Free iterative theories: a coalgebraic view

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Every finitary endofunctor of Set is proved to generate a free iterative theory in the sense of Elgot. This work is based on coalgebras, specifically on parametric corecursion, and the proof is presented for categories more general than just Set.

1. Introduction

Iterative algebraic theories were introduced by Calvin C. Elgot in Elgot (1975) as a concept serving the study of computation (on, say, Turing machines) at a level abstracting from the nature of external memory. The main example presented by Elgot is the theory of rational trees, that is, infinite trees that are solutions of systems of finitary iterative equations. Or, equivalently, that possess only finitely many subtrees. He and his coauthors later proved that this theory is a free iterative theory on a given (finitary) signature (Elgot et al. 1978).

The purpose of the present paper is to generalise Elgot’s result from signatures (in other words, polynomial endofunctors of the category of sets) to finitary endofunctors of Set and some ‘set-like’ categories, for example, the category of posets. Using a very general Solution Theorem, developed in previous work, which shows by coalgebraic methods how iterative equations can be solved in categories, we prove that finitary endofunctors generate free iterative theories (in other words, finitary monads), called rational monads. We construct the rational monad in two steps:

(1) A rational monad of a strongly finitary endofunctor, that is, a finitary endofunctor ‘preserving finiteness’, is constructed in Section 4.

and

(2) A rational monad of a general finitary endofunctor is derived in Section 5.

We work with categories called strongly LFP, which we introduce in Section 2. And we assume that the given endofunctor preserves monomorphisms. For the category Set this last assumption can be omitted, as we show in Section 6. For all these cases the common formulation of a rational monad, \( R \), is: \( R \) assigns to every object \( Y \) (of ‘parameters’) the union \( RY \) of all images of solutions of finitary flat equations with parameter \( Y \).

In the rest of the introduction we explain the concepts mentioned above, and give further references.

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1.1. What is a rational tree?

The algebra of finite and infinite \( \Sigma \)-labelled trees has, for every finitary signature \( \Sigma \), the important property that iterative equational systems of a certain (liberal) type have unique solutions. And when we solve finitary flat iterative systems, we obtain precisely the so-called rational trees. We will now describe this in more detail.

For any set \( X \) of variables, we use \( T_\Sigma X \) to denote the algebra of all finite and infinite \( \Sigma \)-labelled trees with variables from \( X \). That is, trees labelled so that a node with \( n > 0 \) children is labelled by an \( n \)-ary operation symbol (an element of \( \Sigma_n \)), and a leaf is labelled by a variable or a constant symbol (an element of \( X + \Sigma_0 \)). The operations on \( T_\Sigma X \) are given by tree-tupling. Now consider a system of iterative equations

\[
\begin{align*}
  x_0 & \approx t_0(x_0, x_1, x_2, \ldots, y_0, y_1, y_2, \ldots) \\
  x_1 & \approx t_1(x_0, x_1, x_2, \ldots, y_0, y_1, y_2, \ldots) \\
  & \vdots \\
  x_n & \approx t_n(x_0, x_1, x_2, \ldots, y_0, y_1, y_2, \ldots) \\
  & \vdots
\end{align*}
\]

(1)

where \( t_i \) are trees with leaves labelled by variables from \( X = \{x_0, x_1, x_2, \ldots \} \) and parameters from \( Y = \{y_0, y_1, y_2, \ldots \} \), that is,

\[
t_i \in T_\Sigma(X + Y) \quad \text{for } i = 0, 1, 2, \ldots
\]

Such a system is called guarded if none of the trees \( t_i \) is a variable from \( X \). This condition alone guarantees that there exists a unique solution of (1), that is, a unique list \( x_i^\dagger(y_0, y_1, y_2, \ldots) \) of trees in \( T_\Sigma Y \) such that the expected identities

\[
\begin{align*}
  x_0^\dagger & = t_0(x_0^\dagger, x_1^\dagger, x_2^\dagger, \ldots, y_0, y_1, y_2, \ldots) \\
  x_1^\dagger & = t_1(x_0^\dagger, x_1^\dagger, x_2^\dagger, \ldots, y_0, y_1, y_2, \ldots) \\
  & \vdots \\
  x_n^\dagger & = t_n(x_0^\dagger, x_1^\dagger, x_2^\dagger, \ldots, y_0, y_1, y_2, \ldots) \\
  & \vdots
\end{align*}
\]

hold in \( T_\Sigma Y \).

**Theorem 1.1.** Every guarded system of iterative equations has a unique solution.

This is a special case of a much more general Solution Theorem, which we mention in Section 1.2 below.

\[\dagger\] We use \( \approx \) to denote formal equations and \( = \) to denote the identity of the two sides.
**Example 1.2.** Let $\Sigma$ consist of binary operation symbols $+$ and $*$ and a constant symbol $\perp$. The following system of iterative equations

\[
\begin{align*}
&x_0 \approx x_1 \quad x_1 \approx \downarrow \\
&y \quad \downarrow
\end{align*}
\]

is guarded. The solution is given by the following trees in $T_\Sigma Y$:

\[
\begin{align*}
x_0^\dagger &= \quad * \quad \perp \\
&\quad + \quad \perp \\
&\quad y \quad \perp
\end{align*}
\]

\[
x_1^\dagger = \quad * \\
\quad \downarrow \\
x_0^\dagger
\]

A guarded equational system (1) is called *finitary* if it has only finitely many variables and the right-hand sides $t_i$ are finite trees (such as those in Example 1.2). It is easy to see that every finitary system has a simple reduction to a finitary *flat* system, that is, one with finitely many variables, and with right-hand sides $t_i$ that are either

(a) flat trees $\quad \begin{array}{c} \sigma \\ x_0 \ldots \ x_{n-1} \end{array}$ with $\sigma \in \Sigma_n$ and $x_0, \ldots, x_{n-1} \in X$

or

(b) single parameters from $Y$.

Observe that (a) includes single constant symbols $\sigma \in \Sigma_0$.

**Example 1.3.** A reduction of the system of Example 1.2 is obtained by introducing new variables $z_0$, $z_1$ and $z_2$ as follows:

\[
\begin{align*}
x_0 &\approx + \\
x_1 &\quad \downarrow \\
&\quad z_0 \\
z_0 &\approx * \\
z_1 &\quad \downarrow \\
z_1 &\quad \downarrow \\
z_2 &\quad \downarrow
\end{align*}
\]

The unique solution of this reduced system is the original solution $x_0^\dagger$ and $x_1^\dagger$ together with the obvious new trees $z_i^\dagger$ for $i = 0, 1, 2$.

**Definition 1.4.** A tree in $T_\Sigma Y$ is called *rational* if it can be obtained by solving a finitary flat system of equations.
Thus, the trees $x_0^+$ and $x_1^+$ in Example 1.2 are examples of rational trees. Rational trees are fully characterised as those trees in $T_\Sigma Y$ that have, up to tree isomorphism, only finitely many subtrees (Elgot et al. 1978). For example, all subtrees of $x_0^+$ in Example 1.2 are isomorphic to one of the following five (obtained by breadth-first search of the nodes of $x_0^+$):

Now the subalgebra

$R_\Sigma Y$

of $T_\Sigma Y$ of all rational trees has a solution property 'almost as strong' as the algebra $T_\Sigma Y$ itself.

**Theorem 1.5.** Every finite guarded system of iterative equations with rational right-hand sides has a unique solution in $R_\Sigma Y$.

A direct proof of this theorem is not difficult, but we do not have to discuss it here because we are going to prove a much more general result, called the Rational Solution Theorem, see Corollary 5.8.

**1.2. What is a solution in general?**

In this subsection we recall briefly some results, which were obtained independently by Larry Moss (Moss 2001) and the present authors in collaboration with Peter Aczel (Aczel et al. 2002). The reader can find more details in the extended abstract Aczel et al. (2001), which has already been published.

We now generalise infinite trees, which are elements of the final coalgebra $T_\Sigma X$ of the polynomial endofunctor $H_\Sigma(\_)+X$, to final coalgebras of $H(\_)+X$ for an arbitrary endofunctor $H$. Recall that, given a signature $\Sigma$, the corresponding polynomial functor

$H_\Sigma : \text{Set} \longrightarrow \text{Set}$

defined by

$H_\Sigma A = \Sigma_0 + \Sigma_1 \times A + \Sigma_2 \times A^2 + \cdots$

has the property that $H_\Sigma$-algebras are just the classical universal algebras of signature $\Sigma$. A final $H_\Sigma$-coalgebra is well known to be the coalgebra $T_\Sigma \emptyset$ of all finite and infinite $\Sigma$-labelled trees without variables (Adámek and Koubek 1995). Now the functor $H_\Sigma(\_)+X$ is also polynomial (for the signature obtained from $\Sigma$ by adding a constant symbol for every variable in $X$), thus,

$T_\Sigma X$ is a final coalgebra of $H_\Sigma(\_)+X$.

In our previous work (Aczel et al. 2002) we have introduced the concept of an *iteratable* endofunctor of $\text{Set}$ (or, more generally, of any category with binary coproducts): it is an
endofunctor $H$ such that $H(\_)+X$ has a final coalgebra, for every $X$. We use the notation $TX$

to denote a final coalgebra of $H(\_)+X$, and the coalgebra-structure (which, by Lambek’s Lemma (Lambek 1968), is an isomorphism between $TX$ and $HTX + X$) is denoted by giving names to the coproduct injections of $TX$ (as a coproduct of $HTX$ and $X$) as follows:

$\eta_X : X \to TX$ (‘injection of variables’)

and $\tau_X : HTX \to TX$ (‘$TX$ becomes an $H$-algebra’).

Recall that an endofunctor is called finitary if it preserves filtered colimits. Every finitary endofunctor is iterable, see Example 2.11 of Aczel et al. (2002). The above concept of a guarded system of equations can be formalised by a function $e : X \to T(X + Y)$, $x_i \mapsto t_i$. And since $T(X + Y) = (HT(X + Y) + Y) + X$ is a coproduct with injections $[\tau_{X+Y}, \eta_{X+Y} \cdot \text{inr}] : HT(X + Y) + Y \to T(X + Y)$ and $\eta_{X+Y} \cdot \text{inl} : X \to T(X + Y)$, to say that the right-hand sides $t_i$ are never variables from $X$ is precisely to say that $e$ factors through $[\tau_{X+Y}, \eta_{X+Y} \cdot \text{inr}]$. Hence, the notion of a guarded system (1) generalises as follows.

**Definition 1.6.** Let $H$ be an iterable endofunctor. We define a **guarded equation morphism** for $H$ to be a morphism of the form $e : X \to T(X + Y)$

$(X$ is the ‘object of variables’ and $Y$ the ‘object of parameters’) that factors through the coproduct injection $[\tau_{X+Y}, \eta_{X+Y} \cdot \text{inr}] : HT(X + Y) + Y \to T(X + Y)$:

$$
\begin{array}{c}
X \\
\xrightarrow{e} \\
\downarrow_{[\tau_{X+Y}, \eta_{X+Y} \cdot \text{inr}]} \\
HT(X + Y) + Y
\end{array}
$$

The $H$-algebras $TX$ have a rich structure. First, substitution of trees in $T_\Sigma X$ for variables generalises to all iterable endofunctors. Recall that given an interpretation of variables $x \in X$ as trees $s(x)$ over $Y$, that is, a function $s : X \to T_\Sigma Y$, the corresponding substitution of trees from $T_\Sigma Y$ into (leaves of) trees of $T_\Sigma X$ is a homomorphism $\hat{s} : T_\Sigma X \to T_\Sigma Y$

of $\Sigma$-algebras. Moreover, $\hat{s}$ is the unique homomorphic extension of $s$. This can be generalised to all iterable endofunctors.

**Theorem 1.7 (Substitution Theorem).** For every morphism $s : X \to TY$ there exists a unique homomorphism $\hat{s} : TX \to TY$ of $H$-algebras extending $s$ (that is, with $s = \hat{s} \eta_X$).

The proof can be found in Moss (2001) or Aczel et al. (2001) (and a slightly improved version in Aczel et al. (2002)).
Corollary 1.8. \((T, \eta, (\_))\) is a Kleisli triple, that is, the following three axioms are satisfied:

(i) \(\hat{\eta}_X = \text{id}_{TX}\), for every object \(X\),

(ii) \(\hat{s} \eta_X = s\), for every morphism \(s : X \to TY\),

and

(iii) \(\hat{r} \hat{s} = \hat{r} s\), for every pair \(s : X \to TY\) and \(r : Y \to TZ\).

As shown in Manes (1976), axioms (i)–(iii) are equivalent to having a monad \((T, \eta, \mu)\), where \(\mu_X = \text{id}_{TX} : TTX \to TX\). Observe that \(\mu_X\) is a homomorphism of \(H\)-algebras (since every \(\hat{s}\) is). However, this monad is usually not finitary; thus, it cannot be identified with an algebraic theory in the sense of Lawvere.

Example 1.9. For every signature \(\Sigma\) we have a monad \((T\Sigma, \eta, \mu)\), where \(T\Sigma X\) is the algebra of all finite and infinite \(\Sigma\)-labelled trees above, \(\eta\) is the insertion of variables (as one-node trees), and \(\mu\) is the usual substitution of trees into trees. This monad is seldom finitary. (In fact, it is finitary iff all operations of \(\Sigma\) are either unary or nullary.)

Next, we can introduce solutions for equation morphisms \(e : X \to T(X + Y)\) by mimicking the case of trees as follows. A solution of \(e\) is a morphism \(e^\dagger : X \to TY\). It has the property that when the following ‘substitution’ morphism

\[ s = [e^\dagger, \eta_Y] : X + Y \to TY \]

is considered (that is, every variable \(x_i\) is substituted by the tree \(e^\dagger (x_i) = x_i^\dagger \in TY\), while parameters are unchanged), the composite

\[ X \xrightarrow{e} T(X + Y) \xrightarrow{\hat{s}} TY \]

(corresponding to performing the substitution \(s\) on all variables of the right-hand sides of (1)) is equal to \(e^\dagger\).

Definition 1.10. We define a solution of an equation morphism \(e : X \to T(X + Y)\) to be a morphism \(e^\dagger : X \to TY\) such that the triangle

\[ \begin{array}{ccc} X & \xrightarrow{e^\dagger} & TY \\ & \searrow_{[e^\dagger, \eta_Y]} & \downarrow \text{commutes} \\ T(X + Y) & \xrightarrow{e} & T(X + Y) \end{array} \]

commutes.

The following result is called Parametric Corecursion in Moss (2001) and Solution Theorem in Aczel et al. (2001); see also a much improved version of the proof in Aczel et al. (2002).

Theorem 1.11 (Solution Theorem). Given an iteratable endofunctor \(H\), every guarded equation morphism has a unique solution.
1.3. What is a rational monad?

Consider a finitary endofunctor $H$ of $\mathsf{Set}$ (that is, $H$ preserves filtered colimits). The following generalises the above finitary flat systems of equations for $H = H_{\Sigma}$.

**Definition 1.12.** We define a *finitary flat equation morphism* for $H$ to be a morphism of the following form

$$e : X \longrightarrow HX + Y \quad (X \text{ finite}).$$

In brief, finitary flat equation morphisms are just the finite coalgebras of the endofunctor $H(\_ + Y)$. (If $H = H_{\Sigma}$, then a finitary flat equation morphism

$$e : X \longrightarrow \coprod_{n \in \omega} \Sigma_n \times X^n + Y \quad (X \text{ finite})$$

is precisely the concept as introduced above.) Every finitary flat equation morphism gives rise to a guarded equation morphism using $\tau_X : HTX \longrightarrow TX$ above as follows:

$$X \xrightarrow{e} HX + Y \xrightarrow{\epsilon_X HY + Y} TX + Y \xrightarrow{[\text{inl}, \eta_X + Y \cdot \text{inr}]} T(X + Y). \quad (2)$$

By a harmless abuse of notation, we use $e^{\dagger} : X \longrightarrow TY$ to denote the unique solution of (2). We have proved in Aczel et al. (2002) that for flat equation morphisms we have

$$\text{solution} = \text{corecursion}.$$

That is, $e^{\dagger}$ is the unique homomorphism of the coalgebra $X$ into the final coalgebra $TY$.

Mimicking the above definition of $R_{\Sigma}Y$ as the algebra of all solutions of finitary flat equation systems (for $H = H_{\Sigma}$), we introduce below an $H$-algebra

$$RY = \bigcup \text{im}(e^{\dagger})$$

as a union of images of all solutions of finitary flat equation morphisms with the parameter set $Y$ (and an arbitrary finite set $X$ of variables). In Adámek et al. (2002) we show that this definition of $RY$ can be formulated equivalently using all finitary equation morphisms, not just the flat ones.

These $H$-algebras $RY$, which are subalgebras of $TY$, also have a rich structure; we are going to prove the following.

**Rational Substitution Theorem:** Every morphism $s : X \longrightarrow RY$ has a unique extension into a homomorphism $\tilde{s} : RX \longrightarrow RY$ of $H$-algebras (see Theorem 5.14).

**Rational Solution Theorem:** Every guarded equation morphism $e : X \longrightarrow R(X + Y)$ where $X$ is finite has a unique solution $e^{\dagger} : X \longrightarrow RY$ (see Corollary 5.8).

As a corollary of the Rational Substitution Theorem, we conclude immediately that $R$ is also a monad on $\mathsf{Set}$, this time a finitary monad (or, equivalently, Lawvere’s algebraic theory (Manes 1976)). We call this monad the *rational monad* generated by the given finitary endofunctor $H$. 

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*Free iterative theories: a coalgebraic view*
1.4. A rational monad is a free iterative monad

C.C. Elgot has studied algebraic theories (in other words, finitary monads on Set) with the property that certain iterative equations have unique solutions; he called such theories iterative (Elgot 1975). From Section 1.2 above, it is quite natural to call, for a given monad \((S,\eta,\mu)\), any morphism

\[ e : X \longrightarrow S(X + Y) \]

an equation morphism (for \(S\)). Recall that every monad defines substitution: given \(s : X \longrightarrow SY\) we have the corresponding homomorphism \(\hat{s} = \mu_Y \cdot Ss : SX \longrightarrow SY\) of free \(S\)-algebras. Now, a morphism \(e^\dagger : X \longrightarrow SY\) will be called a solution of \(e\) if the triangle

commutes. But how can we express the property of \(e\) being guarded? (Or ideal, which is a slightly stricter concept that Elgot used: for trees this means that the right-hand sides are neither variables, nor single parameters. But this difference is inessential as we prove below – see Remark 4.31.) For that purpose, Elgot introduced the concept of an ideal algebraic theory. Translated into the language of monads, it yields the following definition (see Aczel et al. (2002) for a very simple proof that our formulation is equivalent to Elgot’s):

**Definition 1.13.** A monad \((S,\eta,\mu)\) is called ideal if:

(i) \(S = S' + Id\) and \(\eta : Id \longrightarrow S\) is the right-hand injection (notation: \(\sigma : S' \longrightarrow S\) for the left-hand one)

and

(ii) \(\mu\) restricts to \(\mu' : S'S \longrightarrow S'\), that is, there is a natural transformation \(\mu'\) such that the square

\[
\begin{array}{ccc}
S'S & \longrightarrow & S' \\
\sigma S & \downarrow & \sigma \\
SS & \mu & \longrightarrow \ S \\
\end{array}
\]

commutes.

