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A general final coalgebra theorem

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By the Final Coalgebra Theorem of Aczel and Mendler, every endofunctor of the category of sets has a final coalgebra, which, however, may be a proper class. We generalise this to all 'well-behaved' categories $\mathscr{K}$. The role of the category of classes is played by a free cocompletion $\mathscr{K}^\infty$ of $\mathscr{K}$ under transfinite colimits, that is, colimits of ordinal-indexed chains. Every endofunctor F of $\mathscr{K}$ has a canonical extension to an endofunctor F^∞ of $\mathscr{K}^\infty$ which is proved to have a final coalgebra (and an initial algebra). Based on this, we prove a general solution theorem: for every endofunctor of a locally presentable category $\mathscr{K}$ all guarded equation-morphisms have unique solutions. The last result does not need the extension $\mathscr{K}^\infty$: the solutions are always found within the category $\mathscr{K}$.

1. Introduction

Labelled transition systems, with a set Σ of actions, as used, for example, in the CCS calculus of Robin Milner, require solutions of systems of recursive equations of the form

$$x_i \approx a_i \qquad (i \in I)$$

where $X = \{x_i \mid i \in I\}$ is a set of variables and each a_i is either a subset of $X \times \Sigma$, or a parameter (from a set Y). This led Peter Aczel to introduce the non-well-founded set theory, in which the unique solvability of such equations is an explicit axiom (Aczel 1988). Later he and Nax Mendler proved their Final Coalgebra Theorem (Aczel and Mendler 1989), which shows that one can stay within well-founded set theory, provided that solutions of equations as above are taken, instead of the category **Set** of small sets, in the category **Class** of classes. Their result states that every set-based endofunctor of **Class** has a final coalgebra. In fact, 'set-based' is superfluous: we proved in Adámek *et al.* (2002b) that all endofunctors of **Class** are set-based, thus, every endofunctor F of **Set** has an essentially unique extension to an endofunctor F^∞ of **Class**. Moreover, the functor F^∞ is iteratable, that is, for every object Y, a final coalgebra TY of the endofunctor $F^\infty(-) + Y$ exists. Recursive equations as above have the form of a morphism

$$e : X \to F^\infty X + Y.$$

(In the above case $F = \mathscr{P}(- \times \Sigma)$ and $e : x_i \mapsto a_i$.) It follows from results of Aczel *et al.* (2002) that every such equation morphism has a unique solution $e^\dagger : X \to TY$.

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In the current paper we will prove that all this generalises from endofunctors of **Set** to endofunctors of all 'reasonable' categories $\mathscr{K}$. We use $\mathscr{K}^{\infty}$ to denote the free cocompletion of $\mathscr{K}$ under transfinite colimits, that is, colimits of ordinal-indexed chains. For example, $\mathbf{Set}^{\infty} = \mathbf{Class}$. Every endofunctor F of $\mathscr{K}$ has an essentially unique extension to an endofunctor F^{∞} of $\mathscr{K}^{\infty}$ preserving transfinite colimits. We then prove the following.

General Final Coalgebra Theorem. For every endofunctor F of a cocomplete, cowell-powered, locally small category the extension F^{∞} has a final coalgebra (as well as an initial algebra).

For locally presentable categories, more can be proved: every endofunctor F of $\mathscr{K}$ generates a free completely iterative monad (in the sense of Elgot *et al.* (1978), see also Aczel *et al.* (2002)) on the category $\mathscr{K}^{\infty}$. From this, we show, without reference to the extension $\mathscr{K}^{\infty}$, that every guarded iterative equation morphism in $\mathscr{K}$ has a unique solution. Thus, we obtain a general solution theorem (with no assumption about the endofunctor F). This extends the General Solution Theorem proved in Adámek *et al.* (2002b) for endofunctors of **Set**.

The current paper presents an improvement of results announced in our preprint Adámek *et al.* (2002a).

Set-theoretical assumptions

We assume that a universe of 'small' sets has been chosen so that we can form the category of all small sets. We further assume the Axiom of Choice in a higher universe. Then the universe of smallsets is a (non-small) set in that higher universe, so we can use

$$\aleph_{\infty}$$

to denote the cardinality of that set. This enables us to identify:

— *small sets* with sets of cardinality less than $\aleph_{\infty}$; and
— *classes* with sets of cardinality at most $\aleph_{\infty}$.

More precisely, for a set theorist, the universe of small sets can be $V(\aleph_{\infty})$. However, we will take as

Set

the category of all sets of cardinality less than $\aleph_{\infty}$ (equivalent to $V(\aleph_{\infty})$). And we take as

Class

the category of all sets of cardinality less than or equal to $\aleph_{\infty}$. This is namely equivalent to the category of classes as used in set theory because every class that is not small has cardinality $\aleph_{\infty}$.

2. Initial algebras and final coalgebras

Assumption 2.1. Throughout this section $\mathscr{K}$ denotes a category that is:

(1) Cocomplete – that is, has small colimits;

(2) Locally small – that is, has only a class of morphisms, and all hom-sets are small; and
(3) Cowellpowered – that is, every object has only a small set of quotient objects.

Notation 2.2.

(1) Recall that the well-ordered category of all small ordinals is denoted by **Ord**. Colimits indexed by **Ord** are called *transfinite colimits*, and functors preserving transfinite colimits are called *small-accessible*.
(2) Let $\mathscr{K}$ be any category. We use

$$E : \mathscr{K} \to \mathscr{K}^{\infty}$$

to denote a free cocompletion of $\mathscr{K}$ under transfinite colimits.
Explicitly, $\mathscr{K}^{\infty}$ is a category having transfinite colimits and E is a full embedding with the following universal property:

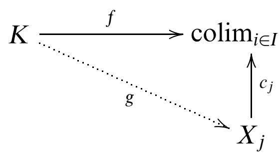
for every functor $F : \mathscr{K} \to \mathscr{L}$ where $\mathscr{L}$ has transfinite colimits there exists a small-accessible extension $F' : \mathscr{K}^{\infty} \to \mathscr{L}$ of F, unique up to a natural isomorphism.

Remark 2.3. Every object K of $\mathscr{K}$ is *small-presentable* in $\mathscr{K}^{\infty}$. This means that the hom-functor of K is small-accessible. Explicitly, for every morphism

$$f : K \to \operatorname*{colim}_{i \in \mathbf{Ord}} X_i$$

from K into a transfinite colimit in $\mathscr{K}^{\infty}$ (with a colimit cocone $c_j : X_j \to \operatorname{colim}_{i \in \mathbf{Ord}} X_i$) we have:

(i) f factors through some c_j:



(ii) The factorisation is essentially unique, that is, given $g' : K \to X_j$ with $f = c_j \cdot g'$, there exists an ordinal $k \geqslant j$ such that the connecting morphism $x_{jk} : X_j \to X_k$ of the given chain fulfils

$$x_{jk} \cdot g = x_{jk} \cdot g'.$$

Conversely, every small-presentable object K of $\mathscr{K}^{\infty}$ is isomorphic to an object of $\mathscr{K}$. (In fact, K is obviously a retract of an object of $\mathscr{K}$. Since $\mathscr{K}$ is cocomplete, idempotents split in $\mathscr{K}$, so all retracts of objects from $\mathscr{K}$ in $\mathscr{K}^{\infty}$ are isomorphic to objects of $\mathscr{K}$.)

Remark 2.4. $\mathscr{K}^{\infty}$ is unique up to an equivalence of categories, and can be characterised (up to equivalence) by the following properties:

(1) $\mathscr{K}^{\infty}$ has transfinite colimits, and every object is a transfinite colimit of objects from $\mathscr{K}$.
(2) Every object of $\mathscr{K}$ is small-presentable in $\mathscr{K}^{\infty}$.

The universal property of $\mathscr{K}^\infty$ mentioned above can be restated as follows: the functor category $[\mathscr{K}, \mathscr{L}]$ is equivalent to the full subcategory $[\mathscr{K}^\infty, \mathscr{L}]_{\mathrm{sa}}$ of $[\mathscr{K}^\infty, \mathscr{L}]$ formed by all small-accessible functors under the equivalence functor

$$(-) \cdot E : [\mathscr{K}^\infty, \mathscr{L}]_{\mathrm{sa}} \to [\mathscr{K}, \mathscr{L}].$$

The pseudoinverse of this equivalence is given by taking a left Kan extension along E.

Examples 2.5.

1. $\mathbf{Set}^\infty = \mathbf{Class}$. This follows from Remark 2.4. First, **Class** has transfinite colimits, in fact, all class-indexed colimits. (This is obvious: a coproduct of a class of classes is a class, since $(\aleph_\infty)^2 = \aleph_\infty$, and coequalisers clearly exist.) Also, every class is a union of a transfinite chain of small subsets. Second, verifying that every small set K is small-presentable is as easy as verifying the well-known fact that every finite set is finitely presentable.
2. An analogous description can be provided for the cocompletions $\mathscr{K}^\infty$ of other 'everyday' categories. For example, if $\mathscr{K} = \mathbf{Pos}$ is the category of all small posets and order-preserving maps, then

   $$\mathbf{Pos}^\infty$$

   is the category of all partially ordered classes and order-preserving maps. The argument is analogous to that given above for **Class**.
   Or for $\mathscr{K} = \mathbf{Cpo}$, the category of all small posets with directed joins and continuous (= directed-joins preserving) maps, we have

   $$\mathbf{Cpo}^\infty$$

   the category of all partially ordered classes having joins of small directed subsets, and functions preserving such joins.
3. $\mathbf{Ord}^\infty$ is the extension of **Ord** by a largest element.

