Operators

In the case special case where $\rhd$ is the identity functor (see Example 2.4(1)), it is well-known that a fixpoint operator satisfying the Conway axioms is equivalent to a trace operator w.r.t. the product on $\mathscr{C}$ (see Hasegawa [16, 15]). In this section we present a similar result for a generalized notion of a guarded trace operator on $(\mathscr{C}, \rhd)$.

Remark 3.1. Recall that the notion of an (ordinary) trace operator was introduced by Joyal, Street and Verity [17] for symmetric monoidal categories. The applicability of the notion of trace to non-cartesian tensor products is in fact one of main reasons of its popularity. Our generalization can also be formulated for symmetric monoidal categories, see the remark preceding Construction 3.4 below. However, the main results in this section, i.e., Theorems 3.5 and 3.7 do not make any use of this added generality. Hence, we keep the Assumption 2.1 like in the remainder of the paper.

Definition 3.2. A (cartesian) *guarded trace operator* on $(\mathscr{C}, \rhd)$ is a natural family of operations

$$\mathsf{Tr}^X_{A,B} : \mathscr{C}(\rhd X \times A, X \times B) \to \mathscr{C}(A,B)$$

subject to the following three conditions:

1. **Vanishing.** (I) For every $f : \rhd 1 \times A \to B$ we have

$$\mathsf{Tr}^1_{A,B}(f) = (A \cong 1 \times A \xrightarrow{p_1 \times A} \rhd 1 \times A \xrightarrow{f} B).$$

   (II) For every $f : \rhd X \times \rhd Y \times A \to X \times Y \times B$ we have

$$\mathsf{Tr}^Y_{A,B}(\mathsf{Tr}^X_{\rhd Y \times A, Y \times A}(f)) = \mathsf{Tr}^{X \times Y}_{A,B}(\rhd(X \times Y) \times A \xrightarrow{\mathsf{can} \times A} \rhd X \times \rhd Y \times A \xrightarrow{f} X \times Y \times A).$$

2. **Superposing.** For every $f : \rhd X \times A \to X \times B$ we have

$$\mathsf{Tr}^X_{A \times C, B \times C}(f \times C) = \mathsf{Tr}^X_{A,B}(f) \times C.$$

3. **Yanking.** Consider the canonical isomorphism $c : \rhd X \times X \to X \times \rhd X$. Then we have

$$\mathsf{Tr}^X_{X, \rhd X}(c) = (X \xrightarrow{p_X} \rhd X).$$

If Tr is a (cartesian) guarded trace operator on $(\mathscr{C}, \rhd)$, $(\mathscr{C}, \rhd, \mathsf{Tr})$ is called a *guarded traced (cartesian) category*.

Of course, when $\rhd$ is taken to be the identity on $\mathscr{C}$ (as in Example 2.4(1)), our notion of guarded trace specializes to the notion of an ordinary trace operator (w.r.t. product) of Joyal, Street and Verity.

In addition, as in the case of ordinary trace operators naturality of Tr can equivalently be expressed by three more axioms:

4. **Left-tightening.** Given $f : \rhd X \times A \to X \times B$ and $g : A' \to A$ we have

$$\mathsf{Tr}^X_{A',B}(\rhd X \times A' \xrightarrow{\rhd X \times g} \rhd X \times A \xrightarrow{f} X \times B) = (A' \xrightarrow{g} A \xrightarrow{\mathsf{Tr}^X_{A,B}(f)} B).$$

5. **Right-tightening.** Given $f : \rhd X \times A \to X \times B$ and $g : B \to B'$ we have

$$\mathsf{Tr}^X_{A,B'}(\rhd X \times A \xrightarrow{f} X \times B \xrightarrow{X \times g} X \times B') = (A \xrightarrow{\mathsf{Tr}^X_{A,B}(f)} B \xrightarrow{g} B').$$

6. **Sliding.** Given $f : \rhd X \times A \to X' \times B$ and $g : X' \to X$ we have

$$\mathsf{Tr}^X_{A,B}(\rhd X \times A \xrightarrow{f} X' \times B \xrightarrow{g \times B} X \times B) = \mathsf{Tr}^{X'}_{A,B}(\rhd X' \times A \xrightarrow{\rhd g \times A} \rhd X \times A \xrightarrow{f} X' \times B).$$

