On Colimits in Categories of Relations

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Abstract

We study (finite) coproducts and colimits of ω-chains in $\mathbf{Rel}(\mathcal{C})$, the 2-category of relations over a given category $\mathcal{C}$. The former exist and are "the same" as in $\mathcal{C}$ provided that $\mathcal{C}$ is extensive. The latter do not exist for example in $\mathbf{Rel}(\mathbf{Set})$. However, the canonical construction of those colimits in the category of sets can be generalized to $\mathbf{Rel}(\mathbf{Set})$. The canonical cocone is shown to satisfy a 2-categorical universal property, namely that of an lax adjoint cooplimit. Sufficient conditions for any base category $\mathcal{C}$ to admit the construction are given.

A necessary and sufficient condition for the construction to yield colimits of ω-chains in the category of maps of $\mathbf{Rel}(\mathcal{C})$ is also given.

Key words: relation, map, (co)limit of ω-chains, (co)products, 2-categories, initial algebra construction.

AMS-Subj.-Class.: 18B10, 18A30, 18A35.

1 Introduction

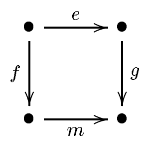
Categories $\mathbf{Rel}(\mathcal{C})$ of relations over a given category $\mathcal{C}$ are not (co)complete—even for such basic cases as $\mathcal{C} = \mathbf{Set}$. The following paper, motivated by the construction of initial algebras as certain ω-colimits (used in Computer Science), investigates

(i) ω-colimits

(ii) coproducts

in $\mathbf{Rel}(\mathcal{C})$. Coproducts are easy: whenever $\mathcal{C}$ has universal and disjoint coproducts (equivalently, $\mathcal{C}$ is an extensive category), then $\mathbf{Rel}(\mathcal{C})$ has "the same" coproducts. Colimits of ω-chains are much more difficult. On the one hand, $\mathbf{Rel}(\mathbf{Set})$ unfortunately does not have these colimits. On the other hand, $\mathbf{Rel}(\mathbf{Set})$ does have "lax adjoint cooplimits" of ω-chains, and this result holds for many other base categories $\mathcal{C}$.

When speaking about categories of relations, we always assume that $\mathcal{C}$ is a finitely complete category with a factorization system $(\mathcal{E}, \mathcal{M})$ in the sense of [1] (i. e. without assuming $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$ or $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$). Thus, $\mathcal{E}$ and $\mathcal{M}$ are classes of morphisms of $\mathcal{C}$ that are closed under composition with isomorphisms, and any arrow f of $\mathcal{C}$ has an $(\mathcal{E}, \mathcal{M})$-factorization, i. e., $f = m \circ e$ for some $m \in \mathcal{M}$ and $e \in \mathcal{E}$. Moreover, this factorization is unique up to isomorphism, which is implied by the famous diagonal fill-in property: for every commutative square



where $e \in \mathcal{E}$ and $m \in \mathcal{M}$ there exists a uniquely determined fill-in morphism t such that $t \circ e = f$ and $m \circ t = g$.

In this setting one can speak about relations in $\mathcal{C}$ with respect to $(\mathcal{E}, \mathcal{M})$ and define the (horizontal) composition of relations. In order to make this composition associative, and to force the existence of identity relations, one has to make certain assumptions on $(\mathcal{E}, \mathcal{M})$. One sufficient condition for associativity and identity relations is that the class $\mathcal{E}$ be stable under pullback, and this is what we shall henceforth assume. Hence, we obtain a 2-category $\mathbf{Rel}(\mathcal{C})$. Since in a 2-category there exists more structure provided by the 2-cells, it makes sense to look for weaker notions of (co)limits even if a strict (2-)limit of a certain diagram does not exist, as is the case for ω-colimits in $\mathbf{Rel}(\mathcal{C})$.

We shall prove here that under certain assumptions on $\mathcal{C}$, $\mathbf{Rel}(\mathcal{C})$ has so-called lax adjoint cooplimits of ω-chains. In general, a *lax adjoint limit* of a 2-diagram $F : \mathcal{D} \to \mathcal{C}$ is a lax cone $(L, \ell : \Delta L \to F)$, where ℓ is a lax natural transformation (cf. [2]) with domain the constant 2-functor with value L, such that for all objects C of $\mathcal{C}$ the functor

$$\mathcal{C}(C, L) \xrightarrow{\ell \circ \Delta(-)} \text{Lax-Cone}(C, F) \tag{1}$$

of composition with the limit projections ℓ is a right adjoint. Lax-Cone(C, F) denotes the category of lax natural transformations $\Delta C \to F$ and modifications between them.

Note that this is a straightforward generalization of the notions of lax limit and lax bilimit where the functor (1) is required to be an isomorphism or an equivalence of categories, respectively. The prefixes "co" and "op" mean that the direction of 2-cells and 1-cells, respectively, is to be reversed.

The assumptions on $\mathcal{C}$ to force the existence of lax adjoint cooplimits of ω-chains in $\mathbf{Rel}(\mathcal{C})$ are somewhat similar to the assumptions for the existence of coproducts. More explicitly, we assume that ω-colimits, as well as the appropriate dual limits, exist in $\mathcal{C}$ and the former are well-behaved with respect to pullbacks, i. e., colimits of ω-chains in $\mathcal{C}$ are universal and commute with pullbacks. An additional assumption that our proof needs is that the class $\mathcal{M}$ consists of monomorphisms.

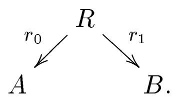
As for the outline of the present paper, we start by recalling basic facts about relations relative to a factorization system in Section 2. We also introduce the subcategory $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ of maps of $\mathbf{Rel}(\mathcal{C})$ (these are the 1-cells with a right adjoint). In Section 3 we describe (finite) coproducts in $\mathbf{Rel}(\mathcal{C})$ and apply our result to $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$. Finally, in Section 4, ω-colimits in $\mathbf{Rel}(\mathcal{C})$ are discussed. We first give an example that shows that these colimits do not exist in $\mathbf{Rel}(\mathbf{Set})$. Then lax cooplimits of ω-chains are constructed. At the end of the section we consider some consequences of the construction, for instance by applying the result to the category $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$. We also state some open problems. In particular, it is interesting to see whether our construction can be used to obtain some sort of (perhaps lax) initial algebras in $\mathbf{Rel}(\mathcal{C})$.

2 Relations relative to a factorization system

Throughout the paper we shall assume that $\mathcal{C}$ is a finitely complete category equipped with an $(\mathcal{E}, \mathcal{M})$ factorization system such that the class $\mathcal{E}$ is stable under pullback.

Relations in such a setting have been studied quite extensively ([7], [9], [6], [5], [11], [10]). Most of these treatments assume a priori that the class $\mathcal{E}$ consists of epimorphisms or $\mathcal{M}$ consists of monomorphisms or both. But in fact, none of this is necessary to obtain a calculus of relations ([11], [10]). Hence, no further assumption on $\mathcal{E}$ and $\mathcal{M}$ other than pullback stability of $\mathcal{E}$ are made at this moment. However, note that those assumptions often guarantee certain nice properties of $\mathbf{Rel}(\mathcal{C})$. For example, the hom-categories $\mathbf{Rel}(\mathcal{C})(A, B) = \mathbf{Rel}(A, B)$ are partially ordered if $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$. If $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$, then certain special classes of relations, such as isomorphisms, have convenient characterizations in terms of the classes $\mathcal{E}$ and $\mathcal{M}$ of $\mathcal{C}$ (cf. [10], Corollary 4.21).

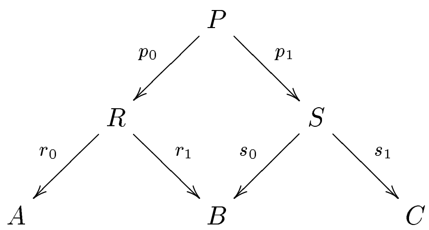
We shall now define the 2-category $\mathbf{Rel}(\mathcal{C})$ of relations over $\mathcal{C}$. Recall that a span in $\mathcal{C}$ is just a pair r_0, r_1 of arrows with common domain:



We call it a span in $\mathcal{M}$ if the arrow $\langle r_0, r_1 \rangle : R \to A \times B$ lies in the class $\mathcal{M}$. For any object C of $\mathcal{C}$ we denote by $\mathcal{M}/C$ the full subcategory of the comma category $\mathcal{C}/C$ formed by the arrows of $\mathcal{M}$. For any objects A and B of $\mathcal{C}$ one defines the hom-category

$$\mathbf{Rel}(A, B) = \mathrm{sub}(A \times B) \simeq \mathcal{M}/(A \times B),$$

i. e. equivalence classes of isomorphic spans in $\mathcal{M}$. Instead of using the equivalence classes we shall refer to objects of $\mathcal{M}/(A \times B)$ as relations in lieu of the equivalence classes represented by them. Relations shall be denoted by $r : A \nrightarrow B$. For two objects r and s of $\mathcal{M}/(A \times B)$ we write $r \simeq s$ to indicate that they represent the same relation. Recall that $(\mathbf{Iso}, \mathbf{All})$, where $\mathbf{Iso}$ denotes the class of isomorphisms of $\mathcal{C}$ and $\mathbf{All}$ the class of all morphisms, is a factorization system for any category, and that relations with respect to this factorization system are (represented by) spans $r = \langle r_0, r_1 \rangle : R \to A \times B$. Further recall that the horizontal composition of spans $\langle r_0, r_1 \rangle : R \to A \times B$ and $s = \langle s_0, s_1 \rangle : S \to B \times C$ is given by forming a pullback



of r_1 along s_0 in $\mathcal{C}$. We denote the composite by $s \diamond r$. Taking $\delta_A = \langle 1_A, 1_A \rangle : A \to A \times A$ as the identities one obtains a 2-category $\mathbf{Span}(\mathcal{C})$, whose hom-categories we denote by $\mathbf{Span}(A,B)$. In order to define compositions of relations w.r.t. an arbitrary factorization system $(\mathcal{E}, \mathcal{M})$ observe that for all objects A and B of $\mathcal{C}$ we have a left adjoint $\mathrm{im} : \mathbf{Span}(A,B) \to \mathbf{Rel}(A,B)$ of the inclusion functor as shown in the following diagram