Remark 2.6. Our 'small-accessible' and 'small-presentable' terminology stems from the theory of accessible categories (Makkai and Paré 1989; Adámek and Rosický 1994). Recall the concept of a λ-filtered category for a given infinite cardinal λ: it is a category $\mathscr{D}$ such that every (non-full) subcategory on less than λ morphisms has a cocone in $\mathscr{D}$. Colimits of diagrams with λ-filtered domains are called *λ-filtered colimits.*

Functors preserving λ-filtered colimits are called *λ-accessible* and objects K such that the hom-functor $\mathscr{K}(K, -)$ is λ-accessible are called *λ-presentable.*

We will call a category $\mathscr{D}$ *small-filtered* if it has a class of morphisms and every small subcategory of $\mathscr{D}$ has a cocone in $\mathscr{D}$; that is, $\mathscr{D}$ is λ-filtered for all small cardinals λ.

Colimits of diagrams with small-filtered domains are called *small-filtered colimits.*

Lemma 2.7. Every small-filtered category $\mathscr{D}$ has a cofinal functor $F : \mathbf{Ord} \to \mathscr{D}$.

Proof. There exists a small-filtered partially ordered class $\mathscr{D}_0$ and a cofinal functor $F_0 : \mathscr{D}_0 \to \mathscr{D}$ – this is proved precisely as Theorem 1.5 in Adámek and Rosický (1994) (where 'finite' is substituted by 'small', and ω by **Ord**). Thus, it is sufficient to find a cofinal transfinite chain in $\mathscr{D}_0$. If the objects of $\mathscr{D}_0$ are indexed as $\{x_i \mid i \in \mathbf{Ord}\}$, a cofinal chain y_i

($i \in \mathbf{Ord}$) can be obtained by transfinite induction, choosing, for every ordinal i, an upper bound y_i of the small set $\{x_i\} \cup \{y_j \mid j < i\}$. □

Corollary 2.8. A category has small-filtered colimits iff it has transfinite colimits. A functor is $\aleph_\infty$-accessible (that is, preserves small-filtered colimits) iff it is small-accessible.

Notation 2.9.

(1) For notational convenience we assume that $\mathscr{K}$ is a full subcategory of $\mathscr{K}^\infty$ and $E : \mathscr{K} \to \mathscr{K}^\infty$ is the inclusion functor.

(2) For every functor $F : \mathscr{K} \to \mathscr{K}$ we use

$$F^\infty : \mathscr{K}^\infty \to \mathscr{K}^\infty$$

to denote the (essentially unique) left Kan extension of EF along E, which is a small-accessible endofunctor. For every natural transformation

$$f : F \to G \qquad \text{in } [\mathscr{K}, \mathscr{K}]$$

we use

$$f^\infty : F^\infty \to G^\infty \qquad \text{in } [\mathscr{K}^\infty, \mathscr{K}^\infty]$$

to denote the unique natural transformation extending f.

Example 2.10. Consider the usual extension of the power-set functor $\mathscr{P} : \mathbf{Set} \to \mathbf{Set}$ to the power-set functor

$$\mathscr{P}^\infty : \mathbf{Class} \to \mathbf{Class}$$

assigning to every class X the class $\mathscr{P}^\infty X$ of all subsets of X. The set-functor $\mathscr{P}(- \times \Sigma)$, whose coalgebras are labelled transition systems, has the extension F^∞ assigning to every class X all subsets of $X \times \Sigma$; coalgebras of F^∞ are labelled transition systems on classes whose fan-outs are sets.

Remark 2.11. Peter Aczel and Nax Mendler (Aczel and Mendler 1989) call an endofunctor F of **Class** *set-based* provided that for every element of F, $x \in FX$, there exists a small subset $m : Y \to X$ of the class X such that x lies in the image of $Fm : FY \to FX$. This is equivalent to F being small-accessible, see the argument in Adámek and Porst (2001) for 'bounded = accessible'. We proved in Adámek *et al.* (2002b) that *every* endofunctor of **Class** is indeed set-based.

Theorem 2.12. Let $\mathscr{K}$ be a locally small, cocomplete, cowellpowered category. Then $\mathscr{K}^\infty$ has:

(a) class-indexed colimits; and

(b) multiple pushouts of epimorphisms (arbitrarily indexed).

Moreover, $\mathscr{K}$ is closed under small colimits in $\mathscr{K}^\infty$.

Remark. In (a), we mean colimits of diagrams with at most a class of morphisms in the domain.

Proof. The closure of $\mathscr{K}$ under small colimits is trivial, since objects of $\mathscr{K}$ are small-presentable in $\mathscr{K}^\infty$ (see Remark 2.3 and recall that in cocomplete categories idempotents split). Statement (a) only requires us to show that $\mathscr{K}^\infty$ has small colimits: since it has small-filtered colimits, it has class-indexed colimits (given a class-indexed diagram D, consider the small-filtered colimit of the diagram of colimits of all small subdiagrams of D; this is a colimit of D).

The existence of small coproducts in $\mathscr{K}^\infty$ is evident since objects of $\mathscr{K}^\infty$ are transfinite colimits of objects of $\mathscr{K}$: given a small collection of chains $D_i\colon \mathbf{Ord} \to \mathscr{K}^\infty$ ($i \in I$), form the small-filtered diagram

$$\mathbf{Ord} \xrightarrow{\langle D_i\rangle_{i\in I}} \mathscr{K}^I \longrightarrow \mathscr{K},$$

where the second part is taking coproducts in $\mathscr{K}$. Its colimit is the coproduct of $\operatorname{colim} D_i$ in $\mathscr{K}^\infty$. Working analogously with coequalisers, given a parallel pair $f, g\colon \operatorname{colim} D \to \operatorname{colim} D'$ in $\mathscr{K}^\infty$, where D, D' are transfinite chains in $\mathscr{K}$, we can find natural transformations $f_i, g_i\colon D_i \to D'_{j_i}$ in $\mathscr{K}$ with $f = \operatorname{colim} f_i$ and $g = \operatorname{colim} g_i$. Then componentwise coequalisers of f_i, g_i in $\mathscr{K}$ form a chain that is a coequaliser of f and g in $\mathscr{K}^\infty$.

To prove statement (b), we first show that every object A of $\mathscr{K}^\infty$ expressed as a colimit $(X_i \xrightarrow{x_i} A)_{i\in\mathbf{Ord}}$ of a transfinite chain (X_i) in $\mathscr{K}$ has the following property: for every epimorphism $e\colon A \to B$ there exists a chain $e_i\colon X_i \to Y_i$ of epimorphisms in $\mathscr{K}$ and morphisms $y_i\colon Y_i \to B$ ($i \in \mathbf{Ord}$) such that $e = \operatorname{colim}_{i\in\mathbf{Ord}} e_i$ in the morphism-category $(\mathscr{K}^\infty)^\to$ with the following colimit cocone

$$\begin{array}{ccc} X_i & \xrightarrow{e_i} & Y_i \\ {\scriptstyle x_i}\big\downarrow & & \big\downarrow{\scriptstyle y_i} \\ A & \xrightarrow[e]{} & B \end{array}$$

In fact, since $\mathscr{K}^\infty$ is a free cocompletion of $\mathscr{K}$ under transfinite colimits, it is easy to deduce that $(\mathscr{K}^\infty)^\to$ is a free cocompletion of $\mathscr{K}^\to$ under transfinite colimits, and thus that for every morphism e of $\mathscr{K}^\infty$ we have a chain of morphisms e_i with $e = \operatorname{colim} e_i$ having the colimit cocone as above. We can factor each e_i as an epimorphism $e'_i\colon X_i \to Y'_i$ followed by an extremal monomorphism $m_i\colon Y'_i \to Y_i$ (for all $i \in \mathbf{Ord}$) – in fact, every cocomplete and cowellpowered category has such factorisations, see Adámek *et al.* (1990, 14.21). Using the diagonal fill-in between epimorphisms and extremal monomorphisms for all ordinals $i \leqslant j$,

$$\begin{array}{ccccc} X_i & \xrightarrow{e'_i} & Y'_i & \xrightarrow{m_i} & Y_i \\ {\scriptstyle x_{ij}}\big\downarrow & & \vdots\,{\scriptstyle y'_{ij}} & & \big\downarrow{\scriptstyle y_{ij}} \\ X_j & \xrightarrow{e'_j} & Y'_j & \xrightarrow{m_j} & Y_j \\ {\scriptstyle x_j}\big\downarrow & & \big\downarrow{\scriptstyle y'_j} & \swarrow{\scriptstyle y_j} & \\ A & \xrightarrow[e]{} & B & & \end{array}$$

we obtain a new chain, e'_i, in $\mathscr{K}^\to$ and a cocone of that chain in $(\mathscr{K}^\to)^\infty$ with the vertex e.