Remark 3.3. The generalization for a symmetric monoidal category $(\mathscr{C}, \otimes, I, c)$ equipped with a pointed endofunctor $\rhd : \mathscr{C} \to \mathscr{C}$ requires the assumption that $\rhd$ is *comonoidal*, i.e., equipped with a morphism $m_I : \rhd I \to I$ and a natural transformation $m_{X,Y} : \rhd(X \times Y) \to \rhd X \times \rhd Y$ satisfying the usual coherence conditions. In fact, in the formulation of Vanishing (II) we used that in every category the product $\times$ is comonoidal via $m_{X,Y} = \mathsf{can}$.

Construction 3.4. 1. Let $(\mathscr{C}, \rhd, \mathsf{Tr})$ be a guarded traced category. Define a guarded fixpoint operator $\dagger_{\mathsf{Tr}} : \mathscr{C}(\rhd X \times A) \to \mathscr{C}(A, X)$ by

$$f^{\dagger_{\mathsf{Tr}}} = \mathsf{Tr}^X_{A,X}(\rhd X \times A \xrightarrow{\langle f, f\rangle} X \times X)) : A \to X.$$

2. Conversely, suppose $(\mathscr{C}, \rhd, \dagger)$ is a guarded fixpoint category. Define $\mathsf{Tr}_{\dagger}{}^X_{A,B} : \mathscr{C}(\rhd X \times A, X \times B) \to \mathscr{C}(A, B)$ by setting for every $f : \rhd X \times A \to X \times B$

$$\mathsf{Tr}_{\dagger}{}^X_{A,B}(f) = (A \xrightarrow{\langle (\pi_\ell \cdot f)^\dagger, A\rangle} X \times A \xrightarrow{\quad p_X \times A \quad} \rhd X \times A \xrightarrow{f} X \times B \xrightarrow{\pi_r} B).$$

The main result in this section states that the category $\mathscr{C}$ is guarded traced iff it is a guarded Conway category:

Theorem 3.5. *1. Whenever $(\mathscr{C}, \rhd, \mathsf{Tr})$ is a guarded traced category, $(\mathscr{C}, \rhd, \dagger_{\mathsf{Tr}})$ is a guarded Conway category. Furthermore, $\mathsf{Tr}_{\dagger_{\mathsf{Tr}}}$ is the original operator Tr.*

2. Whenever $(\mathscr{C}, \rhd, \dagger)$ is a guarded Conway category, $(\mathscr{C}, \rhd, \mathsf{Tr}_\dagger)$ is guarded traced. Furthermore, $\dagger_{\mathsf{Tr}_\dagger}$ is the original operator $\dagger$.

The proof details are similar to the proof details for ordinary fixpoint operators and traced cartesian categories (see Hasegawa [15]). Here one has to stick $\rhd$ in "all the right places" in all the necessary verifications of the axioms for trace and dagger, respectively. However, some of proof steps, in particular the derivation of a guarded version of the so-called Bekič identity require some creativity; it is not a completely automatic adaptation.

Hasegawa related uniformity of trace to uniformity of dagger and we can do the same in the guarded setup. Recall that in iteration theories uniformity (called *functorial dagger implication*) plays an important role. On the one hand, this quasiequation implies the so-called *commutative identities*, an infinite set of equational axioms that are added to the Conway axioms in order to yield a complete axiomatization of fixpoint operators in domains. On the other hand, most examples of iteration theories actually satisfy uniformity, and so uniformity gives a convenient sufficient condition to verify that a given Conway theory is actually an iteration theory.