$$\begin{array}{ccc} \mathbf{Span}(A,B) & \xrightarrow{\ \sim\ } & \mathcal{C}/A \times B \\ {\scriptstyle \mathrm{im}}\downarrow \dashv \uparrow{\scriptstyle \mathrm{in}} & & {\scriptstyle \mathrm{im}_0}\downarrow \dashv \uparrow{\scriptstyle \mathrm{in}_0} \\ \mathbf{Rel}(A,B) & \xrightarrow{\ \sim\ } & \mathcal{M}/A \times B\,, \end{array}$$

where im_0 and in_0 are given by choosing representatives of the equivalence classes in $\mathbf{Span}(A,B)$ and $\mathbf{Rel}(A,B)$ respectively. Horizontal composition of relations $r : A \leftrightarrow B$ and $s : B \leftrightarrow C$ is defined by

$$s \circ r = \mathrm{im}(s \diamond r).$$

This definition is easily seen to define a functor

$$\mathbf{Rel}(B,C) \times \mathbf{Rel}(A,B) \to \mathbf{Rel}(A,C).$$

For details we refer the Reader to [11] or [10]. The only ingredients now missing in order to define the 2-category $\mathbf{Rel}(\mathcal{C})$ are associativity of the composition and identity relations. The stability of $\mathcal{E}$ is a sufficient condition for composition to be associative and for the identities to be given by $\iota_A = \mathrm{im}(\delta_A)$.[1] Moreover,

$$\mathrm{im}(b \diamond a) = \mathrm{im}(b) \circ \mathrm{im}(a) \tag{2}$$

for any spans a and b such that the composition is defined. Thus, in order to compose relations, one manipulates with spans, taking $(\mathcal{E}, \mathcal{M})$-images to obtain the actual relation whenever necessary.

2.1 Maps and the graph functor

Following Lawvere [8], a *map* in a 2-category is a 1-cell with a right adjoint. Pavlović [11] observed that any map $\langle r_0, r_1 \rangle : R \to A \times B$ in $\mathbf{Rel}(\mathcal{C})$ has a canonical right adjoint, namely its opposite relation r^o given by $\langle r_1, r_0 \rangle$. The notion of a map tries to capture the idea of a function being a "total" and "single-valued" relation. In general, a relation $r : A \leftrightarrow B$ is called *total* if there exists a 2-cell $\iota_A \to r^o \circ r$; r is called *single-valued* if there exists a 2-cell $r \circ r^o \to \iota_B$. It turns out that a relation is a map if and only if it is total, single-valued and monomorphic as an arrow of $\mathcal{C}$, i.e., $\langle r_0, r_1 \rangle : R \to A \times B$ is monomorphic ([12], [10]). Moreover, for the 2-category $\mathbf{Rel}(\mathcal{C})$ the maps do not only form a sub-2-category but even a subcategory; any 2-cell between two maps is an isomorphism. We denote this subcategory by $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$.

For every arrow $f : A \to B$ of $\mathcal{C}$ one defines its *graph* to be the relation $\Gamma f = \mathrm{im}\langle 1_A, f \rangle$. Every graph is a map. Hence, this assignment yields a functor

$$\Gamma : \mathcal{C} \to \mathbf{Map}(\mathbf{Rel}(\mathcal{C})),$$

[1] Without any assumption on $(\mathcal{E}, \mathcal{M})$, composition of relations is not associative (cf. [7]). Furthermore, stability of $\mathcal{E}$ is not a necessary condition for associativity. For a detailed discussion of these issues see [10].

the so-called *graph functor*.

Note that one can show by using ideas from the theory of allegories (see [4]) that the category $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ is finitely complete (see [10] for a proof of this fact).

3 Some colimits in $\mathbf{Rel}(\mathcal{C})$

In this section we shall study finite coproducts in the 2-category $\mathbf{Rel}(\mathcal{C})$ as well as in the underlying category. The latter shall henceforth be referred to as the ordinary category $\mathbf{Rel}(\mathcal{C})$. Observe that by self-duality of $\mathbf{Rel}(\mathcal{C})$ we also describe finite products.

First of all note that $\mathbf{Rel}(\mathbf{Set})$ has finite (co)products, which are given by the coproducts in $\mathbf{Set}$. The generalization of this fact to $\mathbf{Rel}(\mathcal{C})$ does apparently not appear in the literature anywhere. We shall give a sufficient condition on $\mathcal{C}$ which allows the construction of coproducts in $\mathbf{Rel}(\mathbf{Set})$ to be carried over to our setting. One assumes that coproducts of $\mathcal{C}$ are well-behaved with respect to pullbacks. More precisely, $\mathcal{C}$ needs to be extensive (cf. [3]) or equivalently, its coproducts need to be disjoint and universal.

3.1 Extensive categories

Carboni, Lack and Walters have studied extensive and distributive categories in [3]. In this subsection we recall those facts from [3] that are needed here.

Definition 3.1. *A category $\mathcal{C}$ with coproducts of pairs of objects is* extensive *if and only if it has pullbacks along injections of coproducts, and if every commutative diagram*

$$\begin{array}{ccccc} A_1 & \xrightarrow{a_1} & A & \xleftarrow{a_2} & A_2 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_2} \\ X_1 & \xrightarrow{x_1} & X_1 + X_2 & \xleftarrow{x_2} & X_2 \end{array}$$

comprises a pair of pullback squares in $\mathcal{C}$ if and only if the top row is a coproduct diagram in $\mathcal{C}$.

Definition 3.2. *Let $\mathcal{C}$ be a category. If $(X_i \to X \mid i \in I)$ is a cocone under a diagram with vertices X_i, we say X is a* universal colimit *of the diagram provided that, for each morphism $Y \to X$, the cocone $(Y \times_X X_i \to Y \mid i \in I)$ is a colimit for the "pulled-back" diagram with vertices $Y \times_X X_i$.*

Note that a universal initial object is an initial object 0 with the following property: For any object A, if $A \to 0$ is an arrow then A is also an initial object; equivalently, any arrow $A \to 0$ is an isomorphism. A universal initial object is often referred to as a *strict* initial object.

Definition 3.3. *Let $\mathcal{C}$ be a category with coproducts and pullbacks. A coproduct $X = \coprod_{i \in I} X_i$ is said to be* disjoint *if*

(i) for every $j \in I$ the coproduct injection $X_i \xrightarrow{i_j} X$ is monomorphic,

(ii) for each pair $j,k \in I$ with $j \neq k$, the pullback of the two injections i_j, i_k is an initial object of $\mathcal{C}$.

Definition 3.4. *A category with binary products and coproducts is said to be* distributive *if the canonical arrow*

$$(A \times C) + (B \times C) \xrightarrow{\text{dist}} (A + B) \times C$$

is an isomorphism for any objects A, B and C.

Proposition 3.5. *(i) In a category with initial objects and universal binary coproducts, initial objects are strict.*

(ii) A category with finite coproducts and pullbacks along their injections is extensive if and only if the coproducts are disjoint and universal.

(iii) An extensive category with binary products is distributive.

3.2 Finite coproducts in $\mathbf{Rel}(\mathcal{C})$

Proposition 3.6. *Let 0 be a strict initial object of $\mathcal{C}$. Then 0 is initial in the 2-category $\mathbf{Rel}(\mathcal{C})$.*

Proof. Let A be any object of $\mathbf{Rel}(\mathcal{C})$. Observe that $\langle 1_0, \mathsf{i} \rangle$, where i denotes the uique arrow from 0 to A, is a relation in $\mathbf{Rel}(0, A)$. Indeed, if $me = \langle 1_0, \mathsf{i} \rangle$ is any $(\mathcal{E}, \mathcal{M})$-factorization, then by strictness of 0, e is an isomorphism, whence $\langle 1_0, \mathsf{i} \rangle$ lies in $\mathcal{M}$. Moreover, it is the only relation $0 \looparrowright A$, for suppose $r : R \to 0 \times A$ is any relation, then R must be initial in $\mathcal{C}$, again by strictness of 0. Finally, note that $\langle 1_0, \mathsf{i} \rangle$ is a graph and therefore monomorphic as arrow of $\mathcal{C}$. Hence, the only 2-cell in $\mathbf{Rel}(0, A)$ is the identity. □

Proposition 3.7. *Suppose that $\mathcal{C}$ is an extensive category. Then $\mathbf{Rel}(\mathcal{C})$ has coproducts, which are formed as in $\mathcal{C}$, i. e. the graph functor $\Gamma : \mathcal{C} \to \mathbf{Rel}(\mathcal{C})$ preserves coproducts.*

Proof. We shall show that

$$\begin{array}{ccccc} & A & & B & \\ /\!\!/ & & \searrow^{i_A} \quad {}^{i_B}\swarrow & & \backslash\!\!\backslash \\ A & & A+B & & B\,, \end{array} \tag{3}$$

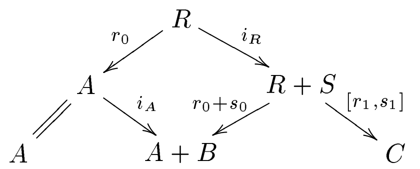
where $A + B$ denotes the coproduct of A and B in $\mathcal{C}$ with injections i_A and i_B, is a coproduct diagram in the ordinary category $\mathbf{Rel}(\mathcal{C})$ with injections given by $i_0 = \mathrm{im}\langle 1_A, i_A \rangle$ and $i_1 = \mathrm{im}\langle 1_B, i_B \rangle$.