Recall that for every factorisation system $(\mathscr{E}, \mathscr{M})$ on $\mathscr{K}$, the class $\mathscr{M}$ considered as a full subcategory of $\mathscr{K}^{\to}$ is reflective. In other words, the functor

$$\mathscr{K}^{\to} \to \mathscr{M}$$

assigning to every f the $\mathscr{M}$-part of its factorisation is a left adjoint. Thus, it preserves all colimits. From this we can conclude that $\operatorname{colim}_{i\in\mathbf{Ord}} m_i = \mathrm{id}_B$, and, therefore, $B = \operatorname{colim}_{i\in\mathbf{Ord}} Y_i'$. It follows easily that $e = \operatorname{colim}_{i\in\mathbf{Ord}} e_i'$.

We are ready to prove that for every collection

$$f^t : A \to B^t \quad (t \in T)$$

of epimorphisms (with the given domain $A = \operatorname{colim} X_i$), there exists a multiple pushout. For each t we have a chain of epimorphisms $e_i^t : X_i \to E_i^t$ ($i \in \mathbf{Ord}$) with a colimit

$$\begin{array}{ccc} X_i & \xrightarrow{e_i^t} & E_i^t \\ {\scriptstyle x_i}\downarrow & & \downarrow{\scriptstyle y_i^t} \\ A & \xrightarrow[f^t]{} & B^t \end{array}$$

For each i, we use

$$e_i^* = h_i^t e_i^t : X_i \to E_i^*$$

to denote a multiple pushout of all e_i^t $(t \in T)$ (recalling again that $\mathscr{K}$ is cocomplete and cowellpowered). Note that each h_i^t is an epimorphism then every connecting morphism $x_{ij} : X_i \to X_j$ of the original chain induces a unique $x_{ij}^* : E_i^* \to E_j^*$ with $x_{ij}^* e_i^* = e_j^* x_{ij}$, that is, such that $(x_{ij}, x_{ij}^*) : e_i^* \to e_j^*$ is a morphism of $\mathscr{K}^{\to}$. We obtain a chain in $\mathscr{K}^{\to}$ that has a colimit in $(\mathscr{K}^{\infty})^{\to}$, say, with the following cocone

$$\begin{array}{ccc} X_i & \xrightarrow{e_i^*} & E_i^* \\ {\scriptstyle x_i}\downarrow & & \downarrow{\scriptstyle y_i^*} \\ A & \xrightarrow[e^*]{} & B^* \end{array}$$

We claim that e^* is a multiple pushout of f^t, $t \in T$. In fact, for each t the morphisms $h_i^t : E_i^t \to E_i^*$ form a chain in $\mathscr{K}^{\to}$, and since $\operatorname{colim}_{i\in\mathbf{Ord}} E_i^t = B^t$ (recall that $\operatorname{colim}_{i\in\mathbf{Ord}} e_i^t = f^t$) and $\operatorname{colim}_{i\in\mathbf{Ord}} E_i^* = B^*$, we have a colimit

$$h^t = \operatorname*{colim}_{i\in\mathbf{Ord}} h_i^t$$

of that chain in $(\mathscr{K}^{\infty})^{\to}$ with the following cocone

$$\begin{array}{ccc} E_i^t & \xrightarrow{h_i^t} & E_i^* \\ {\scriptstyle y_i^t}\downarrow & & \downarrow{\scriptstyle y_i^*} \\ B^t & \xrightarrow[h^t]{} & B \end{array}$$

The equalities $e_i^* = h_i^t e_i^t$ yield, by taking a colimit over $i \in \mathbf{Ord}$, the equality

$$e^* = h^t f^t.$$

Suppose that another cocone

$$k = k^t f^t \qquad \text{for } k^t : B^t \to C$$

is given. For each $i \in \mathbf{Ord}$ we obtain a cocone $k^t y_i^t : E_i^t \to C$ $(t \in T)$ that yields a unique

$$z_i : E_i^* \to C \qquad \text{with } z_i h_i^t = k^t y_i^t.$$

It follows that

$$(x_i, z_i) : e_i^* \to k \qquad (i \in I)$$

is a cocone of the chain of all e_i^*'s: for every connecting morphism $x_{ij} : X_i \to X_j$ we have $z_j \cdot x_{ij}^* = z_i$ since a chase through an obvious diagram yields $z_j x_{ij}^* e_i^* = z_j h_j^t e_j^t x_{ij} = z_i e_i^*$. Let $(r, s) : e^* \to k$ be the unique factorisation of the cocone through $e^* = \operatorname{colim}_{i \in \mathbf{Ord}} e_i^*$. Then it is easy to see that $r = \mathrm{id}_A$ and $s : B^* \to C$ form the desired morphism with $sh^t = k^t$ for all $t \in T$. The uniqueness of s follows from the fact that each h^t, being a colimit of epimorphisms h_i^t, is an epimorphism. This concludes the proof that e^* is a multiple pushout of f^t $(t \in T)$. □

Remark 2.13. For every endofunctor F of $\mathscr{K}$ we have $\mathrm{Coalg}\, F$ as a full subcategory of $\mathrm{Coalg}\, F^\infty$ (every coalgebra $A \to FA = F^\infty A$, $A \in \mathscr{K}$, for F is also a coalgebra for F^∞). This subcategory generates $\mathrm{Coalg}\, F^\infty$. In fact, every F^∞-coalgebra is a small-filtered colimit of F-coalgebras. See Adámek and Porst (2002, Theorem IV.2) (just part II of the proof, applied to $\lambda = \aleph_\infty$). The statement proved in Adámek and Porst (2002) is a bit more general: whenever an endofunctor G of $\mathscr{K}^\infty$ preserves small-filtered colimits, all G-coalgebras are small-filtered colimits of G-coalgebras $A \to GA$ with $A \in \mathscr{K}$.

Notation 2.14. For every F^∞-coalgebra $A \xrightarrow{\alpha} F^\infty A$ we use

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & F^\infty A \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle F^\infty e} \\ A^* & \xrightarrow[\alpha^*]{} & F^\infty A^* \end{array}$$

to denote the *greatest congruence* on A.

Recall that a *congruence* is a homomorphism of coalgebras carried by an epimorphism of the underlying category, and that the greatest congruence is one through which every congruence with the given domain factors. The existence of a greatest congruence follows from Theorem 2.12 and the fact that colimits of F^∞-coalgebras are reflected (in fact, created) by the forgetful functor into $\mathscr{K}^\infty$. Thus, $e : A \to A^*$ is obtained as a pushout of all congruences on A.

Example 2.15. Let $F = \mathscr{P}(- \times \Sigma)$ and let A be a labelled transition system as a coalgebra of F^∞. A congruence on A is an equivalence relation $\sim$ on the class A such that given states $x \sim y$ in A and an action $s \in \Sigma$, every state x' reachable from x via s is equivalent to a state y' reachable from y via s, and *vice versa*. This is the usual bisimulation condition for equivalence relations.

Theorem 2.16 (The General Final Coalgebra Theorem). Let $\mathscr{K}$ be a cocomplete, cowell-powered, locally small category. For every endofunctor F of $\mathscr{K}$, an initial F^∞-algebra, μF, exists, in fact,

$$\mu F = \operatorname*{colim}_{i \in \mathbf{Ord}} F^{(i)}0, \qquad (0 \text{ an initial object of } \mathscr{K}).$$

And a final F^∞-coalgebra, νF, exists, in fact,

$$\nu F = \left(\coprod_{A \in \text{Coalg}\, F} A \right)^*$$

is a quotient of the coproduct of all F-coalgebras modulo the greatest congruence.

Proof.

(1) Following Adámek (1974), define a chain $F^{(i)}0$ ($i \in \mathbf{Ord}$) with connecting morphisms $w_{ij} : F^{(i)}0 \to F^{(j)}0$ ($i, j \in \mathbf{Ord}$, $i \leqslant j$) in $\mathscr{K}$ by the following transfinite induction over $\mathbf{Ord}$:

$$F^{(0)}0 = 0, \qquad F^{(1)}0 = F0, \qquad \text{and } w_{01} : 0 \to F0 \text{ is uniquely determined.}$$

For the isolated step, given $F^{(i)}0$ and w_{ij}, put

$$F^{(i+1)}0 = F(F^{(i)}0) \qquad \text{and} \qquad w_{i+1,j+1} = Fw_{ij}.$$

For the limit step, assume that j is a small limit ordinal such that the chain $(F^{(i)}0)_{i<j}$ has already been defined. Put

$$F^{(j)}0 = \operatorname*{colim}_{i<j} F^{(i)}0$$

with a colimit cocone

$$w_{ij} : F^{(i)}0 \to F^{(j)}0 \qquad (i < j).$$

The requirement that we define a chain makes $w_{j,j+1} : F^{(j)}0 \to F(F^{(j)}0)$ uniquely determined:

$$w_{j,j+1} \cdot w_{i+1,j} = w_{i+1,j+1} = Fw_{ij} \qquad (\text{for all } i < j).$$

We use I to denote a colimit of this (small-filtered) chain in $\mathscr{K}^\infty$. Then F^∞ preserves that colimit, yielding a canonical isomorphism

$$F^\infty I \cong \operatorname*{colim}_{i \in \mathbf{Ord}} F^{(i+1)}0 \cong \operatorname*{colim}_{i \in \mathbf{Ord}} F^{(i)}0 = I.$$

This is an initial F^∞-algebra, as proved in Adámek (1974).