This definition applies to all categories with finite coproducts such that coproduct injections are monomorphic.

**Example 1.14.** The monad \(T\) defined by Corollary 1.8 is ideal: here \(T = HT + Id\) with injections \(\tau (=\sigma)\) and \(\eta\). Also, the monad \(R\) defined by the Rational Substitution Theorem is ideal, as we prove below.
For ideal monads \((S, \eta, \mu)\) we call an equation morphism \(e : X \rightarrow S(X + Y)\) guarded if it factors through \([\sigma_{X+Y}, \eta_{X+Y}\text{inr}] : S'(X + Y) + Y \rightarrow S(X + Y)\):

\[
\begin{array}{ccc}
X & \xrightarrow{e} & S(X + Y) \\
& \downarrow{[\sigma, \eta\text{inr}]} & \downarrow \\
S'(X + Y) + Y & \xrightarrow{} & 
\end{array}
\]

**Definition 1.15.** (Elgot 1975) A finitary monad \(S\) in \(\text{Set}\) is called iterative if it is ideal and every guarded equation morphism \(e : X \rightarrow S(X + Y)\) with \(X\) and \(Y\) finite has a unique solution.

The restriction to finite sets \(Y\), in comparison to the Rational Solution Theorem above, is inessential because the rational monad is finitary. Thus, results using finitely presentable parameter objects \(Y\) extend easily to results on arbitrary \(Y\).

The main result of the present paper is the fact that every finitary endofunctor of \(\text{Set}\) generates a free iterative monad, \(\text{viz.},\) the rational monad. For polynomial endofunctors this is the main result of Elgot et al. (1978), but is proved in a completely different manner.

This situation is analogous to the main result in Aczel et al. (2002) concerning completely iterative monads, as introduced in Elgot et al. (1978): they are defined as above, except that the requirement that \(X\) and \(Y\) be finite is dropped. We have proved that every iteratable endofunctor generates a free completely iterative monad, \(\text{viz.},\) the above monad \(T\).

1.5. *Beyond the category of sets*

The concept of an ideal monad is immediately extended to any category with finite coproducts. And we can also extend Definition 1.15 naturally, as follows.

**Definition 1.16.** A monad \(S\) is called iterative if it is ideal and every guarded equation morphism \(e : X \rightarrow S(X + Y)\) with \(X\) and \(Y\) finitely presentable (see Section 2.2 below) has a unique solution.

However, for the main result of our paper, stating that every finitary endofunctor generates a free iterative monad, we need to make rather strong additional assumptions on the category we work with. This contrasts to the situation of Aczel et al. (2002), where the result that every iteratable functor generates a free completely iterative monad is proved for all categories having finite coproducts with monomorphic injections. In the present paper we will assume that the given category is strongly LFP, that is, it has to be extensive and locally finitely presentable with finitely presentable objects closed under strong quotients and have finite hom-sets for finitely presentable objects. We introduce these concepts in Section 2.

**Examples 1.17.** The category \(\text{Set}^S\) (for \(S\)-sorted operations) is strongly LFP. Also, the category of posets (for order-enriched operations) is strongly LFP.

The only additional assumption on the finitary endofunctor is that it preserves monomorphisms, an assumption not needed for set functors, as we show in Section 6.
In the next section we recall everything about locally finitely presentable categories that we need later. Even for readers only interested in $\mathcal{A} = \mathbf{Set}$, this section brings information about finitary monads that will be used further.

2. Finitary monads

2.1. Finitary Kleisli triples in $\mathbf{Set}$.

Recall the concept of a Kleisli triple $(T, \eta, \hat{\cdot})$ on a category $\mathcal{A}$, which is equivalent to that of a monad on $\mathcal{A}$ (Manes 1976). It consists of:

1. A function assigning to every object $X$ in $\mathcal{A}$ an object $TX$ and a morphism $\eta_X : X \to TX$.

2. A function assigning to every morphism $s : X \to TY$ a morphism $\hat{s} : TX \to TY$ so that the axioms (i)–(iii) of Corollary 1.8, hold.

A finitary Kleisli triple in $\mathbf{Set}$ consists, analogously, of two functions: one assigns to every finite set $X$ an object $TX$ and a morphism $\eta_X : X \to TX$, and the other one assigns to every morphism $s : X \to TY$ with $X$ and $Y$ finite a morphism $\hat{s} : TX \to TY$ so that (i)–(iii) of corollary hold. In Adámek et al. (2002) we proved that there exists a unique finitary monad on $\mathbf{Set}$, that is, a monad $(T, \eta, \mu)$ such that $T$ preserves filtered colimits, generating that triple in the expected sense:

(a) For every finite set $X$, the given $\eta_X : X \to TX$ is the component of the unit $\eta : \text{Id} \to T$.

(b) For every $s : X \to TY$ with $X$ and $Y$ finite, the usual ‘substitution formula’

$$\hat{s} = \mu_Y \cdot Ts : TX \to TY$$

holds.

In other words, a finitary Kleisli triple can be extended to a Kleisli triple corresponding to a finitary monad, and vice versa.

Let us also recall from Manes (1976) that finitary monads on $\mathbf{Set}$ are precisely the same as Lawvere’s algebraic theories: given a finitary monad $(T, \eta, \mu)$, we can consider its theory consisting of natural numbers as objects, and functions $n \to Tk$ as morphisms from $n$ to $k$. The composite of

$$n \xrightarrow{f} Tk \quad \text{and} \quad k \xrightarrow{g} Tl$$

is

$$n \xrightarrow{f} Tk \xrightarrow{Tg} TTTl \xrightarrow{\mu_l} Tl$$

Conversely, every Lawvere’s theory represents a finitary variety $\forall^\ast$ whose monad (describing $\forall^\ast$-free algebras) is finitary.
2.2. Locally finitely presentable (LFP) categories

We generalise the concept of a finite set in \( \text{Set} \) to that of a \textit{finitely presentable object} \( A \) in a category \( \mathcal{A} \): it is an object such that the hom-functor \( \mathcal{A}(A, -) : \mathcal{A} \to \text{Set} \) preserves filtered colimits.

**Definition 2.1.** (Gabriel and Ulmer 1971). A category is called LFP if it is cocomplete and has a set of finitely presentable objects whose closure under filtered colimits is the whole category.

**Remark 2.2.** It follows that every LFP category \( \mathcal{A} \) has, up to isomorphism, only a set of finitely presentable objects. Moreover, if 

\[ \mathcal{A}_{fp} \]

denotes a set of representatives of finitely presentable objects (considered as a full subcategory), then for every object \( X \) of \( \mathcal{A} \) we have:

(i) The comma-category \( \mathcal{A}_{fp}/X \) (consisting of all morphisms \( a : A \to X \) with \( A \) in \( \mathcal{A}_{fp} \)) is filtered.

and

(ii) \( X \) is a canonical colimit of the diagram

\[ D_X : \mathcal{A}_{fp}/X \to \mathcal{A} \]

given by

\[ D_X(a : A \to X) = A. \]

Consequently,

(iii) \( \mathcal{A} \) is a free cocompletion of \( \mathcal{A}_{fp} \) under filtered colimits. That is, every functor \( F : \mathcal{A}_{fp} \to \mathcal{B} \), where \( \mathcal{B} \) has filtered colimits, has an extension, unique up to natural isomorphism, to a functor \( F' : \mathcal{A} \to \mathcal{B} \) preserving filtered colimits.

See Adámek and Rosický (1994).

**Examples 2.3.**

(i) \( \text{Set} \) is LFP, choose \( \text{Set}_{fp} \) to be the full subcategory of natural numbers.

(ii) The category \( \text{Pos} \) of posets and monotone maps is LFP, here \( \text{Pos}_{fp} \) is a full subcategory representing all finite (in other words, finitely presentable) posets.

(iii) The category \( \text{Gra} \) of graphs (sets with a binary relation) and graph homomorphisms is LFP with \( \text{Gra}_{fp} \) representing all finite graphs.

(iv) Every variety \( \mathcal{V} \) of (finitary, many-sorted) algebras is LFP. An algebra \( V \) in \( \mathcal{V} \) is a finitely presentable object iff it can be presented by finitely many generators and finitely many equations.

**Definition 2.4.** We define a \textit{finitary Kleisli triple} on an LFP category \( \mathcal{A} \) to be a triple \( (T, \eta, (\_)) \) where \( T \) is a function assigning to every finitely presentable object \( X \) an object \( TX \), with a morphism \( \eta_X : X \to TX \), and \( (\_): \mathcal{A} \to \text{Set} \) is a function assigning to every morphism
s : X \to TY with X and Y finitely presentable a morphism \( \hat{s} : TX \to TY \) so that:

(i) \( \hat{\eta}_X = \text{id}_{TX} \), for all X in \( \mathcal{A}_{fp} \);
(ii) \( \hat{s} \eta_X = s \), for all \( s : X \to TY \), (X, Y in \( \mathcal{A}_{fp} \));

and
(iii) \( \hat{r} \hat{s} = \hat{r} \hat{s} \), for all \( s : X \to TY \) and \( r : Y \to TZ \) (X, Y, Z in \( \mathcal{A}_{fp} \)).

Example 2.5. Every finitary monad \((T, \eta, \mu)\) on an LFP category \( \mathcal{A} \) generates a finitary Kleisli triple with \( \hat{s} = \mu_Y \cdot Ts \) for all \( s : X \to TY \) with X, Y in \( \mathcal{A}_{fp} \).

Remark 2.6. It is proved in Adámek et al. (2002) that finitary Kleisli triples on an LFP category \( \mathcal{A} \) correspond precisely to finitary monads on \( \mathcal{A} \). In fact, the correspondence extends to the level of morphisms of Kleisli triples and monads.

Recall that, given monads \((T, \eta, \mu)\) and \((T', \eta', \mu')\) on a category, a monad homomorphism is a natural transformation \( \rho : T \to T' \) such that the diagrams

\[
\begin{array}{ccc}
T & \xrightarrow{\rho} & T' \\
\downarrow{\eta} & \searrow{\eta'} & \\
Id & & \rho
\end{array}
\quad \text{and} \quad \begin{array}{ccc}
TT & \xrightarrow{TT\rho} & TT' \\
\downarrow{\mu} & \searrow{\mu'} & \\
T & \xrightarrow{\rho} & T'
\end{array}
\]

commute. If the components of \( \rho \) are monomorphisms, then \((T, \eta, \mu)\) is called a submonad of \((T', \eta', \mu')\).

Expressed by Kleisli triples, a homomorphism is given by assigning, to every object \( X \), a morphism \( \rho_X : TX \to T'X \) such that (a) \( \rho_X \cdot \eta_X = \eta'_X \) and, (b) given \( s : X \to TY \), the square

\[
\begin{array}{ccc}
TX & \xrightarrow{\hat{s}} & TY \\
\downarrow{\rho_X} & & \downarrow{\rho_Y} \\
T'X & \xrightarrow{\rho_Y} & T'Y
\end{array}
\]

commutes (where we use \( \hat{()} \) for both Kleisli triples). When both triples are finitary, we restrict to \( X \) and \( Y \) finitely presentable in the above definition.

The precise formulation of the correspondence mentioned above is given by the following proposition (Adámek et al. 2002).

Proposition 2.7. Let \( \mathcal{A} \) be an LFP category. The category of finitary Kleisli triples on \( \mathcal{A} \) and their homomorphisms is equivalent to the category of finitary monads on \( \mathcal{A} \) and their homomorphisms.

Corollary 2.8. The category of finitary monads on \( \mathcal{A} \) is coreflective in the category of all monads on \( \mathcal{A} \). A coreflection of a monad \( T \) is the finitary monad generating the same finitary Kleisli triple as \( T \).

Remark 2.9. From Corollary 2.8 it follows trivially that for a finitary monad \( R \) and an arbitrary monad \( T \), given any morphism of the corresponding finitary Kleisli triples, there is a unique extension to a monad homomorphism from \( R \) to \( T \).
2.3. **Strongly LFP categories**

In the subsequent sections we have several additional requirements on the LFP category we work with, which we now summarise in an (admittedly *ad hoc*) definition. The main point is that \( \text{Set} \) has all these properties.

Recall that a category with coproducts and pullbacks is said to be *extensive* if every commutative diagram

\[
\begin{array}{ccc}
A_0 & \xrightarrow{a_0} & A & \leftarrow & A_1 \\
\downarrow{f_0} & & \downarrow{f} & & \downarrow{f_1} \\
X_0 & \xrightarrow{\text{inl}} & X_0 + X_1 & \xleftarrow{\text{inr}} & X_1
\end{array}
\]

comprises a pair of pullback squares iff the top row is a coproduct diagram. Equivalently, coproducts are disjoint and universal (Carboni *et al.* 1993).

In particular, extensive categories have the property that

- coproduct injections are monomorphisms,

and

- a coproduct of two monomorphisms is monomorphic.

Recall that an epimorphism \( e \) is called *strong* iff it has the diagonal fill-in property with respect to all monomorphisms \( m \) (that is, given morphisms \( u, v \) with \( ve = mu \), there exists \( d \) with \( u = de \) and \( v = md \)). *Strong quotients* are quotients carried by strong epimorphisms.

**Remark 2.10.** Every LFP category has a factorisation of morphisms as strong epimorphisms followed by monomorphisms (Adámek and Rosický 1994, 1.61).

**Definition 2.11.** A category is called *strongly LFP* if it is LFP and extensive, hom-sets of finitely presentable objects are finite, and a strong quotient of a finitely presentable object is finitely presentable.

**Examples 2.12.**

(i) \( \text{Set} \) is strongly LFP.

(ii) \( \text{Pos} \) and \( \text{Gra} \) are strongly LFP.

(iii) Every category of relational structures (of any finitary relational signature) is strongly LFP: the finitely presentable objects are those with finitely many elements, and with all relations but finitely many empty.

2.4. **Finitary endofunctors of an LFP category**

It follows from Remark 2.2(iii) above that the category \( \text{Fin}[\mathcal{A}, \mathcal{A}] \) of finitary endofunctors of an LFP category \( \mathcal{A} \) is equivalent to the functor category \( [\mathcal{A}_{fp}, \mathcal{A}] \). From this result it follows that \( \text{Fin}[\mathcal{A}, \mathcal{A}] \) is itself an LFP category. We now describe its finitely presentable objects.
For any pair $A, B$ of finitely presentable objects of an LFP category $\mathcal{A}$ we use $A \bullet B : \mathcal{A}_{fp} \to \mathcal{A}$ to denote the following functor:

$$A \bullet B : X \mapsto \coprod_{\mathcal{A}_{fp}(A,X)} B.$$ (3)

These functors were called *step functors* in Adámek (1997) and were proved to be finitely presentable in $[\mathcal{A}_{fp}, \mathcal{A}]$ (see Lemma 1 there).

**Proposition 2.13.** Every functor $F$ in $[\mathcal{A}_{fp}, \mathcal{A}]$ is a filtered colimit of functors that are finite colimits of step functors.

**Proof.** See the proof of Theorem 3 in Adámek (1997). \qed

**Definition 2.14.** A finitary functor $F : \mathcal{A} \to \mathcal{A}$ is called *strongly finitary* if it preserves finitely presentable objects.

**Proposition 2.15.** For an LFP category $\mathcal{A}$ the following are equivalent:

(i) The category $\mathcal{A}_{fp}$ has finite hom-sets.

(ii) Every finitary endofunctor of $\mathcal{A}$ is a filtered colimit of strongly finitary endofunctors.

**Proof.** (i) $\Rightarrow$ (ii): Since the category of finitary endofunctors of $\mathcal{A}$ is equivalent to $[\mathcal{A}_{fp}, \mathcal{A}]$, it suffices to show that every functor $F : \mathcal{A}_{fp} \to \mathcal{A}$ is a filtered colimit of functors that preserve finitely presentable objects.

By (i), every functor $A \bullet B$ is strongly finitary and a finite colimit of strongly finitary functors is strongly finitary. Now use Proposition 2.13.

(ii) $\Rightarrow$ (i): Suppose that $A$ and $B$ are finitely presentable objects in $\mathcal{A}$ with $\mathcal{A}_{fp}(A, B)$ infinite. We show that the object

$$(A \bullet A)B = \coprod_{\mathcal{A}_{fp}(A,B)} A$$ (4)

is not finitely presentable in $\mathcal{A}$, and therefore the functor $A \bullet A$ is not strongly finitary. Since $A \bullet A$ is finitely presentable, it cannot be expressed as a filtered colimit of strongly finitary functors: in fact, $A \bullet A$ would be a retract of one of them, but retracts of strongly finitary functors are strongly finitary.

We prove that $\coprod_{J} A$ is not finitely presentable for any infinite set $J$. In fact, $\coprod_{J} A$ is a filtered colimit of $\coprod_{I} A$ for all finite sets $I \subseteq J$. If $\coprod_{J} A$ were finitely presentable, some of the colimit maps, that is, some canonical map

$$c_{I} : \coprod_{I} A \longrightarrow \coprod_{J} A$$ (I finite)

would be a split epimorphism (because the identity of $\coprod_{J} A$ would factor through $c_{I}$, by definition of finite presentability). However, $c_{I}$ is not an epimorphism: choose distinct morphisms $f, g \in \mathcal{A}_{fp}(A, B)$ and let $\bar{f}, \bar{g} : \coprod_{J} A \longrightarrow B$ be morphisms whose components are equal for all indices in $I$, but whose $j$-components (for some $j \in J \setminus I$) are $f$ and $g$, respectively. Then $\bar{f} \neq \bar{g}$ but $\bar{f}c_{I} = \bar{g}c_{I}$. \qed
Corollary 2.16. Suppose that \( \mathcal{A} \) is a strongly LFP category. Every finitary endofunctor \( H \) preserving monomorphisms can be expressed as a filtered colimit of strongly finitary functors preserving monomorphisms.