Definition 3.6. A guarded trace operator Tr is called uniform if for every morphism $f : \rhd X \times A \to X \times B$, $f' : \rhd X' \times A \to X' \times B$ and $h : X \to X'$ we have

$$\begin{array}{ccc} \rhd X \times A & \xrightarrow{f} & X \times B \\ {\scriptstyle \rhd h \times A}\downarrow & & \downarrow{\scriptstyle h \times B} \\ \rhd X' \times A & \xrightarrow[f']{} & X' \times B \end{array} \qquad \Longrightarrow \qquad \mathsf{Tr}^X_{A,B}(f) = \mathsf{Tr}^{X'}_{A,B}(f') : A \to B.$$

Theorem 3.7. *1. Whenever $(\mathscr{C}, \rhd, \mathsf{Tr})$ is a uniform guarded traced category, $\dagger_{\mathsf{Tr}}$ is a uniform guarded Conway operator.*

2. Whenever $(\mathscr{C}, \rhd, \dagger)$ is a uniform guarded Conway category, $\mathsf{Tr}_\dagger$ is a uniform guarded trace operator.

Remark 3.8. Actually, Hasegawa proved a slightly stronger statement concerning uniformity then what we stated in Theorem 3.7; he showed that a Conway operator is uniform w.r.t. any fixed morphism $h : X \to X'$ (i.e. satisfies uniformity just for h) iff the corresponding trace operator is uniform w.r.t. this morphism h. The proof is somewhat more complicated and in our guarded setting we leave this as an exercise to the reader.

Finally, let us note that the bijective correspondence between guarded Conway operators and guarded trace operators established in Theorem 3.5 yields an isomorphism of the (2-)categories of (small) guarded Conway categories and guarded traced (cartesian) categories. The corresponding notions of morphisms are, of course, as expected:

Definition 3.9.

1. $F : (\mathscr{C}, \rhd^{\mathscr{C}}, \dagger) \to (\mathscr{D}, \rhd^{\mathscr{D}}, \ddagger)$ is a morphism of guarded Conway categories whenever $F : \mathscr{C} \to \mathscr{D}$ is a finite-product-preserving functor satisfying

$$\begin{tikzcd} \mathscr{C} \arrow[r, "\rhd^{\mathscr{C}}"] \arrow[d, "F"'] & \mathscr{C} \arrow[d, "F"] \\ \mathscr{D} \arrow[r, "\rhd^{\mathscr{D}}"'] & \mathscr{D} \end{tikzcd} \quad \text{and} \quad p^{\mathscr{D}}_{FX} = F(p^{\mathscr{C}}_X) : FX \to \rhd^{\mathscr{D}} FX = F(\rhd^C X), \tag{3.1}$$

and preserving dagger, i.e., for every $f : \rhd X \times A \to X$ we have

$$F(f^\dagger) = (\rhd^{\mathscr{D}} FX \times FA \cong F(\rhd^{\mathscr{C}} X \times A) \xrightarrow{Ff} FX)^\ddagger.$$

2. A morphism $F : (\mathscr{C}, \rhd^{\mathscr{C}}, \mathsf{Tr}_{\mathscr{C}}) \to (\mathscr{D}, \rhd^{\mathscr{D}}, \mathsf{Tr}_{\mathscr{D}})$ is a finite-product-preserving $F : \mathscr{C} \to \mathscr{D}$ satisfying (3.1) above and preserving the trace operation: for every $f : \rhd^{\mathscr{C}} X \times A \to X \times B$ in $\mathscr{C}$ we have

$$F(\mathsf{Tr}^{X}_{\mathscr{C}\, A,B}(f)) = \mathsf{Tr}^{FX}_{\mathscr{D}\, FA,FB}(\rhd^{\mathscr{D}} FX \times FA \cong F(\rhd^{\mathscr{C}} X \times A) \xrightarrow{Ff} F(X \times B) \cong FX \times FB).$$

Corollary 3.10. *The (2-)categories of guarded Conway categories and of guarded traced (cartesian) categories are isomorphic.*

4 Conclusions and Future Work

We have made the first steps in the study of equational properties of guarded fixpoint operators popular in the recent literature, e.g., [23, 24, 4, 7, 9, 19, 18, 9, 5]. We began with an extensive list of examples, including both those already discussed in the above references and some whose connection with the "later" modality has not seemed obvious so far—e.g., Example 2.4.6 or completely iterative theories in Section 2.2. Furthermore, we formulated the four Conway properties and uniformity in analogy to the respective properties in iteration theories and we showed them to be sound w.r.t. all models discussed in Section 2. In particular, Theorem 2.16 proved that our axioms hold in all categories with a unique guarded dagger. In Theorem 3.5, we have a generalization of a result by Hasegawa for ordinary fixpoint operators: we proved that to give a (uniform) guarded fixpoint operator satisfying the Conway axioms is equivalent to giving a (uniform) guarded trace operator on the same category.