(2) The collection of all F-coalgebras $A = (X_A, \xi_A : X_A \to FX_A)$ is a class because it is a class-indexed union of the small sets $\mathscr{K}(X, FX)$. The category $\mathscr{K}^\infty$ has class-indexed coproducts, by Theorem 2.12, and thus a coproduct

$$B = \coprod_{A \in \text{Coalg}\, F} A$$

exists in $\mathscr{K}^\infty$ and carries the canonical structure of an F^∞-coalgebra. In fact, the forgetful functor $U\colon \operatorname{Coalg} F^\infty \to \mathscr{K}^\infty$ creates all existing colimits, and thus B is the unique F^∞-coalgebra on the coproduct $\coprod_{A \in \operatorname{Coalg} F} A$ in $\mathscr{K}^\infty$ forming a coproduct in $\operatorname{Coalg} F^\infty$. The proof that B^* is a final F^∞-coalgebra is based on Remark 2.13, from which it is sufficient to prove that for every F-coalgebra $A \xrightarrow{\alpha} FA$ there exists a unique homomorphism into B^*. The existence is obvious: compose the coproduct injection $\mathrm{in}_A : A \to B$ with the quotient homomorphism $e\colon B \to B^*$:

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & FA \\ {\scriptstyle \mathrm{in}_A}\downarrow & & \downarrow{\scriptstyle F\,\mathrm{in}_A} \\ B & \xrightarrow{\beta} & FB \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle Fe} \\ B^* & \xrightarrow[\beta^*]{} & FB^* \end{array}$$

To prove the uniqueness, let $u, v : A \to B^*$ be homomorphisms of F^∞-coalgebras and let $c\colon B^* \to C$ be a coequaliser of u and v in $\mathscr{K}$. Since U creates colimits, c carries the structure of a homomorphism of F^∞-coalgebras, so $ce\colon B \to C$ is a congruence on B. By the definition of $e\colon B \to B^*$, it follows that e factors through ce – in other words, c is an isomorphism. This proves $u = v$. □

Remark 2.17.
The dual of the formula $\mu F = \operatorname{colim}_{i \in \mathbf{Ord}} F^i 0$ (*viz*., the equality $\nu F = \lim_{i \in \mathbf{Ord}} F^i 1$) does not hold in general: $\mathscr{K}^\infty$ need not have the limit. For example, if $F = \mathscr{P}$ is the power-set functor, $\lim_{i \in \mathbf{Ord}} \mathscr{P}^i 1$ is *not* a class, see Adámek *et al.* (2002a).

3. A general solution theorem

Assumptions 3.1. Throughout the present section $\mathscr{K}$ denotes a *locally presentable* category in the sense of Gabriel and Ulmer (see Gabriel and Ulmer (1971) or Adámek and Rosický (1994)). That is, an infinite cardinal λ is given such that $\mathscr{K}$ is cocomplete and has a small set $\mathscr{K}_\lambda$ of λ-presentable objects such that every object of $\mathscr{K}$ is a λ-filtered colimit of objects in $\mathscr{K}_\lambda$. Every locally presentable category fulfils the assumptions of Section 2: it is cocomplete, cowellpowered, and locally small (Adámek and Rosický 1994).

Examples 3.2. Set, **Pos** and ω-**Cpo** (the category of posets with joins of ω-chains) are locally presentable, **Ord**, **Cpo** and **Class** are not.

Remark 3.3. We briefly recall the concepts of the solution of an iterative equation morphism and completely iterative monads. For motivation and examples, the reader may consult Aczel *et al.* (2002) or Moss (2002).

A functor $F : \mathscr{K} \to \mathscr{K}$ is called *iteratable* if for every object X in $\mathscr{K}$ a final coalgebra $T_F X$ of the functor $F(-) + X$ exists. We just write TX if the functor F is obvious.

Example 3.4. Every accessible endofunctor of a locally presentable category is iteratable. In contrast, the power-set functor on **Set** is not iteratable.

Notation 3.5. By Lambek's Lemma (Lambek 1968), the structure morphism $TX \to FTX + X$ of the final coalgebra TX is an isomorphism. That is, TX is a coproduct of FTX and X. We use

$$\tau_X : FTX \to TX \qquad \text{('}TX \text{ is an } F\text{-algebra')}$$

and

$$\eta_X : X \to TX \qquad \text{('}TX \text{ contains } X\text{')}$$

to denote the coproduct injections.

Theorem 3.6 (Substitution Theorem). If F is iteratable, then for every morphism $s\colon X \to TX$ in $\mathscr{K}$ there exists a unique extension to a homomorphism $\hat{s}\colon TX \to TY$ of F-algebras, that is, a unique F-algebra homomorphism with $s = \hat{s}\eta_X$.

For a proof, see either Moss (2002, 2.4) or Aczel *et al.* (2002, 2.17). Theorem 3.6 also follows easily from the results of Milius (2005).

Corollary 3.7. The formation of $\eta_X : X \to T_F X$ (for all objects X) and $\hat{s}$ (for all morphisms $s\colon X \to T_F Y$) is a Kleisli triple. The corresponding monad (T_F, η, μ) has the following multiplication:

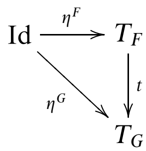
$$\mu_X = \widehat{\mathrm{id}_{T_F X}} : T_F T_F X \to T_F X \qquad \text{for all objects } X.$$

This monad T_F is called the *completely iterative monad generated by F*.

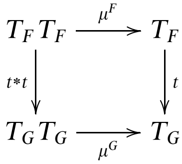
Remark 3.8. We have, for every iteratable functor F, a natural isomorphism $\alpha^F : T_F \to FT_F + \mathrm{Id}$ defined by $(\alpha^F)^{-1} = [\tau, \eta]$. It is easy to check that T_F is a final coalgebra of the endofunctor $H \cdot - + \mathrm{Id}$ on $[\mathscr{K}, \mathscr{K}]$. We can extend that assignment, $F \mapsto T_F$, to a functor from the category of all iteratable endofunctors of $\mathscr{K}$ to the category of all monads on $\mathscr{K}$ as follows: for any iteratable functors F and G and natural transformation $h\colon F \to G$, define a natural transformation $T_h = t\colon T_F \to T_G$ by coinduction, that is, t is the unique coalgebra homomorphism such that the diagram

$$\begin{array}{ccccc} T_F & \xrightarrow{\alpha^F} & FT_F + \mathrm{Id} & \xrightarrow{hT_F + \mathrm{Id}} & GT_F + \mathrm{Id} \\ {\scriptstyle t}\downarrow & & & & \downarrow{\scriptstyle Gt + \mathrm{Id}} \\ T_G & & \xrightarrow[\alpha^G]{} & & GT_G + \mathrm{Id} \end{array} \tag{3.1}$$

commutes. In fact, preservation of identities and composition follow easily from the uniqueness of the morphism t as a coalgebra homomorphism. Thus, it suffices to prove that t is a monad homomorphism, that is, that the two diagrams



and

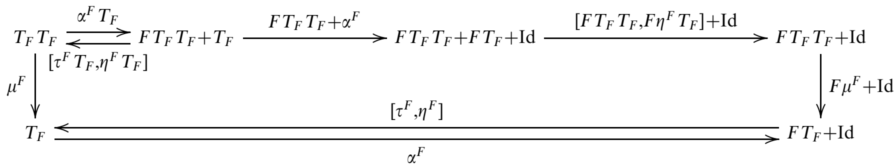


commute. Here $*$ denotes parallel composition of natural transformations.

That the above triangle commutes is clear. Just consider the right-hand component of the coproduct $FT_F + \mathrm{Id}$ in diagram 3.1 above, and recall that $\alpha = [\tau, \eta]^{-1}$.

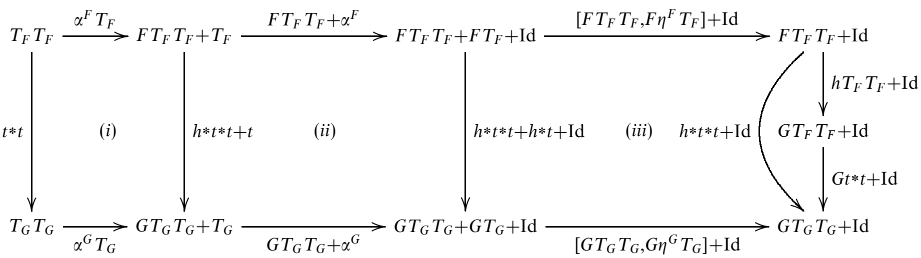
For the commutativity of the above square, we prove that all the morphisms there are coalgebra homomorphisms and then evoke the finality of T_G.