Let $r = \langle r_0, r_1 \rangle : A \looparrowright C$ and $s = \langle s_0, s_1 \rangle : B \looparrowright C$ be relations. We must construct a unique relation $[r, s] : A + B \looparrowright C$ such that $[r, s] \circ i_0 \simeq r$ and $[r, s] \circ i_1 \simeq s$.

The relation $[r, s]$ is the image of the span given by

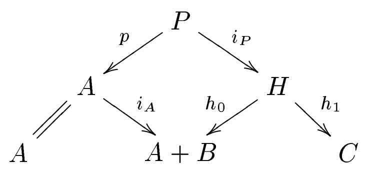
$$\begin{array}{ccc} & R + S & \\ {}^{r_0 + s_0}\swarrow & & \searrow^{[r_1, s_1]} \\ A + B & & C. \end{array}$$

Observe that $[r,s]\circ i_0 \simeq r$ because by extensivity of $\mathcal{C}$, the square in the middle of the diagram

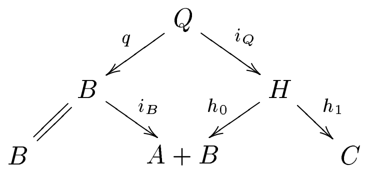


is a pullback. That $[r,s]\circ i_1 \simeq s$ holds true can be shown analogously.

Finally, let us show that $[r,s]$ is unique. In order to prove it let $h = \langle h_0, h_1\rangle : H \to (A+B)\times C$ be a relation for which $h\circ i_0 \simeq r$ and $h \circ i_1 \simeq s$ hold. That means that the composites

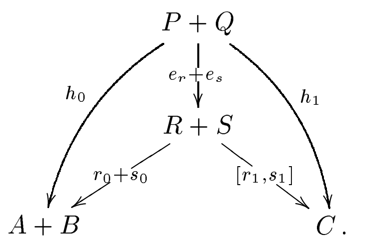


and



represent the same relation as r and s respectively. Therefore there are $\mathcal{E}$-arrows $e_r : P \to R$ and $e_s : Q \to S$ such that $\langle r_0, r_1\rangle e_r$ and $\langle s_0, s_1\rangle e_s$ are $(\mathcal{E},\mathcal{M})$-factorizations of $\langle p, h_1 i_P\rangle$ and $\langle q, h_1 i_Q\rangle$ respectively.

By extensivity of $\mathcal{C}$ we have $H = P + Q$ and $h_0 = p + q$. Observe now that the following diagram commutes:



This proves $h \simeq [r,s]$ since $e_r + e_s \in \mathcal{E}$. The latter holds true because $\mathcal{E}$ (as a full subcategory of $\mathcal{C}$) is closed under colimits. □

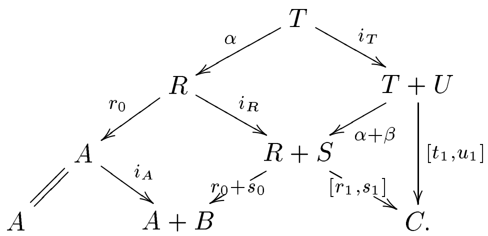
Note that the existence of general (co)products in $\mathbf{Rel}(\mathcal{C})$ follows from the existence of universal and disjoint general coproducts in $\mathcal{C}$ similarly.

Finally, note that even without extensivity of $\mathcal{C}$, the construction of Proposition 3.7 still satisfies a lax universal property, but does not yield a colimit in $\mathbf{Rel}(\mathcal{C})$ (cf. [10], Proposition 8.12). However, if $\mathcal{M}$ is closed under binary coproducts, then we even obtain a 2-co(op)product in $\mathbf{Rel}(\mathcal{C})$ (see [2] for a definition of a general 2-limit). Note that "co" here means that the direction of 2-cells in the definition is reversed and "op" means reversal of the 1-cells. Hence, self duality of $\mathbf{Rel}(\mathcal{C})$ justifies the parentheses in the name. Explicitly, a 2-coopproduct of two objects A and B in $\mathbf{Rel}(\mathcal{C})$ is a coproduct in the usual sense, i.e. an object $A+B$ together with injections $i_0 : A \nrightarrow A+B$ and $i_1 : B \nrightarrow A+B$ such that for any pair $r : A \nrightarrow C$, $s : B \nrightarrow C$ of relations there exists a unique relation $[r,s] : A+B \nrightarrow C$ with $[r,s]\circ i_0 \simeq r$ and $[r,s]\circ i_1 \simeq s$. Moreover, the following additional property must be satisfied. Given a second pair of relations $t : A \nrightarrow C$ and $u : B \nrightarrow C$, and 2-cells $\alpha : t \to r$ and $\beta : u \to s$, then there exists a unique 2-cell $\gamma : [t,u] \to [r,s]$ with $\gamma \circ i_0 = \alpha$ and $\gamma \circ i_1 = \beta$.

Proposition 3.8. *If $\mathcal{C}$ is an extensive category with factorization system $(\mathcal{E},\mathcal{M})$, where $\mathcal{M}$ is closed under binary coproducts, then 2-co(op)products exist in the 2-category* **Rel**$(\mathcal{C})$ *and are given as in Proposition 3.7.*

Proof. The proof of Proposition 3.7 shows that we have a universal cocone. Hence, we must only show the additional property. First note that the hypothesis that $\mathcal{M}$ is closed under coproducts implies that $\langle r_0+s_0,[r_1,s_1]\rangle = \mathrm{dist}\cdot(r+s)$, where dist is the canonical arrow of Definition 3.4, is in $\mathcal{M}$ since $\mathcal{C}$ is extensive with binary products, and therefore distributive (cf. [3]).

Now, given r, s, t, u, α and β as above, consider the diagram



(4)

By extensivity, the upper right square is a pullback. So $\alpha+\beta$ is a 2-cell with $(\alpha+\beta)\circ i_0 = \alpha$. Similarly $(\alpha+\beta)\circ i_1 = \beta$. We have to show that $\alpha+\beta$ is unique. But if we put any 2-cell $\gamma : [t,u] \to [r,s]$ in the place of $\alpha+\beta$ in diagram (4), extensivity of $\mathcal{C}$ implies that $\gamma = \gamma_0+\gamma_1$, and furthermore, that $\gamma_0 = \gamma\circ i_0 = \alpha$ and $\gamma_1 = \gamma\circ i_1 = \beta$. □

3.3 Application to maps

Diagram (3) in Proposition 3.7 yields a coproduct diagram if one restricts 1-cells to maps. For the proof of this result we need the following easily established facts (see [10], Lemma 4.5 and Lemma 8.17 for the proofs).

Lemma 3.9. *In every extensive category pullbacks commute with coproducts, i. e., if $f_ip_i = g_iq_i$, $i = 1,2$, are two pullback squares, so is*

$$\begin{array}{ccc} P_0+P_1 & \xrightarrow{q_0+q_1} & B_0+B_1 \\ {\scriptstyle p_0+p_1}\downarrow & & \downarrow{\scriptstyle g_0+g_1} \\ A_0+A_1 & \xrightarrow[f_0+f_1]{} & C_0+C_1\,. \end{array}$$

Lemma 3.10. *If f is a monomorphic arrow of $\mathcal{C}$ and $f = me$ is an $(\mathcal{E},\mathcal{M})$-factorization, where $\mathcal{E}$ is stable under pullback, then m is monomorphic in $\mathcal{C}$, too.*

Theorem 3.11. *If $\mathcal{C}$ is an extensive category, then coproducts exist in* **Map**(**Rel**$(\mathcal{C})$) *and are given as in Proposition 3.7.*

Proof. We need only check that for any given maps $r = \langle r_0,r_1\rangle : A \leftrightarrow C$ and $s = \langle s_0,s_1\rangle : B \leftrightarrow C$ the induced relation $[r,s] : A+B \leftrightarrow C$ is a map, too. Recall from 2.1 that a relation is a map if and only if it is total, single-valued and monomorphic as an arrow of $\mathcal{C}$.

To see that $[r,s]$ is single-valued we consider the composite $[r,s]\circ[r,s]^o$. In order to obtain this composite we first form the kernel-pair $\ker(r_0+s_0)$. By Lemma 3.9, we have

$$\ker(r_0+s_0)=\langle k_0+m_0, k_1+m_1\rangle,$$

where $\langle k_0,k_1\rangle=\ker(r_0)$ and $\langle m_0,m_1\rangle=\ker(s_0)$. Hence, we obtain

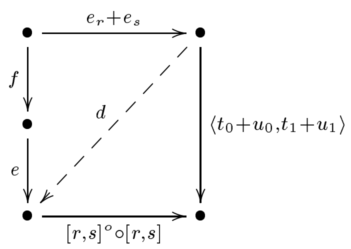
$$[r,s]\circ[r,s]^o=\mathrm{im}\langle[r_1,s_1](k_0+m_0),[r_1,s_1](k_1+m_1)\rangle.$$

Using single-valuedness of r and s we obtain arrows $\varepsilon_r:\langle r_1k_0,r_1k_1\rangle\to\iota_C$ and $\varepsilon_s:\langle s_1m_0,s_1m_1\rangle\to\iota_C$ between spans. Thus $[\varepsilon_r,\varepsilon_s]$ induces a 2-cell $[r,s]\circ[r,s]^o\to\iota_C$.