Proof. Use Proposition 2.15 and express \( H \) as a filtered colimit

\[
H = \text{colim}_d H_d
\]

of strongly finitary functors and use \( \alpha_d : H_d \to H \) to denote a colimit cocone. The category \( \text{Fin}[\mathcal{A}, \mathcal{A}] \simeq [\mathcal{A}_{fp}, \mathcal{A}] \) of finitary endofunctors of \( \mathcal{A} \) is LFP, therefore, it has (StrongEpi, Mono)-factorisations of morphisms, see Remark 2.10. Moreover, both strong epimorphisms and monomorphisms are defined component-wise in any functor category. We can factor each \( \alpha_d : H_d \to H \) in \( \text{Fin}[\mathcal{A}, \mathcal{A}] \) as a strong epimorphism followed by a monomorphism:

\[
\begin{array}{ccc}
H_d & \xrightarrow{\alpha_d} & H \\
\downarrow{\varepsilon_d} & & \downarrow{\nu_d} \\
K_d & \xrightarrow{v_d} & H
\end{array}
\]

Since each \( \varepsilon_d \) is pointwise a strong epimorphism, every functor \( K_d \) is strongly finitary: given a finitely presentable object \( X \), we have \( K_dX \) is a strong quotient of a finitely presentable object \( H_dX \) (we know that \( H_d \) is strongly finitary), thus, \( K_dX \) is finitely presentable.

It is easy to see that the cocone \( (\nu_d : K_d \to H) \) is a filtered colimit, this follows, once again from the fact that \( \varepsilon_d : H_d \to K_d \) are (strong) epimorphisms.

It remains to show that every functor \( K_d \) preserves monomorphisms. Let \( m : X \to Y \) be a monomorphism. From commutativity of the square

\[
\begin{array}{ccc}
HX & \xrightarrow{Hm} & HY \\
\downarrow{v_d} & & \downarrow{v_d} \\
K_dX & \xrightarrow{K_dm} & K_dY
\end{array}
\]

it follows that \((v_d)_Y \cdot K_dm \) is a monomorphism, thus, \( K_dm \) is a monomorphism.

\[\square\]

3. Some properties of solutions

Assumptions 3.1. We use \( H \) to denote a finitary, monomorphism preserving endofunctor of a strongly LFP category \( \mathcal{A} \).

Notice that every finitary functor is iterable, which implies (as in the case \( \mathcal{A} = \text{Set} \) mentioned in Section 1.2 above) that final coalgebras

\[
TX
\]
of \( H(\_)+X \) form a ‘completely iterative’ monad \( T \) (Aczel et al. 2001). As in Section 1.2, \( T \) is a coproduct \( T = HT + Id \) with injections

\[
\tau : HT \to T \quad \text{and} \quad \eta : Id \to T.
\]
Notation 3.2. Put
\[ \tau^* \equiv H \xrightarrow{H\eta} HT \xrightarrow{\tau} T. \]
Observe that \( \tau, \eta \) and \( \tau^* \) have monomorphic components (because \( \mathcal{A} \) has monomorphic coproduct injections, and \( H \) preserves monomorphisms).

The concept of a flat system of equations as introduced in the Introduction (see Sections 1.1 and 1.3) immediately generalises as follows.

Definition 3.3. We define a flat equation morphism to be a morphism
\[ e : X \rightarrow HX + Y, \]
that is, a coalgebra of \( H(\_)+Y \). It is called finitary if \( X \) is finitely presentable.

Remark 3.4. Every flat equation morphism \( e : X \rightarrow HX + Y \) yields a guarded equation morphism in the sense of Section 1.3 by composing with the following monomorphism
\[ m_{X,Y} \equiv HX + Y \xrightarrow{\tau^*+Y} TX + Y \xrightarrow{T[\text{inl},\eta_{X+Y} \cdot \text{inr}]} T(X + Y). \]
We also write, whenever there is no danger of confusion,
\[ e^\dagger : X \rightarrow TY \]
for the unique solution of the corresponding guarded equation. Thus \( e^\dagger \) is the unique morphism such that the square
\[
\begin{array}{c}
X \\
\downarrow e \\
HX + Y \\
\downarrow m_{X,Y} \\
T(X + Y) \\
\downarrow \mu_Y \\
T^2 Y
\end{array}
\xrightarrow{\mu_{X+Y} \cdot e^\dagger} T[X+Y] \xrightarrow{T[e^\dagger,\eta_Y]} TY
\]
commutes. The following proposition was proved in Aczel et al. (2002).

Proposition 3.5 (Solution is Corecursion). For every flat equation morphism \( e : X \rightarrow HX + Y \), the (unique) solution \( e^\dagger : X \rightarrow TY \) is precisely the unique homomorphism of the coalgebra \( e : X \rightarrow HX + Y \) into the final coalgebra \( TY \) of \( H(\_)+Y \).

Remark 3.6. We also need a simple result concerning general iterative monads, see Definitions 1.13 and 1.16.

Let \( S \) be an iterative monad. An equation morphism \( e : X \rightarrow S(X + Y) \) that factors through \( \sigma_{X+Y} \) is, of course, guarded. Moreover, the unique solution \( e^\dagger : X \rightarrow SY \) factors through \( \sigma_Y \): if \( e = \sigma_{X+Y} \cdot e' \), then
\[ e^\dagger = \mu_Y \cdot S[e^\dagger,\eta_Y] \cdot \sigma_{X+Y} \cdot e' = \sigma_Y \cdot (\mu'_Y \cdot S'[e^\dagger,\eta_Y] \cdot e'). \]
**Lemma 3.7.** Let $S$ be an iterative monad and let $e : X \to S(X + Y)$ and $e' : X' \to S(X' + Y)$ be guarded equation morphisms. For every morphism $h : X \to X'$ such that the square

$$
\begin{array}{ccc}
X & \xrightarrow{e} & S(X + Y) \\
\downarrow{h} & & \downarrow{S(h + Y)} \\
X' & \xrightarrow{e'} & S(X' + Y)
\end{array}
$$

commutes, we have

$$e^\dagger = (e')^\dagger \cdot h.$$

**Proof.** The diagram

$$
\begin{array}{ccc}
X & \xrightarrow{h} & X' & \xrightarrow{(e')^\dagger} & SY \\
\downarrow{e} & & \downarrow{e'} & & \downarrow{\mu_Y} \\
S(X + Y) & \xrightarrow{S(h + Y)} & S(X + Y) & \xrightarrow{S[(e')^\dagger, \eta]} & SSY \\
& & & \xrightarrow{S[(e')^\dagger, \eta]} & \\
& & & & SSY
\end{array}
$$

obviously commutes. \(\square\)

**Remark 3.8.**

(i) Returning to flat equation morphisms, the corresponding statement here is a direct corollary of Proposition 3.5: given flat equation morphisms $e : X \to HX + Y$ and $e' : X' \to HX' + Y$, we have, for every homomorphism $h : X \to X'$ of those coalgebras (of $H(\_)+Y$), that

$$e^\dagger = (e')^\dagger \cdot h.$$ 

(ii) In particular, when both $e^\dagger$ and $(e')^\dagger$ happen to be monomorphisms (that is, subobjects of $TY$), then the existence of a homomorphism $h : e \to e'$ implies $e^\dagger \subseteq (e')^\dagger$. The converse is also true: if $e^\dagger \subseteq (e')^\dagger$, then the unique $h : X \to X'$ with $e^\dagger = (e')^\dagger h$ is a homomorphism. In fact, consider the following diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{e} & HX + Y \\
\downarrow{h} & & \downarrow{H(h + Y)} \\
X' & \xrightarrow{e'} & HX' + Y & \xrightarrow{He' + Y} & H(TY + Y)
\end{array}
$$

The outer and lower squares commute by Proposition 3.5. Hence the upper square commutes when extended by $H(e')^\dagger + Y$, which is a monomorphism since $H$ preserves monomorphisms (by assumption) and monomorphisms are closed under coproducts. Thus $h : e \to e'$ is a homomorphism, as desired.
4. Rational monad – strongly finitary case

Assumptions 4.1. Throughout this section, \( \mathcal{A} \) denotes a strongly LFP category, and \( H \) an endofunctor that preserves monomorphisms and is strongly finitary, that is, not only preserves filtered colimits but also finite presentability (if \( X \) is a finitely presentable object, then so is \( HX \)).

In the next section we extend the results to all finitary endofunctors preserving monomorphisms.

Recall from Remark 2.10 that the category \( \mathcal{A} \) has (StrongEpi, Mono)-factorisations of morphisms. We use \( \text{im}(f) \) to denote the monomorphic part of the factorisation of \( f \).

The following definition generalises the definition of the algebra \( R_\Sigma Y \) of all rational trees as solutions of systems of finitary flat equations, see Section 1.1.

Definition 4.2. For every finitely presentable object \( Y \) of \( \mathcal{A} \), we define

\[
RY = \bigcup \text{im}(e^\dagger) \quad (e \text{ a finitary flat equation morphism})
\]

that is, we define an object \( RY \) together with a monomorphism

\[
\varepsilon_Y : RY \longrightarrow TY
\]

to be the subobject of \( TY \) that is the union of images of all solutions \( e^\dagger : X \longrightarrow TY \) of finitary flat equation morphisms \( e : X \longrightarrow HX + Y \) (\( X \) finitely presentable).

Remark 4.3. Explicitly, for every finitary flat equation morphism \( e : X \longrightarrow HX + Y \), we factor \( e^\dagger : X \longrightarrow TY \) as a strong epimorphism \( k : X \longrightarrow X' \) followed by a monomorphism \( m : X' \longrightarrow TY \). Then \( \varepsilon_Y \) is the union of all those monomorphisms \( m \). Union here means just the supremum in the lattice of subobjects, but this is actually a colimit, as we see below.

Observe that each of the images \( m \) is, itself, a solution of a finitary flat equation morphism \( e' : X' \longrightarrow HX' + Y \). In fact, since finitely presentable objects are closed under strong quotients, \( X' \) is finitely presentable, and, since \( Hm \) is a monomorphism (by assumption on \( H \)), so is \( Hm + Y \) (since coproducts are extensive, and thus monomorphisms are stable under finite coproducts, see Section 2.3). Thus we can use the diagonal fill-in property to find a coalgebra structure \( e' \) on \( X' \) such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{e} & HX + Y \\
\downarrow k & & \downarrow Hk + Y \\
X' & \xrightarrow{e'} & HX' + Y \\
\downarrow m & & \downarrow Hm + Y \\
TY & \longrightarrow & HTY + Y
\end{array}
\]

commutes. Now \( m \) is a coalgebra homomorphism, and it follows from Proposition 3.5 that \( m = (e')^\dagger \).
**Notation 4.4.** For every finitely presentable object $Y$ of $\mathcal{A}$, we use

$$\text{EQ}_Y$$

to denote the category of all finitary flat equations with the parameter object $Y$. That is:

- objects are all finitary flat equation morphisms $e : X \to HX + Y$ ($X$ in $\mathcal{A}_{fp}$)

and

- morphisms are coalgebra homomorphisms with respect to $H(\_)+Y$:

$$\begin{array}{ccc}
X & \xrightarrow{e} & HX + Y \\
\downarrow{h} & & \downarrow{Hh+Y} \\
X' & \xrightarrow{e'} & HX' + Y
\end{array}$$

The category $\text{EQ}_Y$ comes with a forgetful functor into the underlying category $\mathcal{A}$. We denote it by

$$\text{Eq}_Y : \text{EQ}_Y \to \mathcal{A}, \quad (X \xrightarrow{e} HX + Y) \mapsto X.$$

**Remark 4.5.** Recall that a full subcategory $\mathcal{D}_0$ of a filtered category $\mathcal{D}$ is cofinal if every object of $\mathcal{D}$ has a morphism into an object of $\mathcal{D}_0$; it follows that (in every category) colimits of $\mathcal{D}$-diagrams coincide with colimits of the restricted $\mathcal{D}_0$-diagrams.

**Proposition 4.6.** For every finitely presentable object $Y$, $\text{Eq}_Y$ is a small, filtered diagram, and $RY$ can be defined as a colimit

$$RY = \operatorname{colim} \text{Eq}_Y.$$

**Proof.** First, notice that $\text{EQ}_Y$ is the full subcategory of $\text{Coalg}(H(\_)+Y)$ of those coalgebras with carrier from $\mathcal{A}_{fp}$. Since $\mathcal{A}_{fp}$ is small and for each object $X$ of $\mathcal{A}$ there is only a set of morphisms $e : X \to HX + Y$, it follows that $\text{EQ}_Y$ is a small category. Moreover, observe that $\text{EQ}_Y$ is filtered because it is finitely cocomplete: in fact, the category of coalgebras has colimits formed on the level of $\mathcal{A}$, and since finite colimits of finitely presentable objects are finitely presentable, it follows that $\text{EQ}_Y$ is closed under finite colimits in $\text{Coalg}(H(\_)+Y)$. Remark 4.3 shows that the filtered diagram $\text{Eq}_Y$ has a cofinal subdiagram formed by all finitary flat equation morphisms with monomorphic solutions. (In fact, for every object $e$ of $\text{Eq}_Y$ we have constructed a morphism $k : e \to e'$ in $\text{Eq}_Y$ with $e'$ having monomorphic solution $m$.)

Now we have defined $RY$ as the union of all $(e')^\dagger = m : X' \to TY$, and since this union is filtered, it is a colimit of the corresponding filtered diagram (whose morphisms are all existing subobject inclusions, or, equivalently, all morphisms of $\text{Eq}_Y$, see Remark 3.8(ii)); this is true in every LFP category, see Adámek and Rosický (1994, 1.63). That diagram is cofinal in $\text{Eq}_Y$, whence $\text{Eq}_Y$ has the same colimit, that is, $RY = \operatorname{colim} \text{Eq}_Y$. $\Box$
Notation 4.7.
(i) We use $e^♯ : X \rightarrow RY$ (for all $e : X \rightarrow HX + Y$ in $EQ_Y$) to denote the colimit cocone of $EQ_Y$.
(ii) Observe that for any finitely presentable object $Y$ of $\mathcal{A}$, $\text{inr} : Y \rightarrow HX + Y$ is an object of $EQ_Y$. We use $\eta^R_Y : Y \rightarrow RY$ to denote the colimit morphism $\text{inr}^♯$.

Example 4.8. Let $H = H_{\Sigma}$ be a polynomial endofunctor of $\text{Set}$. Then $RY$ is the algebra of all rational trees over $Y$, and for every finitary flat equation system $e : X \rightarrow H_{\Sigma}X + Y$ the colimit morphism $e^♯ : X \rightarrow RY$ is a codomain restriction of the solution $e^\dagger : X \rightarrow TY$.

Remark 4.9. Notice that for every finitely presentable object $Y$ of $\mathcal{A}$ the monomorphism $\varepsilon_Y : RY \rightarrow TY$ of Definition 4.2 makes the triangles

\[
\begin{array}{ccc}
X & \xrightarrow{\varepsilon^♯} & RY \\
\varepsilon & \downarrow & \varepsilon_Y \\
RY & \rightarrow & TY
\end{array}
\]

commutative for all $e : X \rightarrow HX + Y$ in $EQ_Y$. This is obvious if $e^\dagger$ is monomorphic; otherwise use the fact established in the proof of Proposition 4.6 that the finitary flat equation morphisms with monomorphic solutions form a cofinal subcategory of $EQ_Y$.

This makes $\varepsilon_Y$ uniquely determined by the equations $\varepsilon_Y e^♯ = e^\dagger$.

Proposition 4.10. $(RY = HRY + Y)$. For every finitely presentable object $Y$ of $\mathcal{A}$ the diagram $EQ_Y$ has the colimit

$$\text{colim } EQ_Y = HRY + Y$$

with the following colimit cocone

$$
\begin{array}{ccc}
X & \xrightarrow{e} & HX + Y \\
& \downarrow & \downarrow_{He^♯ + Y} \\
& HX + Y & \rightarrow \rightarrow \rightarrow \rightarrow \\
& \rightarrow \rightarrow HRY + Y
\end{array}
$$

for $e : X \rightarrow HX + Y$ in $EQ_Y$)

Proof.

(i) The morphisms $\delta_e = (He^♯ + Y) \cdot e$ form a cocone of $EQ_Y$, that is, for every morphism

$$
\begin{array}{ccc}
X & \xrightarrow{e} & HX + Y \\
& \downarrow h & \downarrow Hh + Y \\
X' & \xrightarrow{e'} & HX' + Y
\end{array}
$$

of $EQ_Y$ we have

$$\delta_e = \delta_{e'} \cdot h.$$
Free iterative theories: a coalgebraic view

Indeed, since $h$ is a morphism, we know that

$$e^\sharp = (e')^\sharp \cdot h.$$ 

Therefore, the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{e} & HX + Y \\
\downarrow{h} & & \downarrow{He^\sharp + Y} \\
X' & \xrightarrow{e'} & HX' + Y \\
\end{array}
\]

commutes.

(ii) In order to complete the proof, we shall show that the unique morphism

$$i : RY \longrightarrow HRY + Y \quad \text{with} \quad i \cdot e^\sharp = \delta_e \quad \text{(for} \quad e \in \text{EQ}_Y)$$

is an isomorphism. Define $j : HRY + Y \longrightarrow RY$ as follows. Since $H$ preserves filtered colimits, we have

$$HRY = \colim H\text{Eq}_Y,$$

and therefore

$$HRY + Y = \colim(H\text{Eq}_Y + Y)$$

with colimit cocone $He^\sharp + Y$ ($e$ in $\text{EQ}_Y$). Define for every

$$e : X \longrightarrow HX + Y \quad \text{in} \quad \text{EQ}_Y$$

a new member

$$e_0 \equiv HX + Y \xrightarrow{He^\sharp + Y} H(HX + Y) + Y$$

de $\text{EQ}_Y$. Note that $HX + Y$ lies in $\mathcal{A}_{fp}$ since $H$ is strongly finitary and $\mathcal{A}_{fp}$ is closed under coproducts.

Define a morphism $j : HRY + Y \longrightarrow RY$ by the commutativity of the following triangles:

\[
\begin{array}{ccc}
HRY + Y & \xrightarrow{j} & RY \\
\uparrow{He^\sharp + Y} & & \uparrow{e_0^\sharp} \\
HX + Y & & \\
\end{array}
\]

for all $e$ in $\text{EQ}_Y$.

The morphism $j$ is well-defined since $e_0^\sharp$ (for $e$ in $\text{EQ}_Y$) form a cocone. In fact, for every morphism $h : e \longrightarrow e'$ in $\text{EQ}_Y$, it is easy to show that $Hh + Y : e_0 \longrightarrow e_0'$ is also a morphism of $\text{EQ}_Y$. But then

$$e_0^\sharp = (e_0')^\sharp \cdot (Hh + Y),$$

as desired.