In order to see that μ^F is a coalgebra homomorphism, consider the following commutative diagram:



Consider the components of the coproduct $FT_FT_F + T_F$ separately. For the left-hand component, recall from Theorem 3.6 that μ^F is (componentwise) a morphism of F-algebras; the right-hand component is obvious. Similarly, μ^G is a coalgebra homomorphism.

Finally, we establish that $t * t$ is a coalgebra homomorphism. Consider the following diagram:



It clearly commutes. In part (*i*), invert $\alpha^F T_F = [\tau^K, \tau^F]^{-1}$and consider the components of $FT_FT_F + T_F$ separately: the right-hand component T_F yields only t on both paths and the left-hand component is diagram 3.1. In part (*ii*), the left-hand coproduct component is obvious and the right-hand one is again diagram 3.1. In part (*iii*), the left- and right-hand components of the coproduct $FT_FT_F + FT_F$ obviously commute. For the middle one, use the fact that $\eta^G = t \cdot \eta^F$ and naturality. The triangle on the extreme right is obvious.

It now follows easily that both $\mu^G \cdot (t * t)$ and $t \cdot \mu^F$ are homomorphisms from the coalgebra in the top row of the above diagram to the final coalgebra T_G, and hence they are equal, which completes the proof.

Definition 3.9. By an *equation morphism* with object X of variables and object Y of parameters we mean a morphism $e: X \to T_F(X+Y)$. If more endofunctors are considered, we call e an equation morphism with respect to F.

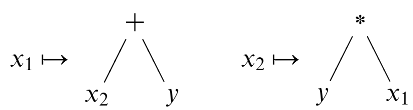
Example 3.10. Let Σ be the signature of two binary operations $+$ and $*$ corresponding to the endofunctor $FX = X \times X \times \{+, *\}$ of **Set**. Here TX is the algebra of all binary (finite or infinite) trees with leaves labelled in X and inner nodes labelled by $+$ or $*$. The iterative system of (formal) equations

$$\begin{aligned} x_1 &\approx x_2 + y \\ x_2 &\approx y * x_1 \end{aligned} \tag{3.2}$$

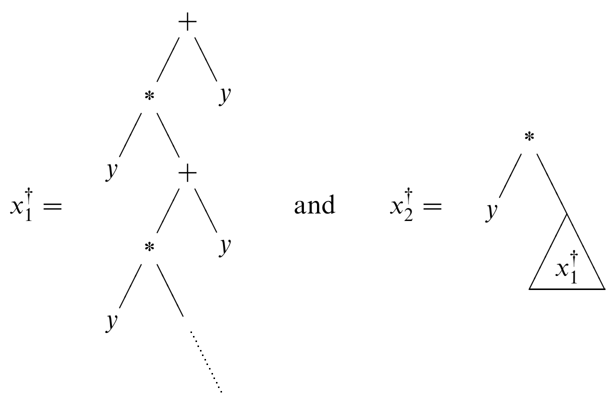
corresponds to the morphism

$$e: \{x_1, x_2\} \to T\{x_1, x_2, y\}$$

defined by



This system has a unique solution $x_1^\dagger$, $x_2^\dagger$, *viz*,



The solution defines a morphism

$$e^\dagger : X \to TY$$

with the following property: if we substitute in the right-hand sides of (3.2) each x_i by $x_i^\dagger$ (and the parameter y by the corresponding tree $\eta_Y(y)$), then $e^\dagger$ is just e under that substitution.

The above substitution morphism is

$$s = [e^\dagger, \eta_Y] : X + Y \to TY,$$

and we extend it, using the Substitution Theorem (Theorem 3.6), to

$$\hat{s} : T(X + Y) \to TY.$$

Thus, solutions $e^\dagger$ are morphisms defined by the property that the triangle

$$\begin{array}{ccc} X & \xrightarrow{\;e^\dagger\;} & TY \\ {\scriptstyle e}\downarrow & \nearrow{\scriptstyle \hat{s}} & \\ T(X+Y) & & \end{array}$$

commutes.

Now, in every monad we have $\hat{s} = \mu_Y Ts$, thus we are led to the following definition.

Definition 3.11. We define a *solution* of an equation morphism $e\colon X \to T(X+Y)$ to mean a morphism $e^\dagger : X \to TY$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{\;e^\dagger\;} & TY \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \mu_Y} \\ T(X+Y) & \xrightarrow[T[e^\dagger, \eta_Y]]{} & TTY \end{array}$$

commutes. If more endofunctors are considered, we call $e^\dagger$ ($: X \to T_F Y$) the solution with respect to F.

Remark 3.12. Some trivial equations, for example, $x \approx x$, have many solutions. But 'almost' all systems of iterative equations turn out to have a unique solution. The cases we want to exclude are the equations $x \approx x'$ where the right-hand side is a variable from X. Now, given an equation morphism $e\colon X \to T(X+Y)$, recall that $T(X+Y)$ is a coproduct of $FT(X+Y)$ and $X+Y$, and thus it is a coproduct of

X with injection $X \xrightarrow{\text{inl}} X+Y \xrightarrow{\eta_{X+Y}} T(X+Y)$

and

$FT(X+Y)+Y$ with injection $FT(X+Y)+Y \xrightarrow{[\tau_{X+Y}, \eta_{X+Y}\,\text{inr}]} T(X+Y)$.

It is the first injection that we want to exclude. More precisely, we want e to factor through the latter one.

Definition 3.13. An equation morphism $e\colon X \to T(X+Y)$ is called *guarded* provided it factors through the coproduct injection $FT(X+Y)+Y \to T(X+Y)$:

$$\begin{array}{ccc} X & \xrightarrow{\;e\;} & T(X+Y) \\ & \searrow & \uparrow{\scriptstyle [\tau_{X+Y}, \eta_{X+Y}\,\text{inr}]} \\ & & FT(X+Y)+Y \end{array}$$

Example 3.14. For $F = \mathscr{P}$, consider the equation

$$x \approx \{x\},$$

or, more precisely, the equation morphism

$$e \equiv 1 \to \mathscr{P}^\infty 1 \to T_{\mathscr{P}^\infty}(1+\varnothing)$$

composed from the embedding $1 = \{x\} \hookrightarrow \{\emptyset, \{x\}\} = \mathscr{P}^\infty 1$ and the component (at 1) of the natural transformation $\mathscr{P}^\infty \xrightarrow{\mathscr{P}^\infty \eta} \mathscr{P}^\infty T_{\mathscr{P}^\infty} \xrightarrow{\tau} T_{\mathscr{P}^\infty}$. In the non-well-founded set theory, the unique solution is the set Ω (characterised by being its own unique element). Here we work, instead, with the final coalgebra of $\mathscr{P}^\infty$, which can be described as the coalgebra of all extensional trees modulo bisimulation (Adámek *et al.* 2004). The solution of $x \approx \{x\}$ is, then, the bisimulation class of the following tree

In contrast to $\mathscr{P}$, the functor $\mathscr{P}_i$ (where i is any ordinal) of all subsets of cardinality at most $\aleph_i$ is accessible, so there is no need to jump from sets to classes. A final coalgebra $\nu\mathscr{P}_i$ exists in **Set** and can be described as the coalgebra of all extensional trees with fan-out less than $\aleph_i$ modulo bisimulation.

Now every equation morphism with respect to $\mathscr{P}$, $e : X \to \mathscr{P}(X + Y)$, can be considered as an equation morphism with respect to $\mathscr{P}_i$ from some i (for example, for $\aleph_i = \mathrm{card}(X + Y)$). Thus, e has a unique solution in $T_{\mathscr{P}_i} Y$, which shows that there is indeed no need to make use of classes. We are going to prove that this is a general phenomenon.

Theorem 3.15 (Solution Theorem). Given an iteratable endofunctor, every guarded equation morphism has a unique solution.

For the proof, see Moss (2002, 2.11), Aczel *et al.* (2002, 3.4–3.8) or Milius (2005).

Proposition 3.16. For every endofunctor F of $\mathscr{K}$, the functor F^∞ is iteratable, and its completely iterative monad T_{F^∞} is small-accessible.

Proof. We use $\mathscr{L}$ to denote the full subcategory of $\mathscr{K}^\infty$ formed by all objects X for which $T_{F^\infty} X$ exists. Observe that $\mathscr{L}$ contains $\mathscr{K}$: in fact, for $X \in \mathscr{K}$, we have a final coalgebra of $F^\infty(-) + X$ by Theorem 2.16. Define a functor

$$\psi : \mathscr{L} \to \mathscr{K}^\infty$$

on objects X by $\psi X = T_{F^\infty} X$. For morphisms $f : Y \to Z$, we use $\psi f : T_{F^\infty} Y \to T_{F^\infty} Z$ to denote the unique homomorphism of coalgebras of $F^\infty(-) + Z$ (with respect to $T_{F^\infty} Y \cong F^\infty(T_{F^\infty} Y) + Y \xrightarrow{\mathrm{id} + f} F^\infty(T_{F^\infty} Y) + Z$, of course). We prove that ψ is small-accessible. Since every object of $\mathscr{K}^\infty$ is a colimit of a chain of objects of $\mathscr{K}$, this will prove that $\mathscr{L} = \mathscr{K}^\infty$, and the proposition follows: T_{F^∞} is small-accessible.