Our paper can be considered as a work in progress report. Our aim is to eventually arrive at completeness results similar to the ones on iteration theories. We do not claim that the axioms we presented are complete. In the unguarded setting, completeness is obtained by adding to the Conway axioms an infinite set of equational axioms called the *commutative identities*, see [10, 25]. We did not consider those here, but we considered the quasi-equational property of uniformity which implies the commutative identities and is satisfied in most models of interest. Only further research can show whether this property can ensure completeness in the guarded setup or one needs to postulate stronger ones.

Other future work pertains to a syntactic type-theoretic presentation of the axioms we studied and a description of a classifying guarded Conway category.

Concerning further models of guarded fixpoint operators, it would be worthwhile to consider fixpoint monads of Crole and Pitts [11] more closely. These generalize our example of the category CPO with the lifting monad. One can prove that any fixpoint monad induces a guarded fixpoint operator satisfying parameter and simplified composition identities as well as uniformity. However, proving the double dagger identity in the general case is an open problem.

It would also be interesting to obtain examples of guarded traced monoidal categories which are not ordinary traced monoidal categories and which do not arise from guarded Conway categories. Traces w.r.t. a trace ideal as considered by Abramsky, Blute and Panangaden [1] might be a good starting point.

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References

[1] Samson Abramsky, Richard Blute & Prakash Panangaden (1999): *Nuclear and Trace Ideals in Tensored $*$-Categories*. J. Pure Appl. Algebra 143(1–3), pp. 3–47, doi:10.1016/S0022-4049(98)00106-6.

[2] Peter Aczel, Jiří Adámek, Stefan Milius & Jiří Velebil (2003): *Infinite Trees and Completely Iterative Theories: A Coalgebraic View*. Theoret. Comput. Sci. 300, pp. 1–45, doi:10.1016/S0304-3975(02)00728-4.

[3] Jiří Adámek & Stefan Milius (2006): *Terminal Coalgebras and Free Iterative Theories*. Inform. and Comput. 204, pp. 1139–1172, doi:10.1016/j.ic.2005.11.005.

[4] Andrew W. Appel, Paul-André Melliès, Christopher D. Richards & Jérôme Vouillon (2007): *A very modal model of a modern, major, general type system*. In Martin Hofmann & Matthias Felleisen, editors: POPL, ACM, pp. 109–122. Available at `http://doi.acm.org/10.1145/1190216.1190235`.

[5] Robert Atkey & Conor McBride (2013): *Productive Coprogramming with Guarded Recursion*. Accepted for ICFP.

[6] Eric Badouel (1989): *Terms and infinite trees as monads over a signature*. Lecture Notes Comput. Sci. 351, pp. 89–103, doi:10.1007/3-540-50939-9_126.

[7] Nick Benton & Nicolas Tabareau (2009): *Compiling functional types to relational specifications for low level imperative code*. In Andrew Kennedy & Amal Ahmed, editors: TLDI, ACM, pp. 3–14. Available at `http://doi.acm.org/10.1145/1481861.1481864`.

[8] Lars Birkedal & Rasmus E. Møgelberg (2013): *Intensional Type Theory with Guarded Recursive Types qua Fixed Points on Universes*. In: Proceedings of LICS, pp. 213–222, doi:10.1109/LICS.2013.27.

[9] Lars Birkedal, Rasmus E. Møgelberg, Jan Schwinghammer & Kristian Støvring (2012): *First Steps in Synthetic Guarded Domain Theory: Step-Indexing in the Topos of Trees*. Logical Methods in Computer Science 8(4:1), pp. 1–45, doi:10.2168/LMCS-8(4:1)2012.

[10] Stephen L. Bloom & Zoltán Ésik (1993): *Iteration Theories: the equational logic of iterative processes*. EATCS Monographs on Theoretical Computer Science, Springer.