(ii a) Proof of $j \cdot i = id$: It is our task to show that

$$jie^\sharp = e^\sharp \quad \text{for all} \quad e \in \text{EQ}_Y.$$
For this observe that $e$ is a morphism in $\text{EQ}_Y$ from $e$ to $e_0$. Therefore

$$j \cdot i \cdot e^\# = j \cdot (He^\# + Y) \cdot e = e_0^\# \cdot e = e^\#.$$

(ii b) Proof of $i \cdot j = \text{id}$: Here we must show that

$$i \cdot j \cdot (He^\# + Y) = He^\# + Y \quad \text{for all } e \in \text{EQ}_Y.$$

But this is again easily seen:

$$i \cdot j \cdot (He^\# + Y) = i \cdot e_0^\# = (He_0^\# + Y) \cdot (He + Y) = He^\# + Y$$

where the last step again uses $e : e \longrightarrow e_0$ in $\text{EQ}_Y$. \qed

**Remark 4.11.** We have established that

$$RY = HRY + Y \quad (Y \text{ in } \mathcal{A}_{fp})$$

with coproduct injection given by $j \cdot \text{inl}$ and $j \cdot \text{inr}$. Note that $j \cdot \text{inl} = \eta_Y^R : Y \longrightarrow RY$. Indeed, the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\text{inr}^x = \eta_Y^R} & RY \\
\downarrow \text{inr} & & \downarrow \text{inr} \\
HY + Y & \xrightarrow{j} & HRY + Y \\
\downarrow \text{inr} & & \downarrow \text{inr} \\
H_\eta_Y^R + Y & \xleftarrow{j} & HY + Y \\
\end{array}
\]

commutes.

**Notation 4.12.** From now on we shall use for any finitely presentable object $Y$ the following notation for the coproduct injections:

$$\rho_Y : HRY \longrightarrow RY \quad \text{and} \quad \eta_Y^R : Y \longrightarrow RY.$$

**Corollary 4.13.** Let $Y$ be a finitely presentable object. For every finitary flat equation morphism $e : X \longrightarrow HX + Y$, the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{e} & HX + Y \\
\downarrow \epsilon^\# & & \downarrow \text{He}^\# + Y \\
RY & = & HRY + Y \\
\end{array}
\]

commutes.

**Proof.** In fact, to show that

$$e^\# = [\rho_Y, \eta_Y^R] \cdot (He^\# + Y) \cdot e,$$
just use the fact that \([\rho_Y, \eta_Y] = j\) and that (since \(e : e \to e_0\) is a morphism in \(\text{EQ}_Y\)),
\[ \epsilon^e = e_0^\sharp \cdot e \cdot e. \]
Thus,
\[ \epsilon^e = e_0^\sharp \cdot e = \rho_Y \cdot e \cdot (H \epsilon^e + Y) \cdot e. \]

\[ \fbox{} \]

**Remark 4.14.**

(i) For any finitely presentable object \(Y\), we have
\[ \epsilon_Y = H \epsilon_Y + Y. \]

More precisely, if \(i : RY \to HRY + Y\) denotes the isomorphism of Proposition 4.10,
then its composite with \(H \epsilon_Y + Y : HRY + Y \to HTY + Y = TY\) is equal to \(\epsilon_Y\).
This follows from the equalities
\[ \epsilon_Y \cdot e^e = (H \epsilon_Y + Y) \cdot i \cdot e^e \quad (e \in \text{EQ}_Y). \]

In fact, consider the following diagrams:

The upper square commutes by definition of \(i\), the two outer triangles by Remark 4.9.
The outer square commutes since the solution \(e^\dagger\) of \(e\) is given by corecursion, see
Proposition 3.5. Thus, the lower square commutes, too.

(ii) Consequently, \(\epsilon_Y\) is a homomorphism of \(H\)-algebras:
\[ \epsilon_Y \cdot \rho_Y = \tau_Y \cdot H \epsilon_Y \]
with
\[ \epsilon_Y \cdot \eta_Y^R = \eta_Y. \quad (5) \]

In fact, \(\epsilon_Y \cdot \eta_Y^R = \text{inr}^\dagger\) for \(\text{inr} : Y \to HY + Y\), and it is easy to verify that \(\text{inr}^\dagger = \eta_Y\).

**Corollary 4.15.** For every finitely presentable object \(Y\), the square
\[
\begin{array}{ccc}
HRY & \xrightarrow{H \epsilon_Y} & HTY \\
\rho_Y & & \downarrow \epsilon_Y \\
RY & \xrightarrow{\epsilon_Y} & TY
\end{array}
\]
is a pullback.
Proof. In fact, since $\mathcal{A}$ is extensive, for every morphism $f : A \to B$ the squares

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{\text{inl}} & & \downarrow{\text{inl}} \\
A + Y & \xrightarrow{f + Y} & B + Y
\end{array}
$$

are pullbacks. Apply this to $f = H\varepsilon_Y$. \qed

**Theorem 4.16 (Rational Substitution Theorem).** For every morphism

$$
s : X \to RY \quad (X, Y \text{ finitely presentable}),
$$

there exists a unique extension to a homomorphism

$$
\widetilde{s} : RX \to RY
$$

of $H$-algebras (that is, a unique homomorphism with $s = \widetilde{s} \cdot \eta_Y^R$).

**Proof.** (1) **Existence:** We are going to find a morphism $\widetilde{s}$ such that the square

$$
\begin{array}{ccc}
RX & \xrightarrow{\widetilde{s}} & RY \\
\downarrow{\varepsilon_X} & & \downarrow{\varepsilon_Y} \\
TX & \xrightarrow{\varepsilon_Y \cdot \widetilde{s}} & TY
\end{array}
$$

(6)

commutes. It follows from Remark 4.14(ii) and Theorem 1.7 that

$$
\varepsilon_Y \cdot \widetilde{s} \cdot \eta_Y^R = \varepsilon_Y \cdot \widetilde{s} \cdot \eta_X = \varepsilon_Y \cdot s
$$

which, since $\varepsilon_Y$ is a monomorphism by definition, proves that $\widetilde{s}$ extends $s$. It also follows that $\widetilde{s}$ is a homomorphism of $H$-algebras: to conclude $\widetilde{s} \cdot \rho_X = \rho_Y \cdot H\widetilde{s}$, use the fact that $\varepsilon_Y$ is a monomorphism and that the diagram

$$
\begin{array}{ccc}
HRX & \xrightarrow{H\widetilde{s}} & HRY \\
\downarrow{\rho} & & \downarrow{\rho} \\
RX & \xrightarrow{\varepsilon_Y} & RY
\end{array}
\quad \begin{array}{ccc}
RX & \xrightarrow{\varepsilon_X} & RY \\
\downarrow{\varepsilon} & & \downarrow{\varepsilon} \\
TX & \xrightarrow{\varepsilon_Y \cdot \widetilde{s}} & TY
\end{array}
\quad \begin{array}{ccc}
TX & \xrightarrow{\varepsilon_X} & TY \\
\downarrow{\varepsilon} & & \downarrow{\varepsilon} \\
HTX & \xrightarrow{H\varepsilon_Y} & HTY
\end{array}
$$

commutes (because the inner square is the above square (6), the outer one is its image and $\varepsilon_Y \cdot s$, $\varepsilon_X$ and $\varepsilon_Y$ are homomorphisms of $H$-algebras).

In order to define $\widetilde{s}$, use the fact that $RY = \operatorname{colim} \operatorname{Eq}_Y$ is a filtered colimit and $X$ is finitely presentable. Thus, there exists an object

$$
f : V \to HV + Y \quad \text{of } \operatorname{Eq}_Y
$$
such that \( s \) factors through its colimit morphism:

\[
\begin{array}{ccc}
X & \xrightarrow{s} & RY \\
\downarrow^{s'} & & \downarrow^{f^\sharp} \\
V & \rightarrow & RY
\end{array}
\]

This allows us to define the desired morphism \( \tilde{s} : \text{colim} \text{Eq}_X \rightarrow RY \) by providing a 'suitable' cocone of the diagram Eq\(_X\) as follows: Every object of Eq\(_X\), say, \( e : Z \rightarrow HZ + X \), will be turned into an object \( \bar{e} \) of Eq\(_Y\) by adding \( V \) (the domain of \( f \)) as new variables: \( \bar{e} : Z + V \rightarrow H(Z + V) + Y \). In more detail, for every object

\[ e : Z \rightarrow HZ + X \quad \text{of Eq}_X \]

we use

\[ \bar{e} : Z + V \rightarrow H(Z + V) + Y \]

to denote the object of Eq\(_Y\) with the following components:

\[
\begin{array}{c}
\bar{e} \cdot \text{inl} \equiv Z \xrightarrow{e} HZ + X \xrightarrow{HZ + f^\sharp} HZ + HV + Y \xrightarrow{[H\text{inl},H\text{inr}]+Y} H(Z + V) + Y \\
\bar{e} \cdot \text{inr} \equiv V \xrightarrow{f} HV + Y \xrightarrow{H\text{inr}+Y} H(Z + V) + Y.
\end{array}
\] (7)

We shall show below that the morphisms

\[
\begin{array}{ccc}
Z & \xrightarrow{\text{inl}} & Z + V \\
\downarrow & & \downarrow^{e^\sharp} \\
\bar{e} & \rightarrow & RY
\end{array}
\] (e in Eq\(_X\))

form a cocone of Eq\(_X\). This establishes the unique

\[
\tilde{s} : RX \rightarrow RY \quad \text{with} \quad \tilde{s} \cdot e^\sharp = \bar{e}^\sharp \cdot \text{inl} \quad (e \text{ in Eq}_X)
\] (9)

This morphism \( \tilde{s} \) fulfills

\[
\varepsilon_Y \cdot \tilde{s} \cdot e^\sharp = (\varepsilon_Y \cdot \bar{e}^\sharp) \cdot \text{inl} = \bar{e}^\dagger \cdot \text{inl}.
\] (10)

We finally establish that

\[
\bar{e}^\dagger \cdot \text{inl} = \varepsilon_Y \cdot \tilde{s} \cdot e^\dagger = \varepsilon_Y \cdot \bar{e} \cdot \varepsilon_X \cdot e^\sharp,
\] (11)

which proves, together with (10), that (6) commutes, since both sides are equal when composed with the injections \( e^\sharp \) of the colimit \( RX \). This concludes the proof.

\textbf{Proof that (8) is a cocone.} Suppose that a morphism of Eq\(_X\) is given:

\[
\begin{array}{ccc}
Z & \xrightarrow{e} & HZ + X \\
h \downarrow & & \downarrow^{Hh+X} \\
Z' & \xrightarrow{e'} & HZ' + X
\end{array}
\]
We are going to prove the commutativity of the following square:

\[
\begin{array}{ccc}
Z + V & \xrightarrow{\bar{e}} & H(Z + V) + Y \\
\downarrow h + V & & \downarrow H(h + V) + Y \\
Z' + V & \xrightarrow{\bar{e}'} & H(Z' + V) + Y
\end{array}
\]

That means that \( h + V \) is a morphism from \( \bar{e} \) to \( \bar{e}' \) in \( \mathcal{EQ}_Y \), proving \( \bar{e}^\sharp = \bar{e}'^\sharp \cdot (h + V) \), which establishes

\[
\bar{e}^\sharp \cdot \text{inl} = \bar{e}'^\sharp \cdot \text{inl} \cdot h,
\]

as desired.

We consider, in (12), the components of \( Z + V \) separately. The equality of the right-hand components of (12) is obvious form the definition of \( \bar{e} \) and \( \bar{e}' \). For the left-hand components, consider the following squares:

\[
\begin{array}{ccc}
Z & \xrightarrow{e} & HZ + X \\
\downarrow h & & \downarrow H(h + X) \\
Z' & \xrightarrow{\bar{e}} & HZ' + X
\end{array}
\]

\[
\begin{array}{ccc}
HZ + X & \xrightarrow{HZ + f^\sharp} & HZ + HV + Y \\
\downarrow [H\text{inl},H\text{inr}] + Y & & \downarrow H(h + V) + Y \\
HZ' + X & \xrightarrow{HZ' + f^\sharp} & HZ' + HV + Y
\end{array}
\]

The left-hand square commutes by assumption and the right-hand one obviously does. Hence, the outer square commutes too.

**Proof of (11).** We shall now show that \( \bar{e}^\dagger = [\hat{\varepsilon}_Y s \cdot e^\dagger, f^\dagger] \), which establishes (11) above. By Proposition 3.5, we have to show that the square

\[
\begin{array}{ccc}
Z + V & \xrightarrow{\bar{e}} & H(Z + V) + Y \\
\downarrow [\hat{\varepsilon}_Y s \cdot e^\dagger, f^\dagger] & & \downarrow H[\hat{\varepsilon}_Y s \cdot e^\dagger, f^\dagger] + Y \\
TY & \xrightarrow{TY} & HTY + Y
\end{array}
\]

commutes.

We consider the components of the coproduct \( Z + V \) separately. For the right-hand component of (13), apply Proposition 3.5 again to obtain

\[
f^\dagger \equiv V \xrightarrow{f} HV + Y \xrightarrow{Hf^\dagger + Y} HTY + Y = TY
\]

thus, the diagram

\[
\begin{array}{ccc}
V & \xrightarrow{f} & HV + Y \\
\downarrow f^\dagger & & \downarrow Hf^\dagger + Y \\
TY & \xrightarrow{HTY + Y} & HTY + Y
\end{array}
\]

commutes.
For the left-hand component of (13), consider the following commutative diagram:

Square (i) commutes once more by Proposition 3.5. For square (ii), use the definition of \( \overline{\epsilon}_Y \), see Substitution Theorem 1.7: we have

\[
\overline{\epsilon}_Y [\tau_X, \eta_X] = [\tau_Y H(\overline{\epsilon}_Y s), \epsilon_Y s] = [\tau_Y, \epsilon_Y s](H(\overline{\epsilon}_Y s) + X).
\]

Square (iii) commutes since

\[
\epsilon_Y \cdot s = \epsilon_Y \cdot f^{\dagger} \cdot s' = f^{\dagger} \cdot s'.
\]  

(14)

The right-hand component of (iv) commutes by Proposition 3.5, and the left-hand one does trivially. The two remaining triangles obviously commute. Hence the outer square commutes as desired.

(II) **Uniqueness:** We are going to prove that for every homomorphism of \( H \)-algebras \( h : RX \to RY \) extending \( s \), the following identity holds:

\[
\overline{\epsilon}_Y [h, RY] = \overline{\epsilon}_Y [\tilde{s}, RY].
\]  

(15)

Since \( \overline{\epsilon}_Y \) is a monomorphism, it follows that \( h = \tilde{s} \). Observe that the diagram

\[
\begin{array}{ccc}
RX + RY & \xrightarrow{HRX + s, RY} & HRX + RY + HRX + HRY + Y \\
\downarrow & & \downarrow \left[H(\text{inl}, \text{inr}) + Y\right] \downarrow \\
RX + X + RY & \xrightarrow{HRX + X + RY} & H(RX + RY) + Y \\
\downarrow [h, RY] & & \downarrow \left[H(h, RY) + Y\right] \\
RY & \xrightarrow{\rho \eta^R} & HRY + Y \\
\uparrow \epsilon & & \uparrow H(\text{inl}, \text{inr}) + Y \\
TY & \xrightarrow{\rho \eta^R} & HTY + Y
\end{array}
\]  

(16)

commutes. In fact, the lower square commutes because \( \tau_Y \cdot H\overline{\epsilon}_Y = \overline{\epsilon}_Y \cdot \rho_Y \) (\( \overline{\epsilon}_Y \) is a homomorphism with \( \epsilon_Y \cdot \eta^R_Y = \eta_Y \) by Remark 4.14(ii)). For the upper square consider the three components of \( HRX + X + RY \): the left-hand component commutes because \( \rho_Y \cdot Hh = h \cdot \rho_X \) (\( h \) is a homomorphism by assumption), the middle component does because \( h \) extends \( s \), that is,

\[
s = h \cdot \eta^R_X,
\]
and the right-hand component commutes trivially. Thus, $\varepsilon_Y [h, RY]$ is a homomorphism into the final coalgebra $TY$. This holds for all $H$-algebra homomorphisms extending $s$, and, in particular, for $\tilde{s}$. By coinduction, we conclude (15). \hfill \Box

**Corollary 4.17.** We obtain a finitary Kleisli triple $(R, \eta^R, (\_))$ on $\mathcal{A}$.

**Proof.** In fact, the axioms (i)–(iii) of Definition 2.4 follow immediately from the uniqueness of $(\_)$:

(i) $\tilde{s} \cdot \eta_X = s$ by definition of $\tilde{s}$.

(ii) $\tilde{\eta}_X = id$ because $id$ is a homomorphism extending $\eta_X$.

(iii) $\tilde{r} \tilde{s} = \tilde{r} s$ because $\tilde{r} \tilde{s}$ is a homomorphism extending $\tilde{r} s$: from $\tilde{s} \eta_X = s$ derive

$$(\tilde{r} \tilde{s}) \eta_X = \tilde{r} s$$

\hfill \Box

**Definition 4.18.** The finitary monad $R$ defined by the above finitary Kleisli triple is called a **rational monad** generated by $H$.

**Notation 4.19.** The unit of the rational monad is denoted by

$$\eta^R : Id \to R,$$

observe that for $Y$ finitely presentable, $\eta^R_Y$ are the above morphisms $inr^\sharp$. We further use

$$\mu^R : RR \to R$$

to denote the multiplication, given by

$$\mu^R_Y = \hat{id}_{RY} : RRY \to RY$$

for finitely presentable $Y$.

The morphisms $\rho_Y : HRY \to RY$ for $Y$ finitely presentable in Notation 4.12 yield a natural transformation

$$\rho : HR \to R.$$ 

In fact, $\mathcal{A}$ is a free cocompletion of $\mathcal{A}_{fp}$ under filtered colimits (Adámek and Rosický 1994), therefore the above natural transformation $\rho_Y : HRY \to RY$ for $Y \in \mathcal{A}_{fp}$ extends uniquely to $\rho : HR \to R$. We have

$$R = HR + Id$$

with the above injections $\rho$ and $\eta^R$. Since coproduct injections are monomorphisms, the components of the last two natural transformations are monomorphisms. And since $H$ preserves monomorphisms, the natural transformation

$$\rho^* \equiv H \xrightarrow{H \eta^R} HR \xrightarrow{\rho} R$$

also has monomorphic components. This is analogous to $\tau^* : H \to T$ in Notation 3.2.