Let $(X_s \xrightarrow{x_s} X)_{s \in \mathbf{Ord}}$ be a transfinite colimit in $\mathscr{K}^\infty$ with $X_s \in \mathscr{L}$. Form a colimit

$$\big(T_{F^\infty} X_s \xrightarrow{y_s} Y\big)_{s \in \mathbf{Ord}}$$

in $\mathscr{K}^\infty$ and define a structure α of a coalgebra of $F^\infty(-)+X$ by the commutativity of the squares

$$\begin{array}{ccc}
T_{F^\infty}X_s & \xrightarrow{\cong} & F^\infty(T_{F^\infty}X_s)+X_s \\
{\scriptstyle y_s}\downarrow & & \downarrow{\scriptstyle F^\infty y_s + x_s} \\
Y & \xrightarrow[\alpha]{} & F^\infty Y + X
\end{array}$$

for all $s \in \mathbf{Ord}$. We prove that Y is a final coalgebra of $F^\infty(-)+X$. We test this on all coalgebras

$$B \xrightarrow{\beta} F^\infty B + X, \qquad B \in \mathscr{K},$$

which is sufficient by Remark 2.13.

1. *Existence of a homomorphism from B to Y:* Since B is small-presentable, see Remark 2.3, the morphism $\beta : B \to \operatorname{colim}_{s\in\mathbf{Ord}} F^\infty B + X_s$ factors as

$$B \xrightarrow{\beta'} F^\infty B + X_s \xrightarrow{\mathrm{id}+x_s} F^\infty B + X$$

for some $s \in \mathbf{Ord}$. The unique homomorphism h of coalgebras of $F^\infty(-)+X_s$ yields the desired homomorphism $y_s h$:

$$\begin{array}{ccccc}
B & \xrightarrow{\beta'} & F^\infty B + X_s & \xrightarrow{\mathrm{id}+x_s} & F^\infty B + X \\
{\scriptstyle h}\downarrow & \circledast & \downarrow{\scriptstyle F^\infty h + \mathrm{id}} & & \downarrow{\scriptstyle F^\infty h + \mathrm{id}} \\
T_{F^\infty}X_s & \xrightarrow[\cong]{} & F^\infty(T_{F^\infty}X_s)+X_s & \xrightarrow[\mathrm{id}+x_s]{} & F^\infty(T^\infty X_s)+X \\
{\scriptstyle y_s}\downarrow & & & \searrow{\scriptstyle F^\infty y_s + x_s} & \downarrow{\scriptstyle F^\infty y_s + \mathrm{id}} \\
Y & & \xrightarrow[\alpha]{} & & F^\infty Y + X
\end{array}$$

It is clear that $y_s h$ is independent of the choice of s.

2. *Uniqueness of a homomorphism from B to Y:* Let $k : B \to Y$ be a homomorphism of coalgebras of $F^\infty(-)+X$. We shall prove that there exists $s \in \mathbf{Ord}$ and a homomorphism $h : B \to T_{F^\infty}X_s$ of coalgebras of $F^\infty(-)+X_s$ such that $k = y_s h$. Since h is uniquely determined by the finality of $T_{F^\infty}X_s$, this proves the uniqueness of k. The small-presentability of B implies that $k : B \to \operatorname{colim}_{s\in\mathbf{Ord}} T_{F^\infty}X_s$ has a factorisation as

$$B \xrightarrow{h} T_{F^\infty}X_s \xrightarrow{y_s} Y$$

for some $s \in \mathbf{Ord}$. If h happens to be a homomorphism, we are finished. If not, consider the above diagram: the outward square commutes since $k = y_s h$ is a homomorphism. Consequently, the morphism $F^\infty y_s + x_s$ merges the two sides of the square $\circledast$. That morphism is, of course, a colimit morphism of the chain $F^\infty(T_{F^\infty}X_s)+X_s$. From the fact that $\hom(B,-)$ preserves this transfinite colimit and that the colimit morphism $F^\infty y_s + x_s$ merges the two sides of $\circledast$, we conclude that some connecting morphism

$$F^\infty \psi x_{ss'} + x_{ss'} : F^\infty(T_{F^\infty}X_s)+X_s \to F^\infty(T_{F^\infty}X_{s'})+X_{s'}$$

of that diagram (where $x_{ss'}: X_s \to X_{s'}$ is a connecting morphism of the given chain for $s \leqslant s'$) also merges the two sides of ⊛. It is now sufficient to substitute s' for s, $\psi(x_{ss'}) \cdot h$ for h, and $(\mathrm{id} + x_{ss'})\beta'$ for β' to obtain a diagram analogous to the one above in which ⊛ commutes. In other words, with this new choice of s' the corresponding morphism $\psi(x_{ss'}) \cdot h : B \to T_{F^\infty} X_{s'}$ is a homomorphism, as required. □

Remark 3.17. Let us apply Solution Theorem 3.15 to the (iteratable) functor F^∞. It tells us that every guarded equation morphism $e: X \to T_{F^\infty}(X + Y)$ has a unique solution $e^\dagger : X \to T_{F^\infty} Y$.

A much better result has been proved for $\mathscr{K} = \mathbf{Set}$ in Adámek *et al.* (2002a): a solution theorem that does not use classes at all! The idea is simple: every endofunctor F of **Set** is a colimit of a transfinite chain

$$F = \operatorname*{colim}_{i \in \mathbf{Ord}} F_i$$

of accessible endofunctors F_i. Each of these endofunctors generates, by Example 3.4, a completely iterative monad T_{F_i} (on **Set**). And a colimit of the functors $T^\infty_{F_i}$ yields the completely iterative monad generated by F^∞:

$$T_{F^\infty} = \operatorname*{colim}_{i \in \mathbf{Ord}} T^\infty_{F_i}.$$

We have proved the following:

(a) Guarded equation morphisms for F with a set X of variables and a set Y of parameters have the form

$$e: X \to T_{F_i}(X + Y)$$

where, for some ordinal i, e is a guarded equation with respect to F_i.

(b) A solution of e with respect to F is just the solution

$$e^\dagger : X \to T_{F_i} Y$$

(in **Set**!) with respect to F_i.

Example 3.18. If $F = \mathscr{P}(- \times \Sigma)$, express it as

$$\mathscr{P}(- \times \Sigma) = \operatorname*{colim}_{i \in \mathbf{Ord}} \mathscr{P}_i(- \times \Sigma)$$

where $\mathscr{P}_i$ is the subfunctor of all subsets of cardinality at most $\aleph_i$. Then, in particular, all flat equations

$$e: X \to \mathscr{P}((X + Y) \times \Sigma)$$

can be solved in **Set** as follows: since X is a set, there exists an upper bound, $\aleph_i$, of the cardinalities of $e(x) \subseteq (X + Y) \times \Sigma$ for $x \in X$. Thus, e is actually a function

$$e: X \to \mathscr{P}_i((X + Y) \times \Sigma),$$

and its solution with respect to $\mathscr{P}_i(- \times \Sigma)$ is the solution for $\mathscr{P}(- \times \Sigma)$ we were looking for.

Lemma 3.19. Every endofunctor F of a locally presentable category is a colimit

$$F = \operatorname*{colim}_{i \in \mathbf{Ord}} F_i$$

of a transfinite chain of accessible endofunctors F_i.

Proof. The category $\mathscr{K}$, being locally presentable, has only a class of morphisms. Thus it can be presented as a union $\mathscr{K} = \bigcup_{i \in \mathbf{Ord}} \mathscr{K}_i$ of a transfinite chain of small subcategories. Given $F : \mathscr{K} \to \mathscr{K}$, we use $F_i : \mathscr{K} \to \mathscr{K}$ to denote a left Kan extension of the restriction $F \cdot E_i : \mathscr{K}_i \to \mathscr{K}$ along the embedding $E_i : \mathscr{K}_i \hookrightarrow \mathscr{K}$. Since $\mathscr{K}_i$ is small, there exists a cardinal λ such that all objects of $\mathscr{K}_i$ are λ-presentable. This implies that F_i is λ-accessible: see the second corollary of 2.5 in Gabriel and Ulmer (1971). Furthermore, $F = \operatorname{colim}_{i \in \mathbf{Ord}} F_i$ because we have the following natural isomorphisms, which, moreover, are natural in the variable $G \in [\mathscr{K}, \mathscr{K}]$:

$$\begin{aligned} [\mathscr{K},\mathscr{K}](\operatorname*{colim}_i F_i, G) &\cong \lim_i [\mathscr{K},\mathscr{K}](F_i, G) \\ &\cong \lim_i [\mathscr{K}_i,\mathscr{K}](FE_i, GE_i) \\ &\cong [\mathscr{K},\mathscr{K}](F, G) \end{aligned}$$

where in the last step we use the fact that for every natural transformation, $F \to G$ lies in $\mathscr{K}_i$ for some ordinal i. □

Proposition 3.20. In every locally presentable category $\mathscr{K}$, all existing colimits of transfinite chains are absolute (that is, are preserved by every functor with domain $\mathscr{K}$).