For totality we consider the composite

$$[r,s]^o\circ[r,s]=\mathrm{im}\langle(r_0+s_0)x_0,(r_0+s_0)x_1\rangle,$$

where $\langle x_0,x_1\rangle=\ker([r_1,s_1])$. Let $\langle h_0,h_1\rangle=\ker(r_1)$ and $\langle n_0,n_1\rangle=\ker(s_1)$. Clearly $[r_1,s_1](h_0+n_0)=[r_1,s_1](h_1+n_1)$. Hence there exists a (unique) arrow $f:\langle h_0+n_0,h_1+n_1\rangle\to\langle x_0,x_1\rangle$ between spans. Now let $r^o\circ r=\langle t_0,t_1\rangle$ and $s^o\circ s=\langle u_0,u_1\rangle$, i.e., we have $\langle t_0,t_1\rangle e_r=\langle r_0h_0,r_0h_1\rangle$ and $\langle u_0,u_1\rangle e_s=\langle s_0n_0,s_0n_1\rangle$ for some arrows e_r and e_s from $\mathcal{E}$. It follows that the outward square of the following diagram



commutes, where e is some arrow in $\mathcal{E}$. Since e_r+e_s lies in $\mathcal{E}$, we can apply the diagonal fill-in property to obtain the arrow d such that the diagram commutes. Now by totality of r and s we have arrows $\eta_r:\delta_A\to\langle t_0,t_1\rangle$ and $\eta_s:\delta_B\to\langle u_0,u_1\rangle$ of spans. It is easy to see that $d(\eta_r+\eta_s):\delta_{A+B}\to[r,s]^o\circ[r,s]$ induces a 2-cell as desired.

Finally, we show that $[r,s]$ is monic as an arrow of $\mathcal{C}$. By Lemma 3.10, it is sufficient to prove that the span

$$m=\langle r_0+s_0,[r_1,s_1]\rangle$$

is monic in $\mathcal{C}$. Suppose that $mf=mg$ for some arrows f and g with common domain X. By extensivity, $X=X_0+X_1$, $f=f_0+f_1$ and $g=g_0+g_1$, where f_i and g_i have domain X_i, $i=0,1$. (Notice that one can assume the same decomposition of X into coproduct components since f and g are coequalized by r_0+s_0). Recall that coproduct injections in $\mathcal{C}$ are monomorphic. Thus it is now easy to see that $rf_0=rg_0$ and $sf_1=sg_1$. Since r and s are monomorphic in $\mathcal{C}$, it follows that $f_i=g_i$ for $i=0,1$, whence $f=g$. □

It is not difficult to show that $\mathbf{Rel}(\mathcal{C})$ is not (co)complete. Of course, the question of (co)completeness of $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ is a completely different one. It can be shown that $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ is finitely complete (see [10]). Moreover, if $\mathcal{C}$ is extensive $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ has coproducts as we have just seen. The question whether these coproducts are extensive in $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ arises immediately. Furthermore, it is interesting to see what conditions must be imposed on $\mathcal{C}$ to force the existence of coequalizers or pushouts in $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$. At this stage we leave these as open problems.

4 Colimits of ω-chains in $\mathbf{Rel}(\mathcal{C})$

Our motivation for studying these special colimits is the construction of initial algebras in $\mathbf{Rel}(\mathcal{C})$ as colimits of certain ω-chains in that category. Unfortunately, it turns out that even in a very reasonable category such as $\mathbf{Rel}(\mathbf{Set})$ these colimits do not exist in general. However, the construction which mimics the appropriate construction in **Set** fails only "by little", at least for total relations. It seems natural to ask whether this canonical construction can be generalized to arbitrary categories $\mathcal{C}$, or what properties of **Set** make the construction possible. But before we turn to the last question, let us present a counterexample for $\mathbf{Rel}(\mathbf{Set})$.

4.1 A counterexample

We shall show that even in $\mathbf{Rel}(\mathbf{Set})$, the category of binary relations between sets, colimits of ω-chains do not exist in general.

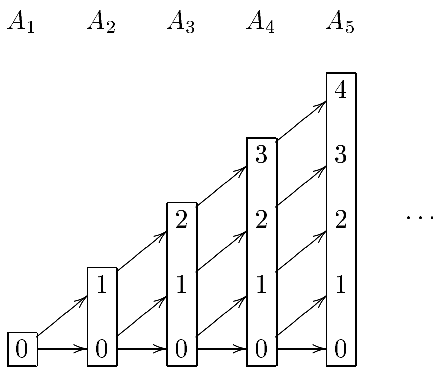
In order to see this, consider the family $\{A_i \mid i \in \mathbb{N} \setminus \{0\}\}$ of sets, where

$$A_i = \{0, \ldots, i-1\},$$

together with the relations $a_{i,i+1} : A_i \nrightarrow A_{i+1}$, defined by

$$a_{i,i+1} = \{(j, j+1) \mid 0 \leq j \leq i-1\} \cup \{(0,0)\}.$$

These sets and relations form an ω-chain in $\mathbf{Rel}(\mathbf{Set})$ as shown by the following figure:



Observe that the pair $(D, \{d_i\}_{i \in \mathbb{N}})$, where $D = \{0, 1\}$, and

$$d_i = \{(0,0)\} \cup \{(j,1) \mid 0 \leq j \leq i-1\},$$

forms a cocone on that chain.

Now assume that the colimit $(C, \{c_i\}_{i\in\mathbb{N}})$ of the above ω-chain exists. Obviously, C is not the empty set since otherwise $c_i = \emptyset$ for all i. Moreover, the unique relation $C \nrightarrow D$ would have to be empty, too, since $C = \emptyset$ is an initial object in **Rel**(**Set**). Thus, $d_i = \emptyset \circ \emptyset = \emptyset$, which is not true. Furthermore, observe that the relations c_i have to be total since the relations d_i are.

Next we will consider chains of elements in our given ω-chain. By a *chain* (of elements) we mean a sequence $\{x_i \mid i \in \mathbb{N}\}$ such that $x_i \in A_i$ and $(x_i, x_{i+1}) \in a_{i,i+1}$ for all $i \geq 1$. An example of this is $(0, 0, 0, \ldots)$, which will be called the 0-*chain*.

Note that there must be a point x in $\mathcal{C}$ such that all elements in the 0-chain are in relation with x but none of the other elements of the A_i are. In order to see this, observe that if any element of the 0-chain is in relation with an $x \in C$, then by commutativity of the cone $(C, \{c_i\})$ all the other elements of that chain also are. Moreover, since $(C, \{c_i\})$ is a colimit of the given ω-chain there is a unique relation $h : C \nrightarrow D$ such that $d_i = hc_i$ for all i. Furthermore, $(0, 0) \in d_i$ for all i, which implies that there is an $x \in C$ such that $(0, x) \in c_i$ and $(x, 0) \in h$ for all i. Now suppose that there is a set A_i and an element $n \in A_i$ with $0 < n \leq i - 1$ such that $(n, x) \in c_i$. Observe that $(n, 0) \notin d_i$. But we have $(n, x) \in c_i$ and $(x, 0) \in h$, which implies $(n, 0) \in h \circ c_i = d_i$, a contradiction.

Thus, taking into account the totality of the relations c_i, we have proved that C has at least two elements, one of which is an element x such that $(0, x) \in c_i$ but $(n, x) \notin c_i$ for all i and $n \in A_i$ for $0 < n \leq i - 1$.

Consider the following two relations with domain C and codomain D:

$$\begin{aligned} h_1 &= \{(x, 0)\} \cup \{(y, 1) \mid y \in C, y \neq x\} \\ h_2 &= h_1 \cup \{(x, 1)\}. \end{aligned}$$

It can easily be checked that $d_i = h_1 c_i$ and $d_i = h_2 c_i$ hold for any i. We shall show this for h_1. So suppose that $(j, 0) \in d_i$. This is the case precisely if $j = 0$. Hence, $(j, x) \in c_i$ and $(x, 0) \in h_1$, which means $(j, 0) \in h_1 \circ c_i$. Conversely, if $(j, 0) \in h_1 \circ c_i$, then we must have $(j, x) \in c_i$ and $(x, 0) \in h_1$, which implies $j = 0$ since no other element of A_i is in relation c_i with $x \in C$.

Finally suppose that $(j, 1) \in d_i$. Since c_i is total there exists $y \in C$ such that $(j, y) \in c_i$. If $j \neq 0$, then $y \neq x$, and $(y, 1) \in h_1$, which shows that $(j, 1) \in h_1 \circ c_i$. If $j = 0$, then we have $(j, 1) \in a_{i,i+1}$, whence there is an element $y \in C$ with $x \neq y$ such that $(1, y) \in c_{i+1}$. So $(0, y) \in c_{i+1} \circ a_{i,i+1} = c_i$. Since $y \neq x$, we have $(y, 1) \in h_1$, which implies that $(j, 1) \in h_1 \circ c_i$. On the other hand, $(j, 1) \in h_1 \circ c_i$ trivially implies that $(j, 1) \in d_i$ just because the latter holds for all j.

Showing that $d_i = h_2 \circ c_i$ for all i is a very similar computation, which is left to the Reader. Since h_1 and h_2 are clearly non-equal, C cannot be a colimit of the given chain.

Observe however, that at least for total relations there is always a largest factorization $h : C \nrightarrow D$.

Note that the above example also shows that bicolimits (cf. [2]) of ω-chains do not exist. Indeed, every equivalence of categories

$$\mathbf{Rel}(\mathbf{Set})(C, D) \to \mathrm{Cocone}(\mathcal{A}, D),$$

where $\mathcal{A} : \mathbb{N} \to \mathbf{Rel}(\mathbf{Set})$ is the obvious diagram, must be an isomorphism because both categories are posets.

4.2 Lax adjoint limits of ω-chains

In the preceding section we have seen that colimits of ω-chains in $\mathbf{Rel}(\mathcal{C})$ do not exist in general. However, a good portion of the canonical construction in **Set** carries over to $\mathbf{Rel}(\mathbf{Set})$. Moreover, this can be generalized to other base categories $\mathcal{C}$, when certain additional conditions are imposed on it. The "almost" universal cocone for an ω-chain in $\mathbf{Rel}(\mathcal{C})$ satisfies the conditions of the next definition. In order to have a handy name we will call this notion a *lax adjoint limit*.