**Remark 4.20.** The fact that $RY$ is a filtered union of all images of solutions of finitary flat equations extends from finitely presentable objects $Y$ to all objects. In fact, both of the formulas

$$RY = \bigcup \text{im}(e^\dagger)$$
and
\[ RY = \text{colim} \, \text{Eq}_Y \]
extend from finitely presentable objects \( Y \) to all objects. Given any object \( Y \) in \( \mathcal{A} \), we use, again,
\[ \text{EQ}_Y \]
to denote the category of all coalgebras of \( H(\,\_\,) + Y \) carried by finitely presentable objects \( X \) and by
\[ \text{Eq}_Y : \text{EQ}_Y \to \mathcal{A} \]
the forgetful functor. This is a filtered diagram and we show that it has colimit
\[ RY = \text{colim} \, \text{Eq}_Y. \]
Express \( Y = \text{colim} \, Y_t \) as a filtered colimit of finitely presentable objects \( Y_t \) with colimit cocone \( y_t : Y_t \xrightarrow{\iota \in I} Y \), \( t \in I \). Then each \( e : X \to HX + Y = \text{colim}(HX + Y_t) \) factors as
\[ X \xrightarrow{e_0} HX + Y_t \xrightarrow{H^X + y_t} HX + Y \]
for some \( t \in I \). We have \( e_0^Y \) as in Notation 4.7(i). Since \( R \) is finitary, we have \( RY = \text{colim} \, RY_t \) with colimit injections \( R_y_t : RY_t \to RY \). We put
\[ e^\# = X \xrightarrow{e_0^Y} RY_t \xrightarrow{R_y_t} RY \]
This is independent of the above factorisation of \( e \). To verify this, we use the fact that the diagram \( (Y_t) \) is filtered: it is sufficient to prove that for every connecting morphism \( y_{t,t'} : Y_t \to Y_{t'} \), if \( e_t \) is defined by the commutative triangle
\[ X \xrightarrow{e_0} HX + Y_t \xrightarrow{H^X + y_t} HX + Y_{t'} \]
we have
\[ e^\# = X \xrightarrow{e^t_0} RY_{t'} \xrightarrow{R_y_{t'}} RY. \]
In fact, since \( R_y_{t'} = R_{y_{t,t'}} \cdot R_y_t \), it is sufficient to prove
\[ e^t_0 = R_{y_{t,t'}} \cdot e_0^Y. \]
Recall that
\[ R_{y_{t,t'}} = \tilde{s} \]
for
\[ s = Y_t \xrightarrow{y_{t,t'}} Y_{t'} \xrightarrow{\eta^R} RY_t. \]
By (9),
\[ R_{y_{t,t'}} \cdot e_0^Y = \tilde{e}^\# \cdot \text{inl} \]
where $\bar{e}$ has, by (7), the following components:

$$X \xrightarrow{e_0} HX + Y_t \xrightarrow{HX + \text{inr} \cdot Y_t} HX + HY_{t'} + Y_t' \xrightarrow{(H\text{inl} + Y_{t'}) \cdot (HX + y_{t'})} H(Y + Y_{t'}) + Y_{t'}$$

Consequently, the diagram

$$X \xrightarrow{e_0} HX + Y_t \xrightarrow{HX + \text{inr} \cdot Y_t} HX + HY_{t'} + Y_t' \xrightarrow{[H\text{inl} \cdot \text{inr}] + Y_{t'}} H(X + Y_{t'}) + Y_{t'}$$

commutes. This proves that inl is a morphism of $\text{EQ}_{Y_t}$, thus, $\bar{e} \cdot \text{inl} = e_1$, which together with (19) proves (18).

It is not difficult to see that the morphisms $e\bar{e}$ form a cocone of $\text{EQ}_{Y_t}$. It follows from Proposition 4.6 that this is a colimit cocone. In fact, given a cocone $(c_e : X \rightarrow C)_{e : X \rightarrow HX + Y}$ of $\text{EQ}_{Y_t}$, we obtain a cocone of $\text{EQ}_{Y_{t'}}$ (for each $t$) by assigning to every $e_0 : X \rightarrow HX + Y_t$ the morphism $c_e$, where $e = (HX + y_t) e_0$. This yields a unique $f_t : RY_{t'} \rightarrow C$, and we obtain a cocone of $(RY_{t'})_{t}$, which uniquely factors by way of $f : RY \rightarrow C$. Then $f e\bar{e} = c_e$ and $f$ is unique with this property. Consequently, $RY = \text{colim} \text{EQ}_{Y_t}$.

The fact that this implies $RY = \bigcup \text{im}(e^t)$ is proved exactly as in Proposition 4.6.

**Lemma 4.21.** For every finitary flat equation morphism $e : X \rightarrow HX + Y$ the diagram

$$X \xrightarrow{e} HX + Y \xrightarrow{He\bar{e} + Y} HRy + Y$$

commutes.

**Proof.** See Corollary 4.13 for finitely presentable objects $Y$. The extension to arbitrary objects $Y$ (by way of $e\bar{e} = Ry_0 \cdot e_0$ above) is routine.

**Remark 4.22.** $R$ is a submonad of $T$ (that is, a subobject in the category of all monads and homomorphisms).

Indeed, we use $e : T_0 \rightarrow T$ to denote a finitary coreflection of $T$, see Corollary 2.8. From (5) and (6) above, the morphisms $\varepsilon_Y$ form a homomorphisms of finitary
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Kleisli-triples, or, equivalently, a homomorphism of finitary monads. Hence we obtain a
monad homomorphism \( \varepsilon = c \cdot \varepsilon_0 : R \rightarrow T \). It is easy to check that for every finitary flat
equation morphism \( e : X \rightarrow HX + Y \) (where \( Y \) is any object of \( \mathcal{A} \)), we have
\[
\varepsilon_Y e^\sharp = e^\flat.
\]

Just use the fact that \( HX + Y \) is a filtered colimit of \( HX + Y' \) for all \( Y' \rightarrow Y \) in \( \mathcal{A}_{fp}/Y \),
and since \( X \) is finitely presentable, \( e \) factors through some \( e' : X \rightarrow HX + Y' \), \( Y' \in \mathcal{A}_{fp} \).
Thus \( \varepsilon_Y \) is a monomorphism since Remark 4.20 shows that \( RY \) is the union of images of
solutions of finitary flat equation morphisms.

We are going to prove that the rational monad is iterative (see Definition 1.16). The main
part of this is the following solution theorem, dealing with rational equation morphisms,
that is, morphisms
\[
e : X \rightarrow R(X + Y) \quad (X \text{ finitely presentable}),
\]
which are guarded, that is, factor through the coproduct injection of \( R(X + Y) = HR (X + Y) + X + Y \):

\[
\begin{array}{ccc}
X & \xrightarrow{e} & R(X + Y) \\
& & \downarrow \rho_{X+Y} \circ q_{X+Y} \cdot \text{inr} \\
& & HR(X + Y) + Y
\end{array}
\]

Theorem 4.23 (Rational Solution Theorem). Every rational equation morphism \( e : X \rightarrow R(X + Y) \) has a unique solution. That is, there exists a unique morphism
\[
e^\sharp : X \rightarrow RY
\]
such that the triangle

\[
\begin{array}{ccc}
X & \xrightarrow{e^\sharp} & RY \\
& & \downarrow \rho_{X+Y} \circ q_{X+Y} \cdot \text{inr} \\
& & HR(X + Y) + Y
\end{array}
\]

commutes.

Proof. (1) Existence: Since \( e \) is guarded we have a factorisation

\[
\begin{array}{ccc}
X & \xrightarrow{e} & R(X + Y) \\
& & \downarrow \rho_{X+Y} \circ q_{X+Y} \cdot \text{inr} \\
& & HR(X + Y) + Y
\end{array}
\]

From the filtered colimit \( RY = \text{colim} \mathcal{E}q_Y \) we obtain a filtered colimit
\[
HR(X + Y) + Y = \text{colim} H\mathcal{E}q_{X+Y} + Y,
\]
since $H(\_)+Y$ preserves filtered colimits. It follows that $e_0$, whose domain is finitely presentable, factors through some colimit arrow $Hg^\sharp + Y$, that is, there exists an object

$$g : W \to HW + X + Y \text{ of } \text{EQ}_{X+Y}$$

and a factorisation

$$
\begin{array}{c}
X \xrightarrow{e_0} HR(X + Y) + Y \\
\downarrow \quad \downarrow w \\
HW + Y \\
\end{array}
$$

(22)

This defines an object

$$h \equiv W + X \xrightarrow{[g,inm]} HW + X + Y \xrightarrow{[\text{inl},w,inr]} HW + Y \xrightarrow{H\text{inl} + Y} H(W + X) + Y$$

of $\text{EQ}_Y$, since $W$ and $X$ are finitely presentable. (Here, $\text{inm}$ denotes the injection into the middle summand.) We define

$$e^\dagger \equiv X \xrightarrow{\text{inr}} W + X \xrightarrow{h^\sharp} RY$$

and prove that (20) commutes. For that, consider the solution $h^\dagger : W + X \to TY$ of $h$.

Note that since $e$ is guarded for $R$, the equation morphism

$$
\begin{array}{c}
X \xrightarrow{e} R(X + Y) \xrightarrow{\varepsilon} T(X + Y) \\
\downarrow \varepsilon_0 \quad \downarrow \quad \downarrow \quad \downarrow \text{[\rho,\eta^{R}:\text{inr}]} \\
HR(X + Y) + Y \xrightarrow{H\varepsilon + Y} HT(X + Y) + Y \\
\end{array}
$$

(23)

is guarded for $T$. In fact, the above square commutes, because $\varepsilon_{X+Y}$ is a homomorphism with $\varepsilon_{X+Y} \cdot \eta_{X+Y}^R = \eta_{X+Y}$, see Remark 4.14(ii). We show that

$$(\varepsilon_{X+Y} \cdot e)^\dagger = h^\dagger \cdot \text{inr},$$

(24)

that is

$$(\varepsilon_{X+Y} \cdot e)^\dagger = \varepsilon_Y \cdot h^\sharp \cdot \text{inr}.$$
commutes. Consequently, the upper left-hand square also commutes (which is the desired equality (20)) because the other inner parts of that diagram clearly commute (recall that \( \varepsilon \) is a monad morphism, see Remark 4.22), and \( \varepsilon \) is a monomorphism.

**Proof of (24).** We shall show that

\[
h^\dagger = [\mu_Y T[(\varepsilon_{X+Y}^\dagger, \eta_Y)^g, (\varepsilon_{X+Y}^\dagger)],
\]

from which the required result follows. Thus it is our task to prove that the diagram

\[
\begin{array}{c}
W + X \\
T(W + X + Y)
\end{array}
\rightarrow
\begin{array}{c}
TY \\
TYY
\end{array}
\]

commutes.

We consider the components of the coproduct \( W + X \) separately. For the right-hand one, we obtain the diagram (27):

\[
\begin{array}{c}
X \\
H(W + X) + Y
\end{array}
\rightarrow
\begin{array}{c}
TY \\
HTY + TY
\end{array}
\]

It commutes: in fact, square (i) commutes by (22) and (23), since \( \varepsilon_{X+Y}^\dagger g^\sharp = g^\dagger \). Square (ii) is just the definition of the solution \( (\varepsilon_{X+Y}^\dagger) \). Square (iii) is easily seen to commute when the components of \( HT(X + Y) + Y \) are considered: for the left-hand components commutativity is obvious; for the right-hand ones we have

\[
T[(\varepsilon_{X+Y}^\dagger, \eta_Y) \cdot \eta_{X+Y} \cdot \text{inr}] = T[(\varepsilon_{X+Y}^\dagger, \eta_Y) \cdot T \cdot \text{inr} \cdot \eta_Y] = T \eta_Y \cdot \eta_Y.
\]

For (iv), recall that \( \mu_Y \circ \tau_{TY}^\dagger = \tau_Y \), and then use the fact that \( \mu_Y \) is an \( H \)-algebra homomorphism (see Corollary 1.8) in order to see that the left-hand components commute; for the right-hand ones commutativity is obvious. All the other parts of the above diagram clearly commute, and thus the whole diagram does.
Finally, for the left-hand component of (26), we must show that the outward square of the diagram (28) commutes. Note that the lower part and the right-hand part of this diagram is the same as in the diagram (27), and therefore they commute. The upper left-hand square commutes by Proposition 3.5. Of the remaining three parts, we only need to consider square (*); the other parts obviously commute. Consider the components of $HW + X + Y$ separately. The first and the last components obviously commute. The middle component need not commute per se, but it does when extended by the passage to $TY$ in the upper right-hand corner of (28), which is sufficient for our purposes. In other words, we are to prove that the outward square of the diagram (29) commutes (here, $\text{inm}$ denotes the injection into the middle summand).

In fact: (i) commutes by definition of $(\_)^\dagger$; (ii) commutes due to (21); (iii) does by (22); (iv) follows from $\varepsilon_{X+Y}$ being a homomorphism of $H$-algebras (see Remark 4.14) and (5). The remaining inner parts commute trivially.

(II) **Uniqueness:** From the Solution Theorem 1.11, we know that $(\varepsilon_{X+Y} e)^\dagger$ is unique, and it is thus sufficient to verify that whenever $e^\ddagger$ is a solution in the sense of (20), we have

$$(\varepsilon_{X+Y} e)^\dagger = \varepsilon_Y e^\ddagger : X \longrightarrow TY.$$ 

Since $\varepsilon_Y$ is a monomorphism by definition, this determines $e^\ddagger$. Thus, we just have to observe the commutativity of diagram (25).

**Remark 4.24.** In the above proof we have seen that every rational equation morphism

$$e : X \longrightarrow R(X + Y)$$

reduces to a finitary flat equation morphism in the following sense: there is a finitely presentable object $W$ (of ‘additional’ variables) and a finitary flat equation morphism

$$h : W + X \longrightarrow H(W + X) + Y$$

whose solution $h^\dagger : W + X \longrightarrow TY$ defines that of $e$ via

$$e^\dagger = h^\dagger \text{inr} : X \longrightarrow TY.$$ 

Or, equivalently, the rational solution $h^\ddagger : W + X \longrightarrow RY$ of $h$ defines that of $e$ via

$$e^\ddagger = h^\ddagger \text{inr} : X \longrightarrow RY.$$ 

**Corollary 4.25.** The rational monad is iterative.

**Proof.** In fact, $R$ is ideal due to Notation 4.19: we have

$$R = HR + \text{Id}$$

with injections $\rho$ and $\eta^R$, and for the natural transformation

$$\mu^R : RR \longrightarrow R$$

with components $\mu^R_X = \text{id}_{RX}$, we consider

$$H\mu^R : HRR \longrightarrow HR,$$
and find that the appropriate diagram

\[
\begin{array}{ccc}
HRR & \xrightarrow{H\mu_R} & HR \\
\rho_R & \downarrow & \rho \\
RR & \xrightarrow{\mu_R} & R
\end{array}
\]

commutes, since \(\mu_R^X\) (being of the form \(\langle,\rangle\)) is a homomorphism of \(H\)-algebras for every finitely presentable \(X\).

Now the Rational Solution Theorem tells us that \(R\) is iterative, see Definition 1.16. 

**Remark 4.26.** Solutions for the monads \(R\) and \(T\) are closely related:

(i) For every guarded equation morphism \(e : X \to R(X + Y)\) for \(R\), we have seen in the proof of Theorem 4.23 that \(\varepsilon_{X+Y} \cdot e : X \to T(X + Y)\) is also guarded, and the rational solution \(e^\dagger\) of \(e\) is determined by

\[
\varepsilon_Y \cdot e^\dagger = (\varepsilon_{X+Y} \cdot e)^\dagger.
\]

(ii) Conversely, guarded equation morphisms \(e : X \to T(X + Y)\), which factor through \(R(X + Y)\),

\[
\begin{array}{ccc}
X & \xrightarrow{e} & T(X + Y) \\
\Downarrow{e_0} & \Downarrow{\varepsilon} & \\
R(X + Y) & \xrightarrow{\varepsilon} & T(X + Y)
\end{array}
\]

have the property that \(e_0\) is guarded for \(R\), and, therefore, the solution of \(e\) is determined by the rational solution of \(e_0\):

\[
e^\dagger = \varepsilon_Y \cdot e_0^\dagger.
\]

In fact, since \(e\) is guarded,

\[
\begin{array}{ccc}
X & \xrightarrow{e} & T(X + Y) \\
\Downarrow{e'} & \Downarrow{[\tau,\eta \cdot \text{inr}]} & \\
HT(X + Y) + Y & \xrightarrow{[\tau,\eta \cdot \text{inr}]} & T(X + Y)
\end{array}
\]

we just have to observe that the square

\[
\begin{array}{ccc}
HR(X + Y) + Y & \xrightarrow{H_{e+Y}} & HT(X + Y) + Y \\
[\rho,\eta^R \cdot \text{inr}] \downarrow & & \downarrow[\tau,\eta \cdot \text{inr}] \\
R(X + Y) & \xrightarrow{e} & T(X + Y)
\end{array}
\]

is a pullback. To verify this, observe that both the components of \(HR(X+Y)+Y\) yield pullbacks: the left-hand one is the pullback in Corollary 4.15, for the right-hand one, use the fact that \(\mathcal{A}\) is extensive, see Section 2.3. The universal property of pullbacks
yields, because of
\[
[\tau_{X+Y}, \eta_{X+Y} \cdot \text{inr}] \cdot e' = e = \varepsilon_{X+Y} \cdot e_0,
\]
the existence of \( f : X \longrightarrow HR(X + Y) + Y \) with \( e' = (H\varepsilon_{X+Y} + Y) \cdot f \) and
\[
e_0 = [\rho, \eta_R \cdot \text{inr}] \cdot f
\]
which proves that \( e_0 \) is guarded with respect to \( R \).

**Remark 4.27.** Every finitary flat equation morphism \( e : X \longrightarrow HX + Y \) can be identified with the corresponding guarded rational equation morphism
\[
X \xrightarrow{e} HX + Y \xrightarrow{\rho^* + Y} RX + Y \xrightarrow{[\text{inl}_R, \eta_R \cdot \text{inr}]} R(X + Y).
\]
The solution of the latter (in the sense of Theorem 4.23) is simply \( e^\# \). In fact, the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{e^\#} & RY \\
\downarrow e & & \downarrow [\rho, \eta] \\
HX + Y & \xrightarrow{He^\# + Y} & HRY + Y \\
\downarrow \rho^* + Y & & \downarrow \rho^* R + Y \\
RY + Y & \xrightarrow{Re^\# + Y} & RRY + Y \\
\downarrow [\text{inl}_R, \eta_R \cdot \text{inr}] & & \downarrow [RRY, \rho_R^\#] \\
R(X + Y) & \xrightarrow{[\text{inl}_R, \eta_R \cdot \text{inr}]} & RRY \\
\end{array}
\]

commutes; see Lemma 4.21 for the upper square.