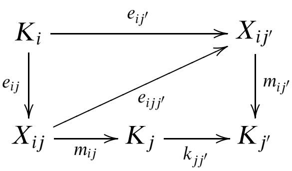
Proof.

(1) All existing colimits of transfinite chains of monomorphisms $k_{ij} : K_i \to K_j$ ($i \leqslant j$ in **Ord**) are absolute. In fact, given a colimit cocone $k_{i,\aleph_\infty} : K_i \to K_{\aleph_\infty}$ ($i \in \mathbf{Ord}$) in $\mathscr{K}$, use the fact that since $\mathscr{K}$ is locally presentable, every object is presentable. Thus, there exists λ such that the object $K_{\aleph_\infty}$ is λ-presentable. Consequently, $\mathrm{id}_{K_{\aleph_\infty}}$ factors through $k_{i,\aleph_\infty}$ for some $i \in \mathbf{Ord}$, which means that $k_{i,\aleph_\infty}$ is a split epimorphism. On the other hand, $k_{i,\aleph_\infty}$ is also a monomorphism, since colimits of transfinite chains of monomorphisms have monomorphic components (Adámek and Rosický 1994, 1.60). Consequently, $k_{i,\aleph_\infty}$ is a isomorphism, which implies that all the connecting morphisms k_{ij} for $i \leqslant j$ are isomorphisms. It is then clear that the (trivial) colimit is absolute.

(2) Given an arbitrary chain $k_{ij} : K_i \to K_j$ ($i \leqslant j < \aleph_\infty$) with a colimit $k_{i,\aleph_\infty} : K_i \to K_{\aleph_\infty}$ ($i < \aleph_\infty$), we use the factorisation system $\mathscr{E}$ = strong epis, $\mathscr{M}$ = monos which exists in every locally presentable category (Adámek and Rosický 1994, 1.61). We use

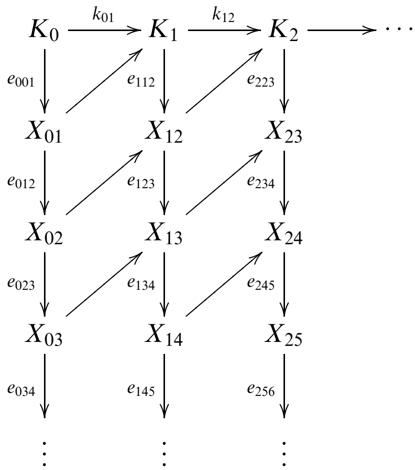
$$e_{ij} : K_i \to X_{ij} \quad (\in \mathscr{E}) \qquad \text{and} \qquad m_{ij} : X_{ij} \to K_j \quad (\in \mathscr{M})$$

to denote a factorisation of k_{ij} (where $X_{ii} = K_i$, $e_{ii} = m_{ii} = \mathrm{id}$) in $\mathscr{K}$. For each $i \in \mathbf{Ord}$ we obtain a chain of strong epimorphisms $e_{ijj'} : X_{ij} \to X_{ij'}$ ($j \leqslant j'$ in **Ord**) by using the diagonal fill-in



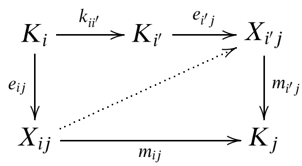
(∗∗)

We thus get the two-dimensional diagram



(3.3)

where the diagonal morphisms are monomorphisms obtained, again, from the diagonal fill-in (for all $i \leqslant i' \leqslant j$)



Since $\mathscr{K}$ is cowellpowered, for every ordinal i the chain $e_{iij} : K_i \to X_{ij}$ ($j \in \mathbf{Ord}$, $j \geqslant i$) of quotients of K_i is stationary, that is, there exists $i^* \geqslant i$ such that all the quotients e_{iij} with $j \geqslant i^*$ are equivalent. In other words, all e_{ii^*j} are isomorphisms ($j \geqslant i^*$), therefore, (∗∗) implies

$$k_{i^*j} \cdot m_{ii^*} \quad \text{are monomorphisms for all } j \geqslant i^*. \tag{3.4}$$

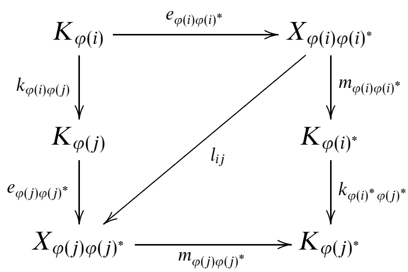
We choose, for each i, our ordinal i^* so that the function $(-)^* : \mathbf{Ord} \to \mathbf{Ord}$ is monotone, and define the iteration of $*$ as the following function $\varphi : \mathbf{Ord} \to \mathbf{Ord}$:

$$\begin{aligned} \varphi(0) &= 0 \\ \varphi(i+1) &= \varphi(i)^* \\ \varphi(j) &= \sup_{i<j} \varphi(i) \quad \text{for limit ordinals } j. \end{aligned}$$

Let $(L_i)_{i\in\mathbf{Ord}}$ be the chain

$$L_i = X_{\varphi(i)\varphi(i)^*}$$

with the connecting morphisms l_{ij} given by the diagonal fill-in for all $i \leqslant j$:



(3.5)

From 3.4, we conclude that

$$l_{ij} \quad \text{are monomorphisms for all } i \leqslant j, \tag{3.6}$$

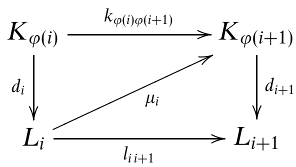
and we have a natural epitransformation

$$d_i = e_{\varphi(i)\varphi(i)^*} : K_{\varphi(i)} \to L_i \qquad (i \in \mathbf{Ord}) \tag{3.7}$$

and a natural transformation

$$\mu_i = m_{\varphi(i)\varphi(i)^*} : L_i \to K_{\varphi(i+1)} \qquad (i \in \mathbf{Ord}).$$

The diagram



(3.8)

commutes for every $i \in \mathbf{Ord}$. In fact the upper triangle commutes by definition of d_i and μ_i, and the lower one commutes because the whole square does (see the upper-left triangle of diagram 3.5), and d_i is an epimorphism.

Consequently, there is a bijective correspondence between cocones $f_i : K_{\varphi(i)} \to B$ of the chain $K_{\varphi(i)}$ and cocones $g_i : L_i \to B$ of the chain L_i, for any object B of $\mathscr{K}^\infty$, given by $(f_i) \mapsto (f_{i+1}\mu_i)$ with the inverse given by $(g_i) \mapsto (g_i d_i)$. This establishes a canonical isomorphism $\operatorname{colim}_{i\in\mathbf{Ord}} K_{\varphi(i)} \cong \operatorname{colim}_{i\in\mathbf{Ord}} L_i$. Now L_i is a mono-chain, thus, a colimit of this chain is absolute. And the above bijective correspondence of cocones is also absolute, since it follows from the commutativity of diagram 3.8, which every functor preserves. Thus, for every functor ϕ with domain $\mathscr{K}$, we have

$$\operatorname*{colim}_{i\in\mathbf{Ord}} \phi L_i \cong \operatorname*{colim}_{i\in\mathbf{Ord}} \phi K_{\varphi(i)} \cong \operatorname*{colim}_{j\in\mathbf{Ord}} \phi K_j.$$

□

Remark 3.21. The above idea of factorising a transfinite chain (see Part (2) of the previous proof) originates from the dissertation of Jan Reiterman, see also Koubek and Reiterman (1979).

Theorem 3.22. For every locally presentable category $\mathscr{K}$ the functor

$$F \mapsto \nu F$$

assigning a final coalgebra to every endofunctor of $\mathscr{K}$ preserves existing colimits of transfinite chains.