Definition 4.1. *Let $D : \mathcal{D} \to \mathcal{C}$ be a 2-functor between 2-categories. A pair (L, ℓ) where L is an object of $\mathcal{C}$ and $\ell : \Delta L \to D$ is a lax natural transformation is called a* lax adjoint limit *of D if for any other such pair (M, m) where m is a lax natural transformation there is a 1-cell $h : M \to L$ and a modification $\eta : m \to \ell \cdot \Delta h$:*

$$\begin{array}{ccc} \Delta M & \xrightarrow{m} & D \\ {\scriptstyle \Delta h}\downarrow & \Downarrow \eta & \nearrow {\scriptstyle \ell} \\ \Delta L & & \end{array},$$

and given any other 1-cell $k : M \to L$ and a modification $\mu : m \to \ell \cdot \Delta k$ there is a unique 2-cell $\alpha : h \to k$ such that the triangle

$$\begin{array}{ccc} m & \xrightarrow{\eta} & \ell \cdot \Delta h \\ & {\scriptstyle \mu}\searrow & \downarrow {\scriptstyle \ell\cdot\Delta\alpha} \\ & & \ell \cdot \Delta k \end{array}$$

is commutative.

Observe that this can be stated more compactly by saying that the functor

$$\ell \circ \Delta(-) : \mathcal{C}(M, L) \to \text{Lax-Cone}(M, D)$$

of composition with ℓ is a right adjoint. This generalizes the notion of (lax) bilimit where this functor is required to be an equivalence of categories. Note that, in fact, we are interested in the dual notion of this, i.e. the notion with all cells reversed. Hence, we discuss lax adjoint cooplimits, which means that the functor F of composition with the canonical cocone is a left adjoint. It turns out that in $\mathbf{Rel}(\mathcal{C})$ the canonical cocone is even strict. Moreover, under a certain assumption on $\mathcal{C}$, the counit of the adjunction will be an isomorphism if we restrict the codomain of F to strict cocones (see Section 4.3); in other words, every cocone factors through the canonical cocone. Despite the fact that this additional assumption is true in **Set** for total relations, it is somewhat strange. But this will be discussed later.

Let us now state our main result.

Theorem 4.2. *The 2-category $\mathbf{Rel}(\mathcal{C})$ has lax adjoint cooplimits of ω-chains whenever the following conditions are satisfied:*

(i) $\mathcal{M} \subseteq \text{Mono}(\mathcal{C})$,

(ii) $\mathcal{C}$ *has limits of ω-cochains*

(iii) $\mathcal{C}$ has universal colimits of ω-chains which commute with pullbacks.

The proof forms the rest of Subsection 4.2.

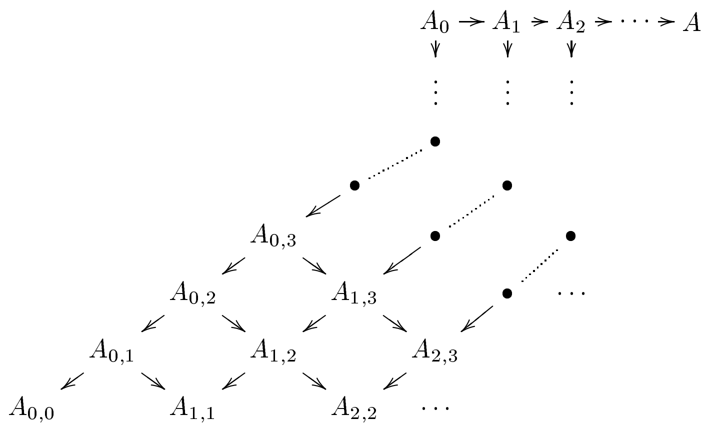
4.2.1 Construction of the canonical cocone

First let us analyze what a lax adjoint cooplimit of an ω-chain in the 2-category $\mathbf{Rel}(\mathcal{C})$ really is, assuming that the class $\mathcal{M}$ consists of monomorphisms. Notice that in this case the hom-categories $\mathbf{Rel}(A, B)$ are posets. Hence, we can denote a 2-cell between relations r and s by $r \leq s$. Suppose now that an ω-chain $\mathcal{A} : \mathbb{N} \to \mathbf{Rel}(\mathcal{C})$ is given. Then a cocone $(A, (a_i)_{i \in \mathbb{N}})$ will be constructed such that for any lax cocone $(B, (b_i)_{i \in \mathbb{N}})$, i. e., $b_i \geq b_j \circ \mathcal{A}(i \leq j)$, there exists a relation $h : A \mathrel{+\!\!\to} B$ with $h \circ a_i \leq b_i$ for all i. Moreover, if $k : A \mathrel{+\!\!\to} B$ is another factorization with $k \circ a_i \leq b_i$, then $k \leq h$. The uniqueness of the last 2-cell and the fact that the given families of 2-cells form modifications are true automatically, since in a poset all diagrams commute. However, note that it is not for the sake of this convenience that we assume $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$. The construction seems to fail much earlier without this assumption.

In order to construct the canonical cocone assume that an ω-chain in $\mathbf{Rel}(\mathcal{C})$, i. e., a zig-zag

$$\begin{array}{ccccccccccc} & A_{0,1} & & A_{1,2} & & \cdots & & A_{i,i+1} & & \cdots \\ \swarrow & & \searrow \ \swarrow & & \searrow & & \swarrow & & \searrow & \\ A_{0,0} & & A_{1,1} & & A_{2,2} & & A_{i,i} & & A_{i+1,i+1} & \end{array}$$

in $\mathcal{C}$, has been given. The following diagram outlines the construction of the canonical cocone for that ω-chain:

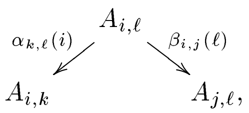


(5)

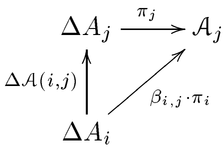
Start by forming the pullbacks $A_{i,i+2}$ of $A_{i,i+1}$ and $A_{i+1,i+2}$ for all $i \in \mathbb{N}$, and then iterate this process to obtain objects $A_{i,j}$ for all $i, j \in \mathbb{N}$. Next take the limits (A_i, π_i) of the cochains

$$A_{i,i} \longleftarrow A_{i,i+1} \longleftarrow \cdots \longleftarrow A_{i,j} \longleftarrow \cdots \tag{6}$$

with projections given by natural transformations $\pi_i : \Delta A_i \to \mathcal{A}_i$, where the $\mathcal{A}_i$ are the obvious diagram functors. Furthermore, denote by $\mathcal{A}_{i,j}$ the diagram given by the tail of the cochain (6) starting at $A_{i,j}$. Note that A_i is the limit of $\mathcal{A}_{i,j}$, because the inclusion of a tail is a final functor. By abuse of notation we will denote the projections in this case by $\pi_i : \Delta A_i \to \mathcal{A}_{i,j}$ for all j, too, since their components are given by the components $\pi_i(n)$ of π_i. Observe that we use a non-standard notation for components of a natural transformation to avoid double indexation. Now note that the arrows pointing down-right in (5) give natural transformations $\beta_{i,j} : \mathcal{A}_{i,j} \to \mathcal{A}_j$ for all $j \geq i \in \mathbb{N}$. We denote the arrows pointing down-left in (5) by $\alpha_{i,j}(k) : A_{k,j} \to A_{k,i}$. Hence, we have



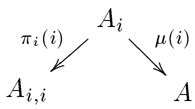
where ℓ is not necessarily equal to k. Clearly $(A_i, \beta_{i,j} \cdot \pi_i)$ is a cone on $\mathcal{A}_j$ so that there exist unique arrows $\mathcal{A}(i,j) : A_i \to A_j$ with



Thus we obtain an ω-chain in $\mathcal{C}$

$$A_0 \longrightarrow A_1 \longrightarrow \cdots \longrightarrow A_i \longrightarrow \cdots$$

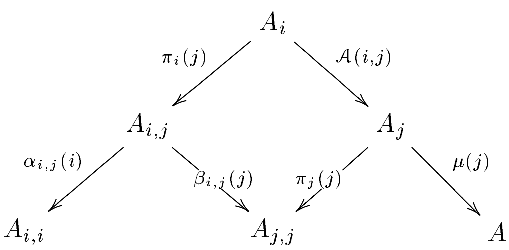
whose colimit (A, μ) is formed in $\mathcal{C}$, where $\mu : \mathcal{A} \to \Delta A$ for the obvious diagram functor $\mathcal{A}$. The claim is now that the images of the spans given by



for $i \in \mathbb{N}$ form the desired cocone for the given chain in $\mathbf{Rel}(\mathcal{C})$.

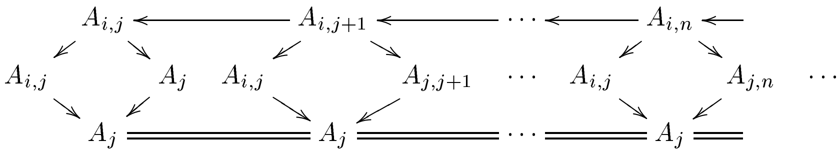
4.2.2 Commutativity

In order to show that we really have constructed a cocone we must show that the square in the following diagram



(7)

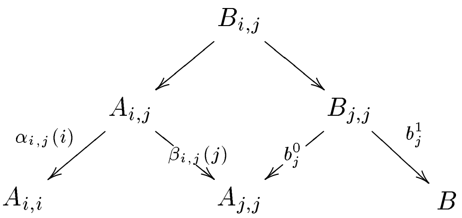
is a pullback for all $j \geq i$. Considering (5), the above square clearly commutes. In order to show the universal property we use the fact that limits commute with limits. Consider the following four cochains all of whose objects at index n form a pullback:



Note that the objects on the left side and on the bottom do not change. If the limits of the four cochains are formed first, we obtain the square in diagram (7), which shows that it is a pullback square.