**Example 4.28.** Let \( \Sigma \) be a signature with \( \Sigma_n \) finite, for every \( n \), and \( \Sigma_n = \emptyset \) for all \( n \geq n_0 \). Let \( H = H_\Sigma \) be a (strongly finitary) polynomial functor in \( \text{Set} \) corresponding to \( \Sigma \), see Section 1.2. Then \( TY \) is the \( \Sigma \)-algebra of finite and infinite \( \Sigma \)-labelled trees. We have constructed \( RY \) above by considering all images of solutions of finitary flat equations \( e : X \longrightarrow H_\Sigma X + Y \) – that is, \( RY \) is the subalgebra of all rational trees in \( TY \).

Thus, in the case of strongly finitary polynomial functors, the rational monad is the monad of rational trees. This was proved in Elgot et al. (1978) to be a free iterative monad on \( H \).

Recall the notions of ideal and iterative monad from Definition 1.13 and Definition 1.16, respectively. We show that the rational monad \( R \) can be characterised as a free iterative monad, when ideal monad morphisms are taken as the ‘right’ morphisms.

**Definition 4.29.** Let \( S \) be an ideal monad.

(a) A natural transformation from a functor \( H \) to \( S \) is called **ideal** if it factors through \( \sigma^S : S' \longrightarrow S \). For example, for the rational monad, the above natural transformation
\[
\rho^* \equiv H \xrightarrow{H\eta^R} HR \xrightarrow{\rho} R
\]
is ideal.
(b) Given another ideal monad \( \tilde{S} \), a monad morphism \( \varphi : S \rightarrow \tilde{S} \) is called \textit{ideal} if it has the form

\[
\varphi = \varphi' + id
\]

for some natural transformation \( \varphi' : S' \rightarrow \tilde{S}' \).

**Theorem 4.30.** The rational monad \( R \) is a free iterative monad on \( H \). That is, given an iterative monad \( S \) and an ideal transformation \( \lambda : H \rightarrow S \), there exists a unique ideal monad morphism \( \tilde{\lambda} : R \rightarrow S \) for which the triangle

\[
\begin{array}{ccc}
H & \xrightarrow{\rho'} & R \\
\downarrow{\lambda} & & \downarrow{\tilde{\lambda}} \\
S & \xrightarrow{\varphi} & \tilde{S} \\
\end{array}
\]

commutes.

**Proof.** (1) \textbf{Existence of} \( \tilde{\lambda} \): By Remark 2.9, it is sufficient to give the components

\[
\tilde{\lambda}_Y : HRY \rightarrow S'Y
\]

of \( \tilde{\lambda}' : HR \rightarrow S' \) (such that \( \tilde{\lambda} = \tilde{\lambda}' + id \)) for finitely presentable objects \( Y \) and to prove that for every arrow \( s : Y \rightarrow RZ \) (\( Y, Z \) finitely presentable) the square

\[
\begin{array}{ccc}
RY & \xrightarrow{\tilde{\lambda}_s} & RZ \\
\downarrow{\tilde{\lambda} = \tilde{\lambda}' + Y} & & \downarrow{\tilde{\lambda} = \tilde{\lambda}' + Z} \\
SY & \xrightarrow{\lambda_Y \cdot He} & SZ
\end{array}
\]  
(30)

commutes (where \( \tilde{\lambda}_Z s = \mu^\natural_Z \cdot S(\tilde{\lambda}_Z s) \), as usual).

To define \( \tilde{\lambda}'_Y \), whose domain is

\[
HRY = \text{colim} \ H\text{Eq}_{Y},
\]

consider an arbitrary object \( e : X \rightarrow HX + Y \) of \( \text{Eq}_{Y} \). Then the following, obviously guarded, equation morphism

\[
\begin{array}{ccc}
\lambda \cdot He \equiv HX & \xrightarrow{He} & H(HX + Y) \\
\downarrow{\mu^\natural_Z} & & \downarrow{\tilde{\lambda}'(HX + Y)} \\
& \xrightarrow{\theta} & S(HX + Y)
\end{array}
\]  
(31)

has the unique solution \( (\lambda_Y \cdot He)^\dagger : HX \rightarrow SY \). Since we have, by Remark 3.6, a factorisation through \( \sigma_Y \), say \( (\lambda \cdot He)^\dagger = \sigma_Y \cdot \tilde{\epsilon} \), we can define \( \tilde{\lambda}'_Y \) by

\[
\tilde{\lambda}'_Y \cdot H\tilde{\epsilon}^\natural = \tilde{\epsilon} \quad \text{(for all} \ e : X \rightarrow HX + Y \ \text{in} \ \text{Eq}_{Y}).
\]  
(32)
This is well-defined because all $\mathcal{e}$ form a cocone of $HEq_Y$. In fact, for every morphism $h : e \rightarrow e'$ in $EQ_Y$, we have a commutative diagram

$$
\begin{array}{ccc}
HX & \xrightarrow{He} & H(HX + Y) \xrightarrow{\lambda} S(HX + Y) \\
\downarrow{Hh} & & \downarrow{S(Hh + Y)} \\
HX' & \xrightarrow{He'} & H(HX' + Y) \xrightarrow{\lambda} S(HX' + Y)
\end{array}
$$

This implies $(\lambda \cdot e)^\dagger = (\lambda \cdot e')^\dagger \cdot Hh$, see Lemma 3.7. Since $\sigma_Y$ is a monomorphism, we conclude

$$\mathcal{e} = \mathcal{e}' \cdot Hh,$$

as required.

In order to prove that (30) commutes, we are to show that for every object $e : X \rightarrow HX + Y$ of $EQ_Y$, we have

$$\tilde{\lambda}_Z \tilde{\lambda}_Y e^\sigma = \tilde{\lambda}_Z \tilde{s} e^\sigma.$$

As in the proof of Theorem 4.16, factor

$$s = f^\sigma s'$$

and define $\bar{e} : Z + V \rightarrow H(Z + V) + Y$ by (7). That is, $\bar{e}$ has the following components:

$$\bar{e} \cdot \text{inl} \equiv X \xrightarrow{e} HX + Y \xrightarrow{HX + fs'} HX + HV + Z \xrightarrow{[\text{inl}, \text{inr}] + Z} H(X + V) + Z$$

and

$$\bar{e} \cdot \text{inr} \equiv V \xrightarrow{f} HV + Z \xrightarrow{H\text{inr} + Z} H(X + V) + Z.$$

Note that for all $e$ in $EQ_Y$, we have

$$\tilde{\lambda}_Y e^\sigma = [(\lambda He)^\dagger, \eta_\sigma^\dagger] \cdot e.$$  (34)

Indeed, consider the following diagram:
The upper square commutes by the definition of the isomorphism \( i : RY \rightarrow HRY + Y \) (see Remark 4.11), the lower triangle is the definition of \( \bar{\lambda}_Y \) on the components of the coproduct \( RY = HRY + Y \), and the right-hand square commutes, since

\[
\sigma_Y \cdot \bar{\lambda}_Y \cdot \hat{e} = (\lambda \hat{e})^t.
\]

Now consider the following diagram:

Observe that the upper part commutes by the definition of \( \bar{e} \), and the right-hand and left-hand parts commute by (34) for \( e \) and \( \bar{e} \), respectively. Of the remaining two inner squares, the upper one commutes by definition of \( \tilde{s} \), see (9). We shall show below that the outer shape commutes. Thus, the lower square commutes when precomposed with \( e^\sharp \), for any \( e \) in \( EQ_Y \), which establishes the commutativity of (30).

In order to show that the outer shape of (35) commutes, we consider the components of \( HX + Y \) separately. For the right-hand one we obtain the following diagram:

Observe that all parts of this diagram, except perhaps the right-hand triangle, clearly commute: we have \( \tilde{\lambda}_Z f^\sharp = [(\lambda Hf)^t, \eta^2_Z] \cdot f \) by (34) and the other two parts are obvious.

Now notice that \( \text{inr} : f \rightarrow \bar{e} \) is a morphism in \( EQ_Z \). Therefore the square
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\[
\begin{array}{cccccc}
HV & \xrightarrow{Hf} & HV + Z & \xrightarrow{\lambda} & S(HV + Z) \\
\downarrow{Hinr} & & \downarrow{S(Hinr+Z)} \\
H(X + V) & \xrightarrow{He} & H(H(X + V) + Z) & \xrightarrow{\lambda} & S(H(X + V) + Z)
\end{array}
\]

commutes. Hence

\[
(\lambda H\bar{e})^\dagger = (\lambda Hf)^\dagger \cdot Hinr
\]

by Lemma 3.7, which establishes the commutativity of (36).

For the left-hand component of the outer shape of (35) we are to establish the commutativity of the following diagram:

\[
\begin{array}{cccccc}
HX & \xrightarrow{Hinl} & H(X + V) \\
\downarrow{(\lambda He)^\dagger} & & \downarrow{(\lambda He)^\dagger} \\
SY & \xrightarrow{\lambda Z s} & SZ
\end{array}
\]

for all \(e \in EQ_Y\)  

(37)

In order to prove this, consider the following arrow \(h : HX + HV \longrightarrow H(HX + HV + Z)\) defined by

\[
h \cdot inl \equiv HX \xrightarrow{He} H(HX + Y) \xrightarrow{H(HX + f's')} H(HX + HV + Z)
\]

and

\[
h \cdot inr \equiv HV \xrightarrow{Hf} H(HV + Z) \xrightarrow{Hinr} H(HX + HV + Z).
\]

Note that we have

\[
H([Hinl, Hinr] + Z) \cdot h = H\bar{e} \cdot [Hinl, Hinr].
\]

(38)

To show the commutativity of (37), we shall prove below that for the solution of the guarded equation morphism \(\lambda_{HX+HV+Z} \cdot h\), the following two claims hold:

\[
(\lambda h)^\dagger = (\lambda H\bar{e})^\dagger \cdot [Hinl, Hinr]
\]

(39)

and

\[
(\lambda h)^\dagger = [\lambda Z s \cdot (\lambda He)^\dagger, (\lambda Hf)^\dagger].
\]

(40)

Observe that the left-hand components yield (37), which concludes the proof of the existence of the ideal monad morphism \(\bar{\lambda} : R \longrightarrow S\).

\textit{Proof} of (39). Consider the commutative diagram (41). The upper left-hand square is just (38), the lower square commutes by naturality of \(\bar{\lambda}\). The right-hand square is the definition of \((\lambda H\bar{e})^\dagger\). Thus \((\lambda H\bar{e})^\dagger \cdot [Hinl, Hinr]\) solves \(\lambda h\), and thus (39) holds.
Proof of (40). As for (39), we use the definition of the solution $(\lambda h)^{\dagger}$, but we consider the components of the coproduct $HX + HV$ separately. For the right-hand component of (40), consider the commutative diagram (42):

\[
\begin{array}{c}
HX + HV \xrightarrow{[\text{Hinr,Hinr}]} H(X + V) \xrightarrow{((\lambda H)^{\dagger})} SZ \\
\downarrow h \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
H(HX + HV + Z) \xrightarrow{H[H[\text{Hinr,Hinr}]+Z]} H(HX + V + Z) \xrightarrow{He} \rho_Z^S \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
S(HX + HV + Z) \xrightarrow{S[H[\text{Hinr,Hinr}]+Z]} S(HX + V + Z) \xrightarrow{S((\lambda H)^{\dagger})^{\dagger},\eta_Z} \SSZ
\end{array}
\]

For the left-hand component of (40), we analyse the diagram (43). All inner parts of this diagram except, perhaps, the lower one obviously commute. For the lower part, remove $S$ and consider the components of $HX + Y$. For the left-hand one, nothing needs to be shown, the right-hand one commutes since by (33) and (34) we have

\[
\begin{aligned}
\tilde{\lambda}_Z \cdot s \cdot \eta_Y^S &= \tilde{\lambda}_Z \cdot s = \tilde{\lambda}_Z \cdot f \cdot s' = [(\lambda Hf)^{\dagger},\eta_Z^S] \cdot f \cdot s'.
\end{aligned}
\]

Therefore (43) commutes, concluding the proof of (40).

(II) Equality $\tilde{\lambda} \cdot \rho^* = \lambda$. It suffices to check that $\tilde{\lambda}_Y \cdot \rho_Y^* = \lambda_Y$ for objects $Y$ in $\mathcal{A}_{fp}$, see Remark 2.2(iii). Notice that for $e = \text{inr} : Y \to HY + Y$ we have $(\lambda_{HY + Y}^{\dagger} H\text{inr})^{\dagger} = \lambda_Y$. Indeed, the diagram (44) commutes. That means we can choose $\tilde{\lambda}_Y^e : HY \to S'Y$ as our $\tilde{e}$ (see (32)). But now we have

\[
\begin{aligned}
\tilde{\lambda}_Y^e H\eta_Y^R &= \tilde{\lambda}_Y^e H\text{inr}^\natural = \lambda_Y^e.
\end{aligned}
\]
Extending by $\sigma_Y$, we obtain the desired equation. In fact, the diagram

\begin{equation}
\begin{array}{c}
H \longrightarrow H \rho^R \longrightarrow HR \rho \longrightarrow R \\
\downarrow \alpha' \downarrow \lambda' \downarrow \lambda \\
S' \sigma \longrightarrow S
\end{array}
\end{equation}

commutes.

(III) **Uniqueness of $\tilde{\lambda}$.** Suppose that $v = v' + id$ is another ideal monad morphism with $v \cdot \rho^* = \lambda$. We have to show that $v'_Y$ satisfies the defining equation of $\tilde{\lambda}'_Y$, that is,

\[ v'_Y \cdot He^\tilde{e} = \tilde{e} : HX \longrightarrow S'Y \]

for all objects $Y$ of $A_{fp}$ and $e$ of $EQ_Y$. Since the coproduct injection $\sigma_Y : S'Y \longrightarrow SY$ is monomorphic, it suffices to show that

\[ \sigma_Y \cdot v'_Y \cdot He^\tilde{e} = \sigma_Y \cdot \tilde{e} = (\lambda He)^\dagger. \]
We use the definition of the solution \((\lambda H e)\) to show that this is indeed the case. All parts of the diagram (46) commute:

\[
\begin{array}{c}
HX \\ H e^x \\
\downarrow H e \\
H (HX + Y) \\
\downarrow \lambda \\
S (HX + Y)
\end{array}
\quad \begin{array}{c}
HR (HX + Y) \\
H R Y \\
\downarrow id \\
\downarrow \rho \\
\downarrow \rho^* \\
R (HX + Y) \\
\downarrow \eta \\
S (HX + Y)
\end{array}
\quad \begin{array}{c}
S Y \\
\downarrow \sigma_Y \\
SS Y
\end{array}
\quad \begin{array}{c}
S Y \\
\downarrow \nu_Y \\
\downarrow \nu \\
\downarrow \nu_S \\
\downarrow \nu_X
\end{array}
\quad \begin{array}{c}
Y \\
\downarrow \eta_Y \\
\downarrow \eta
\end{array}
\]

For the upper left-hand triangle, recall from Corollary 4.13 that \([\rho_Y, \eta_Y^R] \cdot (H e^x + Y) \cdot e = e^x\), for the lower part use naturality of \(\nu\) and \(\nu_Y \eta_Y^R = \eta_Y^S\), for the rest commutativity is obvious. This completes the proof of (45).

\[\tag{46}\]

**Remark 4.31.** Recall that Elgot defined *iterative theories* by the (seemingly weaker) condition that all *ideal equation morphisms*, that is, all \(e : X \rightarrow S(X + Y)\) that factor through \(\sigma_{X+Y}\),

\[
X \xrightarrow{e} S(X + Y) \xrightarrow{\sigma_{X+Y}} S'(X + Y)
\]

have unique solutions. This, however, does not influence the above result. That is, the rational monad is also a free iterative monad in the sense of Elgot. This follows from the above proof: the only guarded equation we used for the iterative monad \(S\) was the equation morphism \(\lambda \cdot He\), see (31), and that is, indeed, ideal.

**Remark 4.32.** Our definition of ideal monad (as a monad \(S\) of the form \(S = S' + Id\)) rests on our assumption that coproduct injections are monomorphic in \(\mathcal{A}\). Without this assumption, we would have to define an ideal monad by means of a functor \(S' : \mathcal{A} \rightarrow \mathcal{A}\) and natural transformations \(\mu' : S'S \rightarrow S'\) such that the functor \(S = S' + Id\) together with the natural transformations \(\text{inr : } Id \rightarrow S\) and \(\mu \equiv SS = SS + S[\mu S] + Id = S\) is
a monad. In that sense it is important to note that in the above theorem we actually proved the following:

For every natural transformation $\lambda' : H \rightarrow S'$, there exists a unique natural transformation $\bar{\lambda}' : HR \rightarrow S'$ such that $\bar{\lambda}' + id$ is an ideal monad homomorphism and the triangle

$$
\begin{array}{ccc}
H & \xrightarrow{H \eta^R} & HR \\
\downarrow{\bar{\lambda}'} & & \downarrow{\bar{\lambda}'} \\
S' & \xrightarrow{id} & S'
\end{array}
$$

commutes.

In fact, both the uniqueness, and the naturality of $\bar{\lambda}'_X : HRX \rightarrow S'X$ follow from the uniqueness and naturality of $\bar{\lambda}_X : RX \rightarrow SX$ (again through our assumptions that the coproduct injections $\rho_X$ and $\sigma_X$ are monomorphisms).

5. Rational monad – finitary case

Assumptions 5.1. Throughout this section $\mathcal{A}$ denotes a strongly LFP category and $H$ a finitary endofunctor preserving monomorphisms.

We are going to prove that, once again, the rule

$$
RY = \bigcup \text{im}(e^\dagger)
$$

defines a free iterative monad on $H$, called the rational monad of $H$.

Remark 5.2. We know from Corollary 2.16 that $H$ is a directed colimit of strongly finitary functors preserving monomorphisms:

$$
H = \underset{i \in I}{\text{colim}} H_i \quad (I \text{ a directed poset}).
$$

We use

$$
\alpha_{i,j} : H_i \rightarrow H_j \quad (i \leq j)
$$

to denote the connecting morphisms and

$$
\beta_i : H_i \rightarrow H \quad (i \in I)
$$

to denote the colimit cocone.