Remark. What we mean, of course, is the functor

$$\Phi : [\mathscr{K}, \mathscr{K}] \to \mathscr{K}^{\infty}$$

assigning to every F the object νF of a final F^{∞}-coalgebra $\nu F \xrightarrow{\tau_F} F^{\infty}\nu F$ and to every natural transformation $f : F \to G$ the unique homomorphism $\Phi f : \nu F \to \nu G$ of G^{∞}-coalgebras:

$$\begin{array}{ccccc} \nu F & \xrightarrow{\tau_F} & F^{\infty}\nu F & \xrightarrow{(f^{\infty})_{\nu F}} & G^{\infty}\nu F \\ {\scriptstyle \Phi f}\downarrow & & & & \downarrow{\scriptstyle G\Phi f} \\ \nu G & & \xrightarrow[\tau_G]{} & & G^{\infty}\nu G \end{array}$$

Proof. Let

$$\left(f_i : F_i \to F\right)_{i \in \mathbf{Ord}}$$

be a colimit of a transfinite chain with connecting maps $x_{ij} : F_i \to F_j$ $(i \leqslant j)$ in $[\mathscr{K}, \mathscr{K}]$. We obtain the corresponding diagram of objects νF_i $(i \in \mathbf{Ord})$, more precisely, we apply Φ to the given diagram. This diagram has a colimit

$$\left(t_i : \nu F_i \to T\right)_{i \in \mathbf{Ord}}$$

in $\mathscr{K}^{\infty}$. There is a unique F^{∞}-coalgebra structure

$$\tau : T \to F^{\infty} T$$

making each t_i a homomorphism of F^{∞}-coalgebras:

$$\begin{array}{ccccc} \nu F_i & \xrightarrow{\tau_{F_i}} & F_i^{\infty}\nu F_i & \xrightarrow{(f_i^{\infty})_{\nu F_i}} & F^{\infty}\nu F_i \\ {\scriptstyle t_i}\downarrow & & & & \downarrow{\scriptstyle F^{\infty} t_i} \\ T & & \xrightarrow[\tau]{} & & F^{\infty} T \end{array}$$

To prove that (T, τ) is a final F^{∞}-coalgebra, we only have to consider an F-coalgebra

$$\beta : B \to FB \qquad (B \in \mathscr{K}),$$

see Remark 2.13. In order to prove the existence and uniqueness of a homomorphism $B \to T$, we first observe that since F^{∞} preserves transfinite colimits, we have

$$F^{\infty} T = \operatorname*{colim}_{i \in \mathbf{Ord}} F^{\infty} \nu F_i$$

with the colimit cocone $F^{\infty} t_i$ $(i \in \mathbf{Ord})$. It is easy to see that

$$\left(f_i^{\infty} : F_i^{\infty} \to F^{\infty}\right)_{i \in I}$$

is a transfinite colimit in $[\mathscr{K}^\infty, \mathscr{K}^\infty]$: the functor $G = \operatorname*{colim}_{i \in \mathrm{Ord}} F_i^\infty$ preserves transfinite colimits, and the limit of universal arrows $F_i \to F_i^\infty$ yields a universal arrow $F \to G$, thus $G = F^\infty$.

Consequently, we also have

$$F^\infty T = \operatorname*{colim}_{i \in I} F_i^\infty v F_i \tag{3.9}$$

with the colimit cocone

$$F_i^\infty v F_i \xrightarrow{(f_i^\infty)_{vf_i}} F^\infty v F_i \xrightarrow{F^\infty t_i} F^\infty T \qquad (i \in \mathbf{Ord}).$$

Now the proof of the unique existence of a homomorphism $B \to T$ is analogous to the proof of Proposition 3.16. $\square$

Corollary 3.23. Let F be an endofunctor of $\mathscr{K}$ expressed as a colimit $(F_i \xrightarrow{h_i} F)_{i \in \mathbf{Ord}}$ of a transfinite chain of accessible endofunctors. Then the completely iterative monad T_{F^∞} is a colimit, in $[\mathscr{K}^\infty, \mathscr{K}^\infty]$, of the transfinite chain $T_{F_i}^\infty$ $(i \in \mathbf{Ord})$.

In fact, for every object X of $\mathscr{K}$, apply Theorem 3.22 to the colimit

$$\Big(F_i(-) + X \xrightarrow{h_i + \mathrm{id}_X} F(-) + X\Big)_{i \in \mathbf{Ord}}$$

to conclude that a final coalgebra $T_{F^\infty} X$ of $F^\infty(-) + X$ is a colimit of the chain of final coalgebras $T_{F_i} X$ of $F_i(-) + X$ with the colimit cocone

$$\big(t_i\big)_X : T_{F_i} X \to T_{F^\infty} X \qquad (i \in \mathbf{Ord})$$

formed by the coalgebra homomorphisms of $F^\infty(-) + X$. Consequently, the two endofunctors of $[\mathscr{K}^\infty, \mathscr{K}^\infty]$ forming the two sides of the intended 'equation'

$$T_{F^\infty} \cong \operatorname*{colim}_{i \in \mathbf{Ord}} T_{F_i}^\infty$$

are canonically naturally isomorphic when restricted to $\mathscr{K}$ on the domain (where $T_{F_i}^\infty X = T_{F_i} X$, of course). To conclude that the above isomorphism takes place, just observe that both sides are small-accessible functors. This is clear for the right-hand side, since colimits commute with colimits, and for the left-hand side, see Proposition 3.16.

Theorem 3.24 (General Solution Theorem). Let F be an endofunctor of a locally presentable category $\mathscr{K}$. Then every guarded equation morphism in $\mathscr{K}^\infty$ with respect to F^∞, where X and Y are in $\mathscr{K}$, has the form

$$\bar{e} : X \to T_{F_i}(X + Y) \qquad (i \in \mathbf{Ord})$$

of a guarded equation morphism with respect to F_i, and the solution

$$\bar{e}^\dagger : X \to T_{F_i} Y$$

with respect to F_i is also a solution with respect to F.

Remark 3.25. More precisely, we assume that $F = \operatorname{colim} F_i$ as in Corollary 3.23; let $t_i : T_{F_i}^\infty \to T_{F^\infty}$ $(i \in \mathbf{Ord})$ be a colimit cocone. Then the statement says that given a guarded equation morphism $e : X \to T_{F^\infty}(X + Y)$ with X and Y in $\mathscr{K}$, there exists a guarded equation morphism $\bar{e}\colon X \to T_{F_i}(X+Y)$ for some $i \in \mathbf{Ord}$ such that the triangles

$$\begin{array}{ccc} X & \xrightarrow{\;e\;} & T_{F^\infty}(X+Y) \\ {\scriptstyle \bar{e}}\downarrow & \nearrow{\scriptstyle (t_i)_{X+Y}} & \\ T_{F_i}(X+Y) & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X & \xrightarrow{\;e^\dagger\;} & T_{F^\infty}Y \\ {\scriptstyle \bar{e}^\dagger}\downarrow & \nearrow{\scriptstyle (t_i)_Y} & \\ T_{F_i}Y & & \end{array}$$

commute.

Proof. For short, we denote the completely iterative monad of F^∞ by $(T^\sharp, \eta^\sharp, \mu^\sharp)$. Suppose that a guarded equation morphism $e\colon X \to T^\sharp(X+Y)$ is given, and consider the factorisation

$$\begin{array}{ccc} X & \xrightarrow{\;e\;} & T^\sharp(X+Y) \\ & \searrow{\scriptstyle e'} & \uparrow{\scriptstyle [\tau^\sharp_{X+Y},\, \eta^\sharp_{X+Y}\,\mathrm{inr}]} \\ & & F^\infty T^\sharp(X+Y)+Y \end{array}$$

Since X is a small-presentable object, by Remark 2.3, e' factors through some $(h^\infty_\lambda * t^\infty_\lambda)_{X+Y} + \mathrm{id}_Y$ for some ordinal λ:

$$\begin{array}{ccc} X & \xrightarrow{\;e'\;} & F^\infty T^\sharp(X+Y)+Y \\ & \searrow{\scriptstyle \bar{e}'} & \uparrow{\scriptstyle (h^\infty_\lambda * t^\infty_\lambda)_{X+Y} + \mathrm{id}_Y} \\ & & F^\infty_\lambda T^\infty_\lambda(X+Y)+Y \end{array}$$

Observe that from Corollary 3.23 and Remark 3.8, the square

$$\begin{array}{ccc} F^\infty_\lambda T^\infty_\lambda(X+Y)+Y & \xrightarrow{\;[(\tau^\infty_\lambda)_{X+Y},\, (\eta^\infty_\lambda)_{X+Y}\,\mathrm{inr}]\;} & T^\infty_\lambda(X+Y) \\ {\scriptstyle (h^\infty_\lambda * t^\infty_\lambda)_{X+Y} + \mathrm{id}_Y}\downarrow & & \downarrow{\scriptstyle (t^\infty_\lambda)_{X+Y}} \\ F^\infty T^\sharp(X+Y)+Y & \xrightarrow[\;[\tau^\sharp_{X+Y},\, \eta^\sharp_{X+Y}\,\mathrm{inr}]\;]{} & T^\sharp(X+Y) \end{array}$$

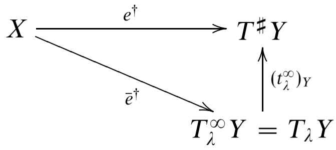
commutes. Thus, by putting

$$\bar{e} = \big[(\tau^\infty_\lambda)_{X+Y}, (\eta^\infty_\lambda)_{X+Y}\,\mathrm{inr}\big] \cdot \bar{e}',$$

we define a guarded equation morphism such that the triangle

$$\begin{array}{ccc} X & \xrightarrow{\;e\;} & T^\sharp(X+Y) \\ & \searrow{\scriptstyle \bar{e}} & \uparrow{\scriptstyle (t^\infty_\lambda)_{X+Y}} \\ & & T^\infty_\lambda(X+Y) = T_\lambda(X+Y) \end{array}$$

commutes. Since t_λ^∞ is a monad morphism, see Remark 3.8, it preserves solutions (Aczel *et al.* 2002, 4.11)), that is, the triangle



commutes. □

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