4.2.3 The universal property

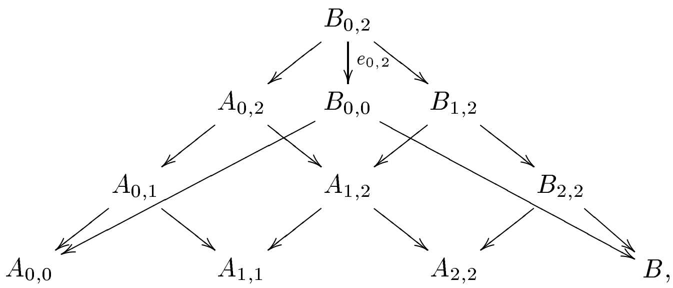
Denote by $a_i : A_{i,i} \nrightarrow A$ the components of the cocone constructed in the last subsection. We shall show that (A, a) satisfies the universal property of a lax adjoint cooplimit. Denote by $a_{i,j} : A_{i,i} \nrightarrow A_{j,j}$ the relations $\mathcal{A}(i \leq j)$ of the given ω-chain. Now suppose that (B, b) is a lax cocone, i.e. $b_i \geq b_j \circ a_{i,j}$ for all natural numbers $i \geq j$, where $b_i : B_{i,i} \to A_{i,i} \times A$. That means that if we form the following pullbacks



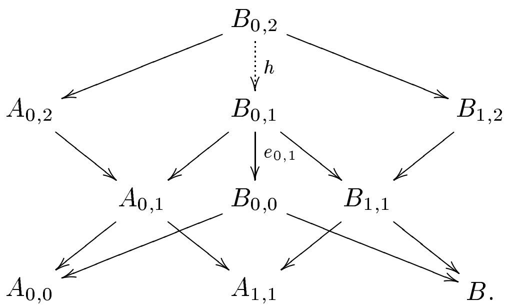
(8)

where $b_j = \langle b_j^0, b_j^1 \rangle$, then there is a 2-cell from this span to the relation b_i given by an arrow $e_{i,j}$. Note that $b_i \simeq b_j \circ a_{i,j}$ if and only if $e_{i,j}$ lies in the class $\mathcal{E}$ of the factorization system of $\mathcal{C}$.

The next step is to construct connecting arrows between the $B_{i,j}$'s to get cochains similar to the ones given by the $A_{i,j}$'s. These arrows arise by using the universal property of the pullbacks. We shall explicitly construct the arrow $B_{02} \to B_{01}$. Consider the composite $b_2 \circ A_{02}$ formed as in the following diagram



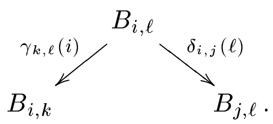
where all squares are pullbacks. Now use the universal property of the pullback in the following diagram:



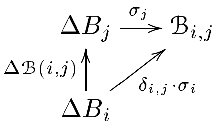
An arrow $h : B_{0,2} \to B_{0,1}$ is induced. One readily checks that $e_{0,1}h = e_{0,2}$ using that b_0 as an $\mathcal{M}$-relation is a monomorphic arrow of $\mathcal{C}$. Iterating this construction, we obtain cochains

$$B_{i,i} \longleftarrow B_{i,i+1} \longleftarrow \cdots \longleftarrow B_{i,j} \longleftarrow \cdots$$

for all $i \in \mathbb{N}$ as before. Form the limits (B_i, σ_i) where $\sigma_i : \Delta B_i \to \mathcal{B}_i$ for the obvious diagrams $\mathcal{B}_i$. Again, the diagrams obtained by taking tails are denoted by $\mathcal{B}_{i,j}$. For all $j \leq i$, their limits are the above B_i with projections given by σ_i. Similar to the $\alpha_{i,j}$ and $\beta_{i,j}$ for the $A_{i,j}$, there are natural transformations $\delta_{i,j} : \mathcal{B}_{i,j} \to \mathcal{B}_j$ and arrows $\gamma_{i,j}$ as follows



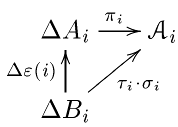
Now clearly $(B_i, \delta_{i,j} \cdot \sigma_i)$ is a cone on $\mathcal{B}_j$ so that there exist unique arrows $\mathcal{B}(i,j) : B_i \to B_j$ with the triangle



commutative. Hence, we obtain an ω-chain

$$B_0 \longrightarrow B_1 \longrightarrow \cdots \longrightarrow B_i \longrightarrow \cdots$$

denoted by $\mathcal{B}$. Now note that, for a fixed $i \in \mathbb{N}$, the arrows $B_{i,j} \to A_{i,j}$ that are the pullback projections obtained in diagram (8), are the components of a natural transformation $\tau_i : \mathcal{B}_i \to \mathcal{A}_i$. Therefore $(B_i, \tau_i \cdot \sigma_i)$ is a cone on $\mathcal{A}_i$. This implies the existence of unique arrows $\varepsilon(i) : B_i \to A_i$ with



for all $i \in \mathbb{N}$, which clearly yield a natural transformation $\varepsilon : \mathcal{B} \to \mathcal{A}$.

Lemma 4.3. *The square*

$$\begin{array}{ccc} B_i & \xrightarrow{\mathcal{B}(i,j)} & B_j \\ {\scriptstyle \varepsilon(i)}\downarrow & & \downarrow{\scriptstyle \varepsilon(j)} \\ A_i & \xrightarrow[\mathcal{A}(i,j)]{} & A_j \end{array}$$

is a pullback for all $j \geq i$ *in* $\mathbb{N}$.

Proof. Again, use the limit interchange rule considering the following cochain of pullback squares

$$\begin{array}{ccccccccc} & B_{i,j} & \longleftarrow & & & B_{i,j+1} & & & \\ \swarrow & & \searrow & & \swarrow & & \searrow & & \\ A_{i,j} & & B_{j,j} & A_{i,j+1} & & & & B_{j,j+1} & \cdots \\ \searrow & & \swarrow & & \searrow & & \swarrow & & \\ & A_{j,j} & \longleftarrow & & & A_{j,j+1} & & & \end{array}$$

which "converges" to the desired square. □

In order to complete the construction of the canonical factorization $h : A \leftrightarrow B$, form the colimit (H, ν) where $\nu : \mathcal{B} \to \Delta H$. Clearly $(A, \mu \cdot \varepsilon)$ and (B, η), where the components of η are given by $\eta(i) = b_i^1 \cdot \sigma_i(i)$, form cocones on $\mathcal{B}$. Therefore there are unique arrows

$$\begin{array}{ccc} & H & \\ {\scriptstyle h_0}\swarrow & & \searrow{\scriptstyle h_1} \\ A & & B \end{array}$$

for which the triangles

$$\begin{array}{ccc} \Delta H & \xleftarrow{\nu} & \mathcal{B} \\ {\scriptstyle \Delta h_0}\downarrow & \swarrow{\scriptstyle \mu\cdot\varepsilon} & \\ \Delta A & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \Delta H & \xleftarrow{\nu} & \mathcal{B} \\ {\scriptstyle \Delta h_1}\downarrow & \swarrow{\scriptstyle \eta} & \\ \Delta B & & \end{array} .$$

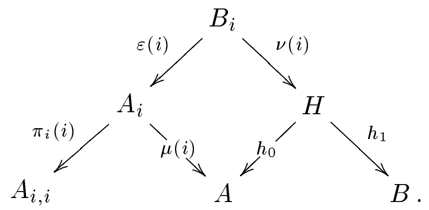
commute. The claim is that $\mathrm{im}\langle h_0, h_1\rangle$ is the canonical factorization $h : A \leftrightarrow B$. In order to prove this, we must first show that

$$b_i \geq h \circ a_i$$

for all $i \in \mathbb{N}$. Here we use condition (iii) of Theorem 4.2, namely that colimits of ω-chains commute with pullbacks in $\mathcal{C}$. For $i \in \mathbb{N}$ consider the following four ω-chains, which together form pullback squares at each index, by Lemma 4.3

$$\begin{array}{ccc} & B_1 & \\ \swarrow & & \| \\ A_1 & & B_1 \\ \| & & \swarrow \\ & A_1 & \end{array} \quad \cdots \quad \begin{array}{ccc} & B_i & \\ \swarrow & & \| \\ A_i & & B_i \\ \| & & \swarrow \\ & A_i & \end{array} \quad \cdots \quad \begin{array}{ccc} & B_i & \\ \swarrow & & \searrow \\ A_i & & B_j \\ \searrow & & \swarrow \\ & A_j & \end{array} .$$

Note that from index i on the left-hand objects and the upper ones do not change any more. Taking colimits, we see that the composite $h \circ a_i$ is given by the pullback square in the following diagram



But clearly, this span factorizes through $b_i : A_{i,i} \nrightarrow B$, since

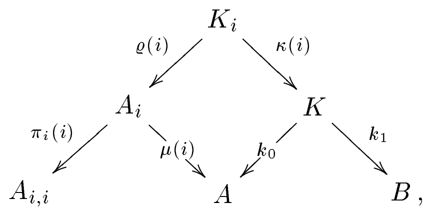
$$b_i\sigma_i(i) = \langle \pi_i(i)\varepsilon(i), h_1\nu(i) \rangle. \tag{9}$$

It is now left to show that for any other factorization $k : A \nrightarrow B$ with $b_i \geq k \circ a_i$ there exists a 2-cell $k \to h$.