Let

$$
\rho_i^* : H_i \rightarrow R_i \quad (i \in I)
$$

be the rational monad of $H_i$ as constructed in Section 4. For each $i \leq j$ we have an ideal natural transformation

$$
H_i \xrightarrow{\alpha_{i,j}} H_j \xrightarrow{\rho_j^*} R_j
$$
for which there exists, by Theorem 4.30, a unique extension to an ideal monad homomorphism \( \bar{\alpha}_{i,j} : R_i \rightarrow R_j \):

\[
\begin{array}{ccc}
H_i & \xrightarrow{\alpha_{i,j}} & H_j \\
\rho_i' & \downarrow & \rho_j' \\
R_i & \xrightarrow{\alpha_{i,j}} & R_j
\end{array}
\]

The unicity makes it clear that we obtain a directed diagram of iterative monads \((R_i \mid i \in I)\) (with connecting morphisms \( \bar{\alpha}_{i,j} \)), and we put

\[ R = \text{colim}_{i \in I} R_i. \]

That is, we define an endofunctor \( R \) of \( \mathcal{A} \) as a colimit (object-wise) of the endofunctors \( R_i \), that is, \( RY = \text{colim}_{i \in I} R_iY \) for all \( Y \) in \( \mathcal{A} \). Next we prove that a colimit of a filtered diagram of iterative monads is an iterative monad (Proposition 5.5). An easy consequence is that the above monad \( R \) is actually a free iterative monad on \( H \). Finally, we prove that \( RY = \text{colim} \text{Eq}_Y \).

**Proposition 5.3 (Aczel et al. 2002).** Ideal monad homomorphisms \( \lambda : S \rightarrow \bar{S} \) between iterative monads preserve solutions. That is, if \( e : X \rightarrow S(X + Y) \) is guarded for \( S \) and has the solution \( e^\dagger : X \rightarrow SY \), then \( \lambda X + Y \cdot e \) is guarded for \( \bar{S} \) and has the solution \( \lambda Y \cdot e^\dagger \) with respect to \( \bar{S} \).

**Corollary 5.4.** For all \( i \leq j \) in \( I \) and any finitary flat equation \( f : X \rightarrow H_iX + Y \), put

\[ g \equiv X \xrightarrow{f} H_iX + Y \xrightarrow{\alpha_{i,j} + Y} H_jX + Y. \]

Then the solution of \( f \) with respect to \( R_i \) denoted by \( f^\sharp_i : X \rightarrow R_iY \) (see Remark 4.27), is related to the solution \( g_j \) of \( g \) with respect to \( R_j \) by

\[
\begin{array}{ccc}
X & \xrightarrow{f^\sharp_i} & R_iY \\
\downarrow & & \downarrow \\
R_iY & \xrightarrow{\alpha_{i,j}} & R_jY
\end{array}
\]

**Proof.** Apply Proposition 5.3 to \( \lambda = \bar{\alpha}_{i,j} \). \( \square \)

**Proposition 5.5.** For every directed collection of iterative monads \( S_i \) \((i \in I \text{ a directed poset})\) and ideal monad morphisms \( \lambda_{i,j} : S_i \rightarrow S_j \) \((i \leq j \text{ in } I)\), there is a unique structure of an iterative monad on the functor

\[ S = \text{colim}_{i \in I} S_i \]

turning the colimit morphisms into ideal monad homomorphisms.

**Remark.** The above colimit is, of course, formed objectwise: we define \( SX \) by \( SX = \text{colim}_{i \in I} S_iX \) for all objects \( X \) in \( \mathcal{A} \), and analogously for morphisms.
Proof. (I) We are given ideal monads \((S_i, \eta_i, \mu_i)\) with appropriate \(\sigma_i : S'_i \rightarrow S_i\) and \(\mu'_i : S'_i S_i \rightarrow S'_i\) for \(i \in I\) and we form a colimit

\[
(\lambda_i : S_i \rightarrow S \mid i \in I)
\]

in the functor category \([\mathcal{A}, \mathcal{A}]\). It is proved in Section 4 of Kelly and Power (1993) that there is a unique structure \((S, \eta, \mu)\) of a monad on the functor \(S\) such that all the transformations \(\lambda_i\) are monad homomorphisms, forming a colimit cocone in the category of all finitary monads on \(\mathcal{A}\).

(II) \((S, \eta, \mu)\) is an ideal monad. In fact, we have a directed diagram of all \(S'_i, i \in I\) and all \(\lambda'_{i,j} : S'_i \rightarrow S'_j, i \leq j\), such that

\[
S_i = S'_i + \text{Id}
\]

and

\[
\lambda_{i,j} = \lambda'_{i,j} + \text{id}
\]

for all \(i \leq j\) in \(I\). (To show that \((\lambda'_{i,j})\) form a directed diagram, use the fact that \((\lambda_{i,j})\) do and that all \(\sigma_i\) are monomorphisms, being coproduct injections of \(S_i = S'_i + \text{Id}\).) Put

\[
S' = \text{colim}_{i \in I} S'_i
\]

with the colimit cocone denoted by \(\lambda'_i : S'_i \rightarrow S', i \in I\). Define \(\sigma : S' \rightarrow S\) as the unique natural transformation such that diagrams

\[
\begin{array}{ccc}
S' & \xrightarrow{\sigma} & S \\
\downarrow{\lambda'_i} & & \downarrow{\lambda_i} \\
S'_i & \xrightarrow{\sigma_i} & S_i
\end{array}
\]

commute for every \(i \in I\). (This is, once more, forced on us by the requirement that \(\lambda_i\) is an ideal monad homomorphism.)

Since the diagonal functor \(I \rightarrow I \times I\) is cofinal, the composite \(S'S\) is a colimit of the diagram of \(S'_i S_i, i \in I\), with connecting morphisms

\[
S'_i S_i \xrightarrow{S'_i \lambda_{i,j}} S'_j S_i \xrightarrow{\lambda'_{i,j} S_i} S'_j S_j \quad \text{for } i \leq j
\]

and colimit cocone

\[
S'_i S_i \xrightarrow{S'_i \lambda_i} S'_i S \xrightarrow{\lambda' i S} S'S \quad \text{for } i \in I.
\]

Then \(\mu' : S'S \rightarrow S'\) is defined by commutativity of the squares

\[
\begin{array}{ccc}
S'S & \xrightarrow{\mu'} & S' \\
\downarrow{\lambda' i S} & & \downarrow{\lambda'_i} \\
S'_i S & \xrightarrow{\sigma'_i} & S'_i
\end{array}
\]

for all \(i \in I\). The verification that \((S, \eta, \mu)\) is an ideal monad is quite mechanical.
(III) \((S, \eta, \mu)\) is iterative. In fact, let

\[
\begin{array}{ccc}
X & \xrightarrow{e} & S(X + Y) \\
& \searrow & [\sigma, \eta; \text{inr}] \\
& & S'(X + Y) + Y
\end{array}
\]

be a guarded equation morphism. Since \(X\) is a finitely presentable object, \(e'\) factors through one of the colimit morphisms of the filtered colimit

\[
S'(X + Y) + Y = \text{colim}_{i \in I} S'_i(X + Y) + Y
\]

with colimit cocone formed by \((\lambda'_i)_{X+Y} + Y, i \in I\). That is, we have \(i \in I\) and \(f'\) such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{e} & S(X + Y) \\
& \searrow & [\sigma, \eta; \text{inr}] \\
& & S'(X + Y) + Y \\
& \searrow & \lambda'_i + Y \\
& & S'_i(X + Y) + Y \\
\end{array}
\]

commutes. This defines a guarded equation morphism \(f : X \longrightarrow S_i(X + Y)\) for \(S_i\). We use \(f^\dagger : X \longrightarrow S_iY\) to denote the unique solution with respect to \(S_i\). Then the composite

\[
\begin{array}{ccc}
X & \xrightarrow{f^\dagger} & S_iY \\
\downarrow & \searrow & \lambda_i \\
S_i(X + Y) & \xrightarrow{S_i[f^\dagger, \eta]} & S_iS_iY \\
\downarrow & \searrow & \lambda_iS \\
S(X + Y) & \xrightarrow{S[f^\dagger, \eta]} & SS_iY \\
\end{array}
\]

is a solution of \(e\) with respect to \(S\). In fact, the diagram (48) obviously commutes. It remains to show that solutions are unique for \(S\). In fact, let

\[
h : X \longrightarrow SY
\]

be a solution of \(e\). We prove \(h = \lambda_i \cdot f^\dagger\). Observe first that the solution \(\lambda_i \cdot f^\dagger\) is independent of the above factorisation \(e' = ((\lambda')_{X+Y} + Y) \cdot f'\). In fact, choose any other factorisation,
through \( j \in I \):

\[
e' = ((\lambda'_j)_{X+Y} + Y) \cdot g' \quad \text{for some } g' : X \to S'_j(X + Y) + Y.
\]

Since \( I \) is directed, we can assume the existence of some \((\lambda'_{i,j})_{X+Y} : S'_i(X+Y) \to S'_j(X+Y)\) such that the diagram

\[
\begin{array}{ccc}
S'_i(X + Y) + Y & \\[-1em] f' & \Downarrow \lambda'_{i,j} + Y & S'_j(X + Y) + Y \\
X & g' \Downarrow \lambda'_{i,j} + Y & \\
S'_i(X + Y) + Y & \end{array}
\]  

commutes.

Put \( g = [(\sigma_j)_{X+Y}, (\eta_j)_{X+Y} \cdot \text{inr}] \cdot g' \), analogously to (47). Notice that the triangle

\[
\begin{array}{ccc}
S_i(X + Y) & \\[-1em] f & \Downarrow \lambda_{i,j} & S_j(X + Y) \\
X & g \Downarrow \lambda_{i,j} & \\
S_i(X + Y) & \end{array}
\]

commutes. By Proposition 5.3 applied to \( \lambda = \lambda_{i,j} \), the triangle

\[
\begin{array}{ccc}
S_iY & \\[-1em] f^\dagger & \Downarrow \lambda_{i,j} & S_jY \\
X & g^\dagger \Downarrow \lambda_{i,j} & \\
S_iY & \end{array}
\]

commutes, where, \( g^\dagger : X \to S_jY \) denotes the unique solution with respect to \( S_j \). This proves

\[
(\lambda_{i,j})_Y \cdot g^\dagger = (\lambda_{j})_Y \cdot (\lambda_{i,j})_Y f^\dagger = (\lambda_{i})_Y \cdot f^\dagger,
\]

as desired.

We are ready to prove that solutions for \( S \) are unique. Let

\[
h : X \to SY
\]
be a solution, that is, let the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{h} & SY \\
\downarrow{f} & & \downarrow{\mu} \\
S_i(X + Y) & \xrightarrow{\lambda_i} & SSY \\
\end{array}
\]

commute. We can factor \( h \) through one of the colimit maps of the filtered colimit defining \( SY \); without loss of generality we can assume that it is \((\lambda_i)_Y\) for the given \( i \in I \) (recall the independence of \((\lambda_i)_Y \cdot f^\dagger\) of the given factorisation of \( e \)). We obtain the diagram (52):

\[
\begin{array}{ccc}
X & \xrightarrow{\bar{h}} & SY \\
\downarrow{f} & & \downarrow{\mu} \\
S_i(X + Y) & \xrightarrow{\bar{\lambda}_i} & SSY \\
\end{array}

\]

The proof would be finished if we knew that \((*)\) in that diagram commutes: then \( \bar{h} = f^\dagger \) implies \( h = \lambda_i \cdot f^\dagger \), as required. However, we do not claim that commutativity. All we claim is that all the other inner parts of (52) commute (trivially) and the outer shape does by (51), thus,

\((\lambda_i)_Y\) merges the two sides of \((*)\).

This implies, since the domain \( X \) of \((*)\) is finitely presentable and \((\lambda_i)_Y\) is a colimit morphism of the filtered colimit defining \( SY \), that some connecting morphism \((\lambda_{i,j})_Y\) for \( i \leq j \) in \( I \) also merges the two sides of \((*)\). Put

\[
g' \equiv X \xrightarrow{f'} S'_i(X + Y) + Y \xrightarrow{\lambda_{i,j} + Y} S'_j(X + Y) + Y
\]

and

\[
g \equiv X \xrightarrow{g'} S'_j(X + Y) + Y \xrightarrow{[\sigma_j,\eta_j]_{\text{intr}}} S_j(X + Y)
\]
We claim that $g$, which obviously is a guarded equation morphism for $S_j$, has the solution
\[ g^+ = (\lambda_{i,j})_Y \cdot \bar{h} : X \rightarrow S_j Y. \] (53)
To prove this, we observe that the diagram (54) commutes:
\[ X \xrightarrow{\bar{h}} S_j Y \xrightarrow{\lambda_{i,j}} S_j Y \]
\[ S_j(X + Y) \xrightarrow{S_j[\bar{h},\eta_i]} S_j S_j Y \]
\[ S_j(X + Y) \xrightarrow{\lambda_{i,j}} S_j(S_j Y) \]
\[ S_j[S_j[\bar{h},\eta_i]] \xrightarrow{S_j[\lambda_{i,j},\eta_j]} S_j S_j Y \]
all the inner parts commute trivially except the upper left-hand square – and, by our choice of $j$, that square commutes when composed with $(\lambda_{i,j})_Y$. This proves (53). Therefore, we have, by (50),
\[(\lambda_i)_Y \cdot f^+ = (\lambda_j)_Y \cdot g^+ = (\lambda_j)_Y \cdot (\lambda_{i,j})_Y \bar{h} = (\lambda_i)_Y \bar{h} = h,\]
which concludes the proof. 

**Definition 5.6.** For every finitary functor $H$, expressed as a filtered colimit of strongly finitary functors $H_i$ we define the *rational monad* $R$ of $H$ as the colimit of the corresponding diagram of the rational monads $R_i$ of $H_i$,
\[ R = \text{colim}_i R_i, \quad \text{with injections } \bar{\beta}_i : R_i \rightarrow R \]
(see Remark 5.2).

We have yet to show that $R$ is well-defined, that is, independent of the given representation of $H$ as $H = \text{colim} H_i$. This follows from the following proposition.

**Proposition 5.7.** The rational monad $R$ of $H$ is a free iterative monad on $H$. That is:
(a) $R$ is an iterative monad;
(b) the natural transformation $\rho^* = \text{colim}_{i \in I} \rho_i^* : H \rightarrow R$ is ideal;
and
(c) given an iterative monad $S$ and an ideal natural transformation $\lambda : H \rightarrow S$, there exists a unique ideal monad homomorphism $\tilde{\lambda} : R \rightarrow S$ with $\lambda = \tilde{\lambda} \cdot \rho^*$.

**Proof.** For (a) see Proposition 5.5. Since each $\rho_i^* : H_i \rightarrow R_i$ is an ideal transformation, that is, $\rho_i^* = \rho_i \cdot H\eta^{R_i}$, it follows that
\[ \text{colim}_{i \in I} \rho_i^* = \left(\text{colim}_{i \in I} \rho_i\right) \cdot H\left(\text{colim}_{i \in I} \eta^{R_i}\right) \]
is also ideal. It remains to prove (c).
For each \( i \in I \) we have an ideal transformation
\[
H_i \xrightarrow{\beta_i} H \xrightarrow{\lambda} S,
\]
which, by Theorem 4.30, yields a unique ideal monad homomorphism \( \tilde{\lambda}_i : R_i \to S \) such that the square
\[
\begin{array}{ccc}
H_i & \xrightarrow{\rho_i} & R_i \\
\downarrow \beta_i & & \downarrow \bar{\lambda}_i \\
H & \xrightarrow{\lambda} & S
\end{array}
\]
(55)
commutes. The unicity makes it clear that \( \tilde{\lambda}_i, i \in I, \) is a cocone of the diagram \((R_i | i \in I)\). Thus, we have a unique ideal monad homomorphism \( \tilde{\lambda} : R \to S \) with
\[
\tilde{\lambda} \cdot \beta_i = \tilde{\lambda}_i \quad \text{for } i \in I.
\]
(56)
Then, from \( \rho^* \cdot \beta_i = \bar{\rho}_i \cdot \rho_i^* \) (recall \( \rho^* = \text{colim} \rho_i^* \)), we obtain
\[
(\tilde{\lambda} \cdot \rho^*) \cdot \beta_i = \tilde{\lambda} \cdot \bar{\rho}_i \cdot \rho_i^* = \tilde{\lambda}_i \cdot \rho_i^* \quad \text{by (56)}
\]
\[
= \tilde{\lambda} \cdot \beta_i \quad \text{by (55)}
\]
for all \( i \in I \), which proves
\[
\tilde{\lambda} \cdot \rho^* = \tilde{\lambda}.
\]
The unicity of \( \tilde{\lambda} \) is obvious: given an ideal homomorphism \( \tilde{\lambda} \) with \( \tilde{\lambda} \cdot \rho^* = \lambda \), we have
\[
\tilde{\lambda} \cdot \bar{\beta}_i : R_i \to S
\]
is an ideal homomorphism with
\[
\tilde{\lambda} \cdot \bar{\beta}_i \cdot \rho_i^* = \tilde{\lambda} \cdot \rho^* \cdot \beta_i = \lambda \cdot \beta_i,
\]
which (since \( R_i \) is free on \( H_i \)) determines \( \tilde{\lambda} \cdot \bar{\beta}_i \) for \( i \in I \), and this determines \( \tilde{\lambda} \).

Recall that guarded equation morphisms \( e : X \to R(X + Y) \) with \( X \) finitely presentable are called rational.

**Corollary 5.8. (Rational Solution Theorem)** Every rational equation morphism \( e : X \to R(X + Y) \) has a unique solution. That is, there exists a unique morphism
\[
e^\sharp : X \to RY
\]
such that the triangle
\[
\begin{array}{ccc}
X & \xrightarrow{e^\sharp} & RY \\
\downarrow e & & \downarrow [e^\sharp, \eta_Y^R] \\
R(X + Y)
\end{array}
\]
commutes (where for \( s = [e^\sharp, \eta_Y^R] \) we denote \( \hat{s} = \mu_Y^R \cdot Rs \), as always).

In fact, if \( Y \) is finitely presentable, this follows immediately from Proposition 5.7(a). If \( Y \) is arbitrary, we express it as a filtered colimit of finitely presentable objects \( Y = \text{colim}_{i \in I} Y_i \),
with a colimit cone $y_i : Y_i \to Y$. Since $R$ is finitary, we obtain $R(X + Y) = \colim_{i \in I} R(X + Y_i)$. By the finite presentability of $X$, $e$ factors through one of the colimit morphisms $R(X + Y_i)$ through a guarded equation morphism $e_i : X \to R(X + Y_i)$. It is a routine verification that the existence and uniqueness of the solution of $e_i$ implies the existence and uniqueness of the solution of $e$.

**Remark 5.9.** The rational monad $R$ fulfills

$$R = HR + Id$$

with coproduct injections

$$\rho : HR \to R$$

(turning each object $RY$ into an $H$-algebra) given by

$$\rho = \colim_i \rho_i : \colim_i H_i R_i \to \colim_i R_i$$

and

$$\eta : Id \to R$$

(the unit of the monad).

In fact, filtered colimits commute with finite coproducts, so all this follows from $R_i = H_iR_i + Id$ with coproduct injections $\rho_i$ and $\eta_i$.