[11] Roy L. Crole & Andrew M. Pitts (1992): *New Foundations for Fixpoint Computations: FIX-Hyperdoctrines and FIX-Logic*. Inform. and Comput. 98(2), pp. 171–210, doi:10.1016/0890-5401(92)90018-B.

[12] Pietro Di Gianantonio & Marino Miculan (2004): *Unifying Recursive and Co-recursive Definitions in Sheaf Categories*. In Igor Walukiewicz, editor: Foundations of Software Science and Computation Structures, Lecture Notes in Computer Science 2987, Springer Berlin / Heidelberg, pp. 136–150. Available at `http://dx.doi.org/10.1007/978-3-540-24727-2_11`. 10.1007/978-3-540-24727-2_11.

[13] Calvin C. Elgot (1975): *Monadic Computation and Iterative Algebraic Theories*. In H. E. Rose & J. C. Sheperdson, editors: *Logic Colloquium '73*, 80, North-Holland Publishers, Amsterdam, pp. 175–230, doi:10.1007/978-1-4613-8177-8_6.

[14] Calvin C. Elgot, Stephen L. Bloom & Ralph Tindell (1978): *On the algebraic structure of rooted trees*. *J. Comput. System Sci.* 16, pp. 362–399, doi:10.1007/978-1-4613-8177-8_7.

[15] Masahito Hasegawa (1999): *Models of Sharing Graphs: A Categorical Semantics of let and letrec*. Distinguished Dissertation Series, Springer, doi:10.1007/978-1-4471-0865-8.

[16] Masihito Hasegawa (1997): *Recursion from Cyclic Sharing: Traced Monoidal Categories and Models of Cyclic Lambda Calculi*. In: *Proc. 3rd International Conference on Typed Lambda Calculi and Applications*, *Lecture Notes Comput. Sci.* 1210, Springer-Verlag, pp. 196–213, doi:10.1007/3-540-62688-3_37.

[17] André Joyal, Ross Street & Dominic Verity (1996): *Traced Monoidal Categories*. *Math. Proc. Cambridge Philos. Soc.* 119(3), pp. 447–468, doi:10.1017/S0305004100074338.

[18] Neelakantan R. Krishnaswami & Nick Benton (2011): *A semantic model for graphical user interfaces*. In Manuel M. T. Chakravarty, Zhenjiang Hu & Olivier Danvy, editors: *ICFP*, ACM, pp. 45–57. Available at `http://doi.acm.org/10.1145/2034773.2034782`.

[19] Neelakantan R. Krishnaswami & Nick Benton (2011): *Ultrametric Semantics of Reactive Programs*. In: *LICS*, IEEE Computer Society, IEEE Computer Society, pp. 257–266. Available at `http://dx.doi.org/10.1109/LICS.2011.38`.

[20] Stefan Milius (2005): *Completely Iterative Algebras and Completely Iterative Monads*. *Inform. and Comput.* 196, pp. 1–41, doi:10.1016/j.ic.2004.05.003.

[21] Robin Milner (1989): *Communication and Concurrency*. International Series in Computer Science, Prentice Hall.

[22] Philip S. Mulry (1994): *Lifting Theorems for Kleisli Categories*. In S. Brookes, M. Main, A. Melton, M. Mislove & D. Schmidt, editors: *Proc. Mathematical Foundations of Programming Semantics (MFPS'93)*, *Lecture Notes Comput. Sci.* 802, Springer, pp. 304–319, doi:10.1007/3-540-58027-1_15.

[23] Hiroshi Nakano (2000): *A Modality for Recursion*. In: *LICS*, IEEE Computer Society, pp. 255–266, doi:10.1109/LICS.2000.855774.

[24] Hiroshi Nakano (2001): *Fixed-Point Logic with the Approximation Modality and Its Kripke Completeness*. In Naoki Kobayashi & Benjamin C. Pierce, editors: *TACS*, *Lecture Notes in Computer Science* 2215, Springer, pp. 165–182, doi:10.1007/3-540-45500-0_8.

[25] Alex Simpson & Gordon D. Plotkin (2000): *Complete axioms for categorical fixed-point operators*. In: *Proc. 15th Symposium on Logic in Computer Science (LICS'00)*, IEEE Computer Society, pp. 30–41, doi:10.1109/LICS.2000.855753.