4.2.4 The weak uniqueness

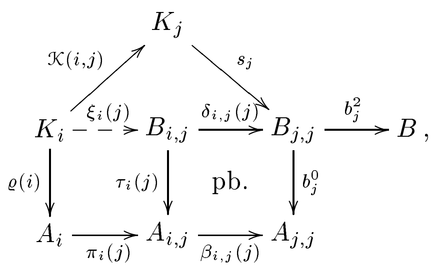
Suppose we are given a relation $k : A \nrightarrow B$ such that $b_i \geq k \circ a_i$ for all $i \in \mathbb{N}$. We shall construct a 2-cell $k \to h$. First let us consider the composites $k \circ a_i$, which are formed as in the following diagram:

(10)



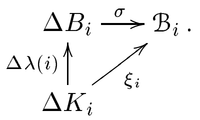
where the square is a pullback and, furthermore, the composite span factorizes through b_i, which means that $b_i s_i = \langle \pi_i(i)\varrho(i), k_1\kappa(i) \rangle$ for some arrow s_i. Now we use condition (ii) of Theorem 4.2. Hence, by universality of the colimit (A, μ), we know that (K, κ), where the components of $\kappa : \mathcal{K} \to \Delta K$ are given as in (10) for an obvious functor $\mathcal{K}$, is a colimit of the ω-chain given by the objects K_i. Moreover, the $\varrho(i)$ yield a natural transformation $\varrho : \mathcal{K} \to \mathcal{A}$ with $\mu \cdot \varrho = \Delta k_0 \cdot \kappa$. We shall construct an arrow $s : K \to H$. This must obviously be done by using the universal property of the colimit (K, κ). Hence, we need a natural transformation $\lambda : \mathcal{K} \to \mathcal{B}$. In order to get its components, we can use the universal property of the limits (B_i, σ_i) (recall that $\mathcal{B}(i) = B_i$). So all amounts to constructing cones $\xi_i : \Delta K_i \to \mathcal{B}_i$. Components of ξ_i can be obtained using the following diagram

(11)

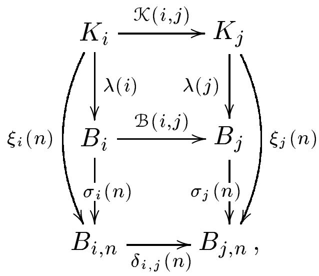


where i is fixed and $j \geq i$. Notice that $\xi_i(i) = s_i$. Proving the naturality of ξ_i is now a straightforward chase through an admittedly rather huge diagram using only that the pullback projections in (11) are jointly monomorphic and that b_i is a monomorphism, too. For the sake of brevity this task must be left to the Reader. Note that here the assumption that $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$ (thus, b_i is monomorphic) is used heavily.

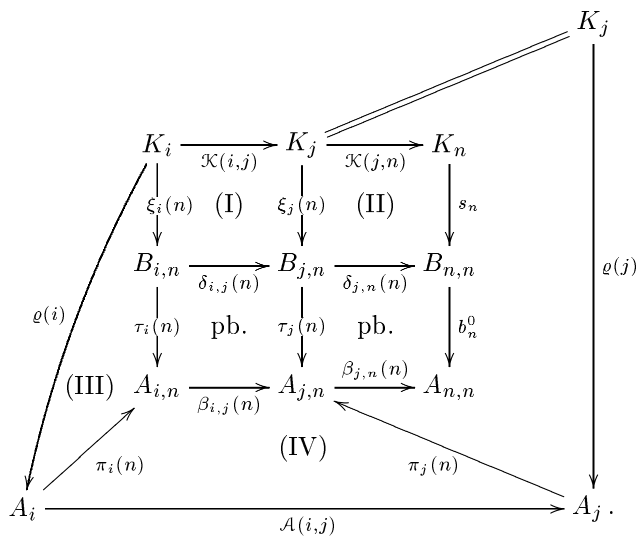
The universal property of the limit (B_i, σ_i) now induces unique factorizations $\lambda(i) : K_i \to B_i$ with



We need to show that the λ_i are components of the desired natural transformation λ. In order to do this we consider the following diagram

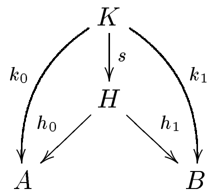


where $n \geq j \geq i$. Note that the diagram is known to be commutative except for the outer and the upper inner square. If we can show that the outer square commutes, then the fact that $\sigma_j : \Delta B_j \to \mathcal{B}_j$ is a monic family shows the naturality of λ. But the commutativity of the outer square can be seen by a quick chase through the following diagram



Notice that the lower two squares are pullbacks by definition (see Diagram (7)), and (II) commutes by definition of ξ_i (see (11)). It is sufficient to show that (I) commutes when extended by the projections of the lower-right pullback. For the projection $\delta_{j,n}(n)$ this is clear since (I) and (II) pasted together commute by definition of ξ_i. For the other projection $\tau_j(n)$ use that $\tau_j(n) \cdot \xi_j(n) = \pi_j(n) \cdot \varrho(j)$, that (III) and (IV) commutes by definition of $\xi_i(n)$ and $\mathcal{A}(i,j)$, and that the outer shape commutes by naturality of ϱ.

We are now ready to construct an arrow $s : K \to H$, which will induce the desired 2-cell $k \to h$. But s can be obtained easily enough. We just have to evoke the universal property of the colimit (K, κ) to obtain a factorization of the cocone $(H, \nu \cdot \lambda)$ through κ; that means we define s by $\nu \cdot \lambda = \Delta s \cdot \kappa$. We must show that we indeed have a 2-cell, i. e., that the following diagram



commutes.

One readily sees that $\pi_i \cdot \varrho = \pi_i \cdot \varepsilon \cdot \lambda$, which implies that $\varrho = \varepsilon \cdot \lambda$. Now

$$\Delta(h_0 s)\kappa = \Delta h_0 \Delta s \cdot \kappa = \Delta h_0 \cdot \nu\lambda = \mu\varepsilon\lambda = \mu\varrho = \Delta k_0 \cdot \kappa,$$

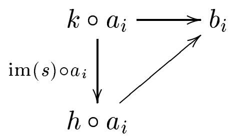
which shows that $h_0 s = k_0$, by uniqueness. Moreover,

$$\Delta(h_1 s)\kappa = \Delta h_1 \Delta s \cdot \kappa = \Delta h_1 \cdot \nu\lambda = \eta\lambda = \Delta k_1 \cdot \kappa,$$

where the last step follows if we unfold the definition of η:

$$(\eta\lambda)(i) = b_i^1 \sigma_i(i)\lambda(i) = b_i^1 \xi_i(i) = b_i^1 s_i = k_1 \kappa(i).$$

Hence, by uniqueness, $h_1 s = k_1$. Finally, note that we need not prove anything towards the uniqueness of the 2-cell $im(s) : k \to h$ and that



commutes for all $i \in \mathbb{N}$ automatically, since $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$. This completes the proof of Theorem 4.2.

4.3 Consequences and Open Problems

As promised in Section 4.2, we shall now give a condition that ensures that any (strict) cocone $(B, b : \mathcal{A} \to \Delta B)$, where $\mathcal{A} : \mathbb{N} \to \mathbf{Rel}(\mathcal{C})$ is the given ω-chain, factors through the canonical cocone (A, a), i. e., $b \simeq \Delta h \circ a$, where $h : A \nrightarrow B$ is the induced relation given by the above construction. Note that this can be expressed more compactly as follows: the counit of the adjunction

$$\mathbf{Rel}(\mathcal{C})(A,B) \underset{G}{\overset{F}{\rightleftarrows}}_{\perp} \mathrm{Cocone}(\mathcal{A}, B)$$

is an isomorphism, where $F = \Delta(-) \circ a$ is composition with the canonical cocone, and G assigns to every cocone (B, b) the induced arrow $h : A \nrightarrow B$.

Unfortunately, even in **Rel**(**Set**) the class of morphisms must be restricted because the condition does not hold true in general. But here it is:

$$\text{For all objects } C \text{ of } \mathcal{C} \text{ the full subcategory } \mathcal{E}/C \text{ of the slice } \mathcal{C}/C \text{ is closed under limits of } \omega\text{-cochains.} \tag{12}$$

More explicitly, if $\mathcal{A} : \mathbb{N} \to \mathcal{C}$ is an ω-cochain with $\mathcal{A}(i, 0) \in \mathcal{E}$ for all $i \in \mathbb{N}$, and if (L, ℓ) is the limit of $\mathcal{A}$, then $\ell(0)$ lies in $\mathcal{E}$, too.

The task is now to check whether Condition (12) implies that $b_i \simeq h \circ a_i$. Recall that if (B, b) is a strict cocone, then the arrows $e_{i,j} : B_{i,j} \to B_{i,i}$ (see page 15) lie in $\mathcal{E}$. Hence, for the cochains $\mathcal{B}_i$, satisfying (12), the projection $\sigma_i(i)$ lies in $\mathcal{E}$. But we have seen that these projections induce the 2-cells $h \circ a_i \to b_i$ (see equation (9) on page 18). Thus $b_i \simeq h \circ a_i$.

Next let us discuss Condition (12) in **Rel**(**Set**). As pointed out earlier, it is not true in general. Here is a counterexample.

Example 4.4. Consider the following cochain in **Set**:

$$\{0\} \xleftarrow{\;!\;} \mathbb{N} \xleftarrow{\text{succ}} \mathbb{N} \xleftarrow{\text{succ}} \mathbb{N} \xleftarrow{\text{succ}} \cdots$$

Its limit must be given by $\emptyset$, for suppose (L, ℓ) is any cone, and there exists an $x \in L$. Let $n := \ell_1(x)$. Then $\ell_{n+1}(x) = 0$, and there cannot be an element in $\mathbb{N}$ with $\ell_{n+2}(x) + 1 = 0 = \ell_{n+1}(x)$. Hence, (L, ℓ) is not a cone, a contradiction.

Clearly the projection $\text{!} : \emptyset \to \{0\}$ is not surjective. However, the map $\mathbb{N} \to \{0\}$ is surjective, whence (12) does not hold in this example.

The situation is different if we only consider those cochains that arise when the relations $a_{i,j} : A_{i,i} \nrightarrow A_{j,j}$ are total. One readily checks that in **Rel**(**Set**),

$$A_i = \{(x_i, x_{i+1}, \ldots, x_j, \ldots) \mid x_j \in A_{j,j}, (x_j, x_{j+1}) \in A_{j,j+1} \forall j \geq i\}. \tag{13}$$

Hence, clearly the projections $\sigma_i(i) : A_i \to A_{i,i}$ are surjective, if all $a_{j,j+1}$ are total. This leads to the following open questions:

- Can the fact that (12) holds for total relations be generalized to a category $\mathcal{C}$ as in Theorem 4.2?
- What additional conditions must be imposed on $\mathcal{C}$ such that (12) holds?