**Remark 5.10.** We are going to establish the formulas $RY = \bigcup \text{im}(e^\sharp)$ and $RY = \colim \text{Eq}_Y$ for all objects $Y$ of $\mathcal{A}$ (cf. 4.2. and 4.6.). It is sufficient to prove the second one for all finitely presentable objects $Y$, the extension to all objects is exactly as in Remark 4.20.

For $Y$ in $\mathcal{A}_{fp}$, the diagram $\text{Eq}_Y^{(i)}$ of all finitary flat equations $e : X \to H_i X + Y$ ($X$ finitely presentable) has a colimit

$$R_i Y = \colim \text{Eq}_Y^{(i)}$$

with colimit cocone $e_i^\sharp : X \to R_i Y$, see Section 4.

Consider an arbitrary finitary flat equation morphism $e : X \to HX + Y$ for $H$. Since $X$ is finitely presentable and $HX + Y$ is a directed colimit of $H_i X + Y$, $i \in I$, we have a factorisation as follows:

$$\begin{array}{c}
X \xrightarrow{e} HX + Y \\
\downarrow f \quad \downarrow \beta_i + Y \\
H_i X + Y
\end{array}$$

for some $i \in I$. Here $f$ is a finitary flat equation morphism for $H_i$, and we use $f_i^\sharp : X \to R_i Y$ to denote the corresponding colimit map of $R_i Y = \colim \text{Eq}_Y^{(i)}$. Define $e_i^\sharp$ by

$$\begin{array}{c}
X \xrightarrow{e_i} RY \\
\downarrow f_i^\sharp \quad \downarrow \beta_i \\
R_i Y
\end{array}$$
This is well-defined (that is, independent of the factorisation) and forms a colimit of the diagram $\mathrm{Eq}_Y$, as we prove now.

**Proposition 5.11.** For every finitely presentable object $Y$ we have

$$RY = \operatorname{colim} \mathrm{Eq}_Y$$

with the above colimit cocone $\varepsilon : X \longrightarrow RY$ ($e$ in $\mathrm{Eq}_Y$).

**Proof.** (I) **Independence of the factorisation:** Since $I$ is a directed poset, all we have to prove is that given $i \leq j$ in $I$, and, using $g : X \longrightarrow HjX + Y$ to denote the composite of $f$ and $(\alpha_{i,j})_X + Y$,

\[
\begin{array}{c}
X \\
\downarrow g \\
HjX + Y \\
\downarrow f \\
\downarrow \alpha_{i,j} + Y \\
H_iX + Y
\end{array}
\]

we have

$$\bar{\beta}_i \cdot f_i^\# = \bar{\beta}_j \cdot g_j^\#.$$

Since the diagram

\[
\begin{array}{c}
X \\
\downarrow g \\
HjX + Y \\
\downarrow f \\
\downarrow \alpha_{i,j} + Y \\
H_iX + Y
\end{array} \xrightarrow{\rho_i^++Y} \begin{array}{c}
R_iX + Y \\
\downarrow \alpha_{i,j} + Y \\
R_i + Y
\end{array} \xrightarrow{\rho_i^+Y} \begin{array}{c}
R(i + Y) \\
\downarrow \bar{\alpha}_{i,j}
\end{array}
\]

commutes, it follows from Corollary 5.4 that the solutions $f_i^\# : X \longrightarrow R_i Y$ (with respect to $R_i$) and $g_j^\# : X \longrightarrow R_j Y$ (with respect to $R_j$) also form a commutative triangle:

\[
\begin{array}{c}
X \\
\downarrow g_j^\# \\
R_j Y \\
\downarrow f_i^\# \\
\downarrow \bar{\alpha}_{i,j} \\
R_i Y
\end{array}
\]

Combined with the fact that $\bar{\beta}_i = \bar{\beta}_j \cdot \bar{\alpha}_{i,j}$ (because $\beta_i = \beta_j \cdot \alpha_{i,j}$), this yields the desired equality $\bar{\beta}_i \cdot f_i^\# = \bar{\beta}_j \cdot g_j^\#$.

(II) **The morphisms $\varepsilon$ form a cocone of** $\mathrm{Eq}_Y$: That is, given a morphism

\[
\begin{array}{c}
X \\
\downarrow h \\
X' \\
\downarrow e
\end{array} \xrightarrow{e} \begin{array}{c}
HX + Y \\
\downarrow Hh + Y \\
HX' + Y
\end{array}
\]
in \( \text{EQ}_Y \), we have \( e^\sharp = (e')^\sharp \cdot h \) holds. Since \( X \) and \( X' \) are finitely presentable, there exists \( i \in I \) such that \( e \) factors as \( e = ((\beta_i)_X + Y) \cdot f \) and \( e' \) as \( e' = ((\beta_i)_{X'} + Y) \cdot f' \). Next, observe that

\[
HX + Y = \operatorname{colim}_{i \in I} H_i X + Y
\]

is a filtered colimit with colimit cocone \((\beta_i)_X + Y\). The parallel pair \( f' \cdot h, (H_i h + Y) \cdot f : X \to H_i X' + Y \) gets merged by the colimit morphism \((\beta_i)_{X'} + Y\):

\[
\begin{array}{ccc}
X & \xrightarrow{f} & H_i X + Y \\
\downarrow{h} & & \downarrow{H_i h + Y} \\
X' & \xrightarrow{f'} & H_i X' + Y \\
\end{array}
\]

\[
\begin{array}{ccc}
& \xrightarrow{e} & \\
H_i X + Y & \xrightarrow{\beta_i + Y} & H_i X' + Y \\
\end{array}
\]

Since the domain \( X \) of that parallel pair is finitely presentable, it follows that some connecting morphism \((\alpha_{i,j})_{X'} + Y\) of the above filtered diagram also merges that parallel pair. That is, in the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & H_i X + Y \\
\downarrow{h} & & \downarrow{H_i h + Y} \\
X' & \xrightarrow{f'} & H_i X' + Y \\
\end{array}
\]

\[
\begin{array}{ccc}
& \xrightarrow{\alpha_{i,j} + Y} & \\
H_i X + Y & \xrightarrow{\alpha_{i,j} + Y} & H_i X' + Y \\
\end{array}
\]

the outward square commutes. Consequently, we have a morphism \( h \) in the filtered diagram defining \( R_j Y \), therefore,

\[
(((\alpha_{i,j})_X + Y) \cdot f)^\sharp_j = (((\alpha_{i,j})_{X'} + Y) \cdot f')^\sharp_j \cdot h.
\]

This proves that the desired triangle

\[
\begin{array}{ccc}
X & \xrightarrow{h} & X' \\
\downarrow{((\alpha_{i,j}+Y)f)^\sharp_j} & & \downarrow{((\alpha_{i,j}+Y)f')^\sharp_j} \\
R_j Y & \xrightarrow{e^\sharp} & R_j Y \\
\end{array}
\]

commutes, since \( e^\sharp \) and \( (e')^\sharp \) are by (1) independent of the above factorisation.
(III) Universal property. Let a cocone $e^\circ : X \to C$ (for $e$ in $\text{EQ}_Y$) of the diagram $\text{Eq}_Y$ be given. For each $i \in I$, we obtain a cocone of $D^{(i)}_Y$ (the diagram of all finitary flat equation morphisms of $H_i$ with respect to $Y$) as follows: to every $f : X \to H_iX + Y$, we assign

$$(((\beta_i)X + Y) \cdot f)^\circ : X \to C.$$ 

This is indeed a cocone, since for every morphism

$$X \xrightarrow{f} H_iX + Y \xrightarrow{\beta_i + Y} HX + Y \xrightarrow{h} H_iX + Y$$

and

$$X' \xrightarrow{f'} H_iX' + Y \xrightarrow{\beta_i + Y} HX' + Y$$

in $\text{Eq}_Y$, we have a morphism

$$X \xrightarrow{f} H_iX + Y \xrightarrow{\beta_i + Y} HX + Y \xrightarrow{h} H_iX + Y$$

in $\text{Eq}_Y$, thus, $(((\beta_i)X + Y) \cdot f)^\circ = (((\beta_i)X' + Y) \cdot f')^\circ \cdot h$. Therefore, we get a unique $c_i : R_iY \to C$

with

$$c_i \cdot f^\circ = (((\beta_i)X + Y) \cdot f)^\circ$$

for all $f$ in $\text{Eq}_Y^{(i)}$.

The uniqueness of $c_i$, for each $i \in I$, makes it obvious that these morphisms form a cocone of the diagram $(R_iY \mid i \in I)$. Thus, there is a unique $c : RY \to C$

with

$$c \cdot (\tilde{\beta}_i)Y = c_i \quad \text{for all } i \in I.$$ 

This morphism fulfills

$$c \cdot e^\circ = e^\circ$$

for each $e$ in $\text{EQ}_Y$.

In fact, we can factor $e$ via $f : X \to H_iX + Y$ as above, and then

$$c \cdot e^\circ = c \cdot (\tilde{\beta}_i)_Y \cdot f^\circ \quad \text{by the definition of } e^\circ$$

$$= c_i \cdot f^\circ \quad \text{by the definition of } c$$

$$= (((\beta_i)X + Y) \cdot f)^\circ \quad \text{by the definition of } c_i$$

$$= e^\circ.$$ 

To prove that $c$ is unique, we only have to observe that given a morphism $c : RY \to C$ with $c \cdot e^\circ = e^\circ$ for each $e$ in $\text{EQ}_Y$, it follows that $c \cdot (\tilde{\beta}_i)_Y = c_i$ for each $i \in I$ (which determines $c$). In fact, $R_iY$ is a colimit of $\text{Eq}_Y^{(i)}$ with colimit morphisms $f^\circ_i$, and from
\[ c \cdot e^\sharp = e^\odot, \] we conclude, for each \( f : X \rightarrow H_i X + Y \) in \( \text{EQ}^{(i)}_Y \), that
\[
\begin{align*}
c \cdot (\bar{\beta}_i)_Y \cdot f^\sharp &= c \cdot ((\beta_i)_X + Y) \cdot f^\sharp \\
&= ((\beta_i)_X + Y) \cdot f^\odot \\
&= c_i \cdot f^\sharp
\end{align*}
\]

**Remark 5.12.** Recall that since \( H \) is iteratable, it generates the monad \( T \) of Corollary 1.8. Together with
\[
\tau^* \equiv H \xrightarrow{H \eta} HT \xrightarrow{\tau} T,
\]
this is a free completely iterative monad on \( H \). We use
\[
\varepsilon : R \rightarrow T
\]
to denote the unique ideal monad homomorphism with
\[
\begin{tikzcd}
R & H & T \\
& \varepsilon & \\
& \tau^* &
\end{tikzcd}
\]
commutative, see Proposition 5.7. By Proposition 5.3, \( \varepsilon \) preserves solutions, thus, it satisfies (due to Remark 4.27)
\[
\varepsilon_Y \cdot e^\sharp = e^\dagger \quad \text{for all} \quad e : X \rightarrow HX + Y \quad \text{in} \quad \text{EQ}_Y,
\]
for all objects \( Y \) of \( \mathcal{A} \). Since \( RY \) is the union of all images of finitary flat equation morphisms (see Remark 5.10), it follows that \( \varepsilon_Y \) is a monomorphism (\( \text{viz.} \), the monomorphism representing that union). This proves the following corollary.

**Corollary 5.13.** \( R \) is a submonad of the monad \( T \) via \( \varepsilon \).

**Theorem 5.14 (Rational Substitution Theorem).** For every morphism
\[
s : X \rightarrow RY,
\]
there exists a unique extension to a homomorphism
\[
\bar{s} : RX \rightarrow RY
\]
of \( H \)-algebras (that is, a unique homomorphism with \( s = \bar{s} \cdot \eta^R_Y \)).

**Proof.** Existence is clear: \( \bar{s} = \mu_Y \cdot Rs \). Uniqueness is proved precisely as in Theorem 4.16.

\[
\square
\]

6. The rational monad of a set functor

We are going to define a rational monad of every finitary functor \( H \) of \( \text{Set} \). In the last section, this has been done whenever \( H \) preserves monomorphisms.
Now all monomorphisms \( m : X \to Y \) in \( \text{Set} \) with \( X \neq \emptyset \) are split – choose \( x_0 \in X \) and define \( e : Y \to X \) by
\[
e(y) = \begin{cases} x, & \text{if } m(x) = y \\ x_0, & \text{otherwise.} \end{cases}
\]
Then \( em = id \). Thus, every functor preserves these monomorphisms. The only trouble-makers are, thus, the empty functions
\[
\psi_X : \emptyset \to X \quad (X \neq \emptyset).
\]
Not all finitary functors preserve monomorphisms. For example, put \( HX = \{1\} \) (terminal object) for \( X \neq \emptyset \) and \( H\emptyset = 1 + 1 \). This defines \( H \) uniquely, and \( H\psi_1 : 1 + 1 \to 1 \) is, of course, no monomorphism. Luckily, iterative monads in \( \text{Set} \) do preserve monomorphisms.

**Proposition 6.1.** Every iterative monad in \( \text{Set} \) preserves monomorphisms.

**Proof.** Let \( S \) be an iterative monad. Observe that \( S\emptyset = S'\emptyset \) (since \( S = S' + \text{Id} \)).

If \( S'\emptyset = \emptyset \), then \( S\psi_X : \emptyset \to SX \) is a monomorphism and the proof is concluded. Suppose \( S'\emptyset \neq \emptyset \), choose \( u' : 1 \to S'\emptyset \) and put \( u = \sigma_{\emptyset} \cdot u' : 1 \to S\emptyset \). Then the composite
\[
e = 1 \xrightarrow{u} S\emptyset \xrightarrow{S\psi_1} S1
\]
is a guarded equation morphism for the empty set \( Y = \emptyset \) of parameters. And \( u \) is a solution of \( e \), that is, the diagram
\[
\begin{array}{ccc}
1 & \xrightarrow{u} & S\emptyset \\
\downarrow{v} & & \uparrow{\mu_{\emptyset}} \\
S\emptyset & \downarrow{S\psi_1} & \downarrow{Su} \\
S1 & \xrightarrow{S\psi_1} & SS\emptyset
\end{array}
\]
(where \( u = [u, \eta_{\emptyset}] : 1 + \emptyset \to S\emptyset \)) commutes. In fact, we have \( u \cdot \psi_1 = \eta_{\emptyset} : \emptyset \to S\emptyset \) because \( \text{Set}(\emptyset, S\emptyset) \) is a singleton set; the rest is clear. This proves that \( S\psi_1 \) is a monomorphism: given \( u, v : 1 \to S\emptyset \) with \( S\psi_1 \cdot u = S\psi_1 \cdot v = e \), then both \( u \) and \( v \) are solutions of \( e \), and thus, \( u = v \). Consequently, \( S\psi_X \) is a monomorphism for every \( X \).

**Remark 6.2.** For every endofunctor \( H \) of \( \text{Set} \), the equivalence relation on \( H\emptyset \) given by
\[
u \sim v \iff H\psi_X(u) = H\psi_X(v)
\]
where \( X \) is a fixed non-empty set, is independent of the choice of \( X \).

In brief,
\[
H\psi_X(u) = H\psi_X(v) \iff H\psi_1(u) = H\psi_1(v).
\]
In fact, since $X \neq \emptyset$, we have functions $f : 1 \to X$ and $g : X \to 1$, and the triangles

\begin{align*}
H\emptyset \xrightarrow{H\psi} HX & \quad H\emptyset \xrightarrow{H\psi} HX \\
\downarrow Hg & \quad \downarrow Hf \\
H1 & \quad H1
\end{align*}

commute (because $\emptyset$ is the initial object).

**Definition 6.3.** For every endofunctor $H$ of $\mathbf{Set}$, we define a quotient

$$\gamma : H \to H^+$$

of $H$ as follows.

For all non-empty sets $X$, we put $H^+X =HX$ and $\gamma_X = id$; for the empty set, let

$$\gamma_\emptyset : H\emptyset \to H\emptyset/\sim = H^+\emptyset$$

be the canonical function of the equivalence $\sim$ of Remark 6.2.

For all functions $f : X \to Y$ with $X \neq \emptyset$, put $H^+f = Hf$, and for the empty function put $H^+id_\emptyset = id_{H^+\emptyset}$ and

$$H^+\psi_X : [u] \mapsto \psi_X(u) \quad \text{for all } u \in H\emptyset$$

if $X$ is non-empty.

The above definition is a small modification of a procedure used by V. Trnková (Trnková 1971).

We now prove that $H^+$ is a reflection of $H$ in the category of all endofunctors of $\mathbf{Set}$ preserving monomorphisms.

**Lemma 6.4.** For every endofunctor $H$ of $\mathbf{Set}$, the functor $H^+$ preserves monomorphisms, and $\gamma : H \to H^+$ has the following universal property:

For every natural transformation $\delta : H \to K$, where $K$ preserves monomorphisms, there exists a unique natural transformation $\delta^+ : H^+ \to K$ with $\delta = \delta^+ \gamma$.

**Proof.** (I) $H^+$ preserves monomorphisms (equivalently, it maps $\psi_1$ to a monomorphism), since $[u] \neq [v]$ holds iff $H\psi_1(u) \neq H\psi_1(v)$, for all $u,v \in H\emptyset$.

(II) Let $\delta : H \to K$ be given. The equation $\delta = \delta^+ \gamma$ forces us to define

$$\delta_X^+ = \delta_X : HX \to KX \quad \text{for all } X \neq \emptyset.$$

It remains to discuss $\delta^+_{\emptyset} : H\emptyset/\sim \to K\emptyset$. All we have to prove is that for $u,v \in H\emptyset$, we have

$$u \sim v \implies \delta_{\emptyset}(u) = \delta_{\emptyset}(v).$$

(57)
Then \( \delta^+ \) is uniquely determined by \( \delta^+_\varnothing = \delta^+ \cdot \gamma_\varnothing \) \((\text{viz.}, \delta^+_\varnothing([u]) = \delta_\varnothing(u))\), and the naturality of \( \delta^+ \) is obvious. To show (57), use naturality of \( \delta \) on \( \psi_1 \):}

\[
\begin{array}{ccc}
H\varnothing & \xrightarrow{\delta^+_\varnothing} & K\varnothing \\
\downarrow{H_{\psi_1}} & & \downarrow{K_{\psi_1}} \\
H1 & \xrightarrow{\delta_1} & K1
\end{array}
\]

From \( u \sim v \) we conclude, since \( H_{\psi_1}(u) = H_{\psi_1}(v) \), that \( K_{\psi_1}(\delta_\varnothing(u)) = K_{\psi_1}(\delta_\varnothing(v)) \). Since \( K_{\psi_1} \) is a monomorphism, it follows that \( \delta_\