Now let us turn to another open problem. A quick look at diagram (5) on page 13 shows that if the spans $a_{i,i+1}$ are graphs, i.e. given as $\langle 1, f_{i,i+1} \rangle$ for arrows $f_{i,i+1} : A_{i,i} \to A_{i+1,i+1}$ of $\mathcal{C}$, then A is just the colimit of the chain induced by the f's. Hence, the graph functor maps colimits of ω-chains to lax adjoint cooplimits. So the lax adjoint cooplimit of ω-chains becomes a colimit when we restrict the arrows in **Rel**($\mathcal{C}$) to graphs. It seems reasonable to ask:

- Does a lax adjoint cooplimit give a colimit of ω-chains **Map**(**Rel**($\mathcal{C}$))?

There is really not so much missing to answer this affirmatively. Recall that two maps are equal as soon as there exists a 2-cell between them since **Map**(**Rel**($\mathcal{C}$)) is a category (see Section 2.1). Thus, one needs only to check whether the injections $a_i : A_{i,i} \nrightarrow A$ and the canonical factorization $h : A \nrightarrow B$ are maps, if all the $a_{i,j}$ and $b_i : A_{i,i} \nrightarrow B$ are so. Of course, this is obvious if all the maps in $\mathbf{Rel}(\mathcal{C})$ are given by graphs (equivalently $\mathcal{E} \subseteq \mathrm{RegEpi}(\mathcal{C})$, see for example [10], Theorem 4.23 or [11]). Unfortunately, in general this seems not to be obvious at all.

However, the question can be answered positively if $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$. The answer does not come completely for free, though. Before we state the result and give its proof note the following lemma whose proof we omit. It can be found in [10] as Proposition 6.1 and Corollary 4.21, respectively, or in [6] (for items (i) and (ii)).

Lemma 4.5. *Suppose that $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$ and let $\Sigma = \mathcal{E} \cap \mathrm{Mono}(\mathcal{C})$. Then the following hold:*

(i) $gf \in \mathcal{E}$ and g monomorphic $\Rightarrow f \in \mathcal{E}$; in particular $gf \in \Sigma$ and $g \in \Sigma \Rightarrow f \in \Sigma$,

(ii) gf monomorphic and $f \in \mathcal{E} \Rightarrow g$ is monomorphic; in particular $gf \in \Sigma$ and $f \in \Sigma \Rightarrow g \in \Sigma$,

(iii) a relation $r = \langle r_0, r_1 \rangle$ is a map if and only if $r_0 \in \Sigma$.

Theorem 4.6. *If $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$ and if Conditions (ii) and (iii) of Theorem 4.2 hold in $\mathcal{C}$, then the construction of Theorem 4.2 yields colimits of ω-chains in $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ if and only if*

for all objects C of $\mathcal{C}$ the full subcategory Σ/C of the slice $\mathcal{C}/C$ is closed under limits of ω-cochains. (14)

Proof. First note that the condition $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$ is not needed here since maps are already monic in $\mathcal{C}$ (see [10], Theorem 4.19). Furthermore, observe that, by Lemma 4.5, condition (14) can be restated more explicitly as follows. For all ω-cochains $\mathcal{D} : \mathbb{N} \to \mathcal{C}$ with $\mathcal{D}(j, i) \in \Sigma$ for all $j \geq i$, the projections of its limit are in Σ, too.

Suppose condition (14) holds and assume that the relations $a_{i,i+1} : A_{i,i} \nrightarrow A_{i+1,i+1}$ are maps. Equivalently, $\alpha_{i,i+1}(i) : A_{i,i+1} \to A_{i,i}$ lies in Σ. But Σ is a pullback stable class. Hence, all $\alpha_{i,j}(k) : A_{k,j} \to A_{k,i}$, $j \geq i \geq k$ are in Σ, too. Applying condition (14), we see that $\pi_i(i) : A_i \to A_{i,i}$ lies in Σ for all i, whence $a_i : A_{i,i} \nrightarrow A$ is a map (recall that if $\langle s_0, s_1 \rangle$ is a span with $s_0 \in \Sigma$, then $r_0 \in \Sigma$ for $\langle r_0, r_1 \rangle = \mathrm{im}\langle s_0, s_1 \rangle$, by 4.5).

Now suppose that a second cocone (B, b) on the ω-chain in $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ is given. Clearly the natural transformations $\tau_i : \mathcal{B}_i \to \mathcal{A}_i$ are in Σ componentwise. By 4.5, all $\gamma_{i,j}(k) : B_{k,j} \to B_{k,i}$, $j \geq i \geq k$, lie in Σ. Hence, σ_i must lie in Σ componentwise as before π_i. This implies that $\varepsilon : \mathcal{B} \to \mathcal{A}$ lies in Σ componentwise, by 4.5 again, since $\pi_i \cdot \Delta\varepsilon(i) = \tau_i \cdot \sigma_i$. Since $\mathcal{E}$ commutes with all colimits, $h_0 : H \to A$ must therefore lie in $\mathcal{E}$, too. In order to see that h_0 is monic, we use condition (iii) of Theorem 4.2, recalling that an arrow is monic in $\mathcal{C}$ if and only if its kernel pair, which is given by the pullback of that arrow along itself, is a diagonal in $\mathcal{C}$.

Finally, to see that condition (14) is necessary, suppose that $\mathcal{D} : \mathbb{N} \to \mathcal{C}$ is an ω-cochain with $\mathcal{D}(j, i) \in \Sigma$ for all $j \geq i \in \mathbb{N}$. This gives an ω-chain $\langle \mathcal{D}(i+1, i), 1 \rangle$ in $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$. We know that its colimit injections are formed by taking the limit (A, π) of $\mathcal{D}$ (and its tails). Note that these injections obtained in $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ are of the form $\langle \pi(i), 1\rangle$; taking images is not necessary since $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$. Therefore, all $\pi(i)$ must lie in Σ. □

Note that if Σ (as a full subcategory of $\mathcal{C}^2$) is closed under limits of ω-cochains, then condition (14) in the previous result holds true.

Moreover, $\mathrm{Mono}(\mathcal{C})$ (as full subcategory of $\mathcal{C}^2$) is always closed under all limits. Hence, if $\mathcal{E}$ is closed under limits of ω-cochains, then condition (14) holds true. Also note that the earlier condition (12) implies condition (14).

Observe that condition (14) is much weaker than condition (12). For example, (14) is true in every regular category, since the class Σ consists of isomorphisms there, which implies that it is closed under all limits and colimits. Other (non-regular) examples are:

(i) the category **Top** of topological spaces and continuous maps with the usual $(\mathrm{Epi}, \mathrm{RegMono})$-factorization,

(ii) the category $\mathbf{Top}_1$ of T1 spaces equipped with the following factorization system: $\mathcal{E}$ consists of *monotone quotient* mappings (a continuous mapping $f : X \to Y$ is monotone if all fibres $f^{-}(y)$ are connected) and $\mathcal{M}$ consists of so-called *light* mappings (a continuous mapping is light if all its fibres are totally disconnected) (for details see [10], Example 4.24).

(iii) the category **CAT** of (small) categories with the factorization system given by taking as $\mathcal{E}$ the class of functors that are bijective on objects and as $\mathcal{M}$ fully faithful functors.

Note that the last example does not satisfy $\mathcal{E} \subseteq \mathrm{Epi}(\mathbf{CAT})$, though.

This raises the question whether a result like Theorem 4.6 can be obtained for a category with factorization system $(\mathcal{E}, \mathcal{M})$, where $\mathcal{E}$ may contain non-epimorphic arrows.

Finally, let us define $\mathcal{K}$ to be the 2-category whose objects are finitely complete categories $\mathcal{C}$ equipped with a factorization system where $\mathcal{E}$ is stable under pullback. An arrow $F : \mathcal{C} \to \mathcal{C}'$ is a left-exact functor (i. e., a functor preserving finite limits) with $F\mathcal{E} \subseteq \mathcal{E}'$ and $F\mathcal{M} \subseteq \mathcal{M}'$. The 2-cells are given by natural transformations between such functors. Clearly any $F : \mathcal{C} \to \mathcal{D}$ in $\mathcal{K}$ induces a 2-functor $\mathbf{Rel}(F) : \mathbf{Rel}(\mathcal{C}) \to \mathbf{Rel}(\mathcal{D})$ sending objects A to FA and relations $r = \langle r_0, r_1\rangle : A \nrightarrow B$ to $\langle Fr_0, Fr_1\rangle : FA \nrightarrow FB$. Note that this lies in $\mathcal{M}$, since $F\langle r_0, r_1\rangle : FR \to F(A \times B)$ lies in $\mathcal{M}'$ and F preserves binary products. A 2-cell $\alpha : r \to s$ is sent to $F\alpha : \mathbf{Rel}(F)(r) \to \mathbf{Rel}(F)(s)$. It is easy to check that $\mathbf{Rel}(F)$ is indeed a 2-functor.

If $\mathcal{C}$ and $\mathcal{C}'$ both satisfy the conditions of Theorem 4.2, and therefore admit lax adjoint cooplimits of ω-chains, then clearly $\mathbf{Rel}(F)$ preserves these if F is ω-cocontinuous and preserves limits of ω-cochains in $\mathcal{C}$. Moreover, Theorem 4.6 shows that the restriction of $\mathbf{Rel}(F)$ to maps is ω-cocontinuous in case $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$ and $\mathcal{E}' \subseteq \mathrm{Epi}(\mathcal{C}')$. Hence, looking back to the motivation at the beginning of the paper one may now proceed and at least construct initial $\mathbf{Rel}(F)$-algebras in the category of maps of $\mathbf{Rel}(\mathcal{C})$ iteratively.

As a last open problem the following question remains: Is it possible to construct some kind of (perhaps lax) initial algebras in $\mathbf{Rel}(\mathcal{C})$ with the help of the lax adjoint colimits of ω-chains described in Section 4.2? We leave this problem for future work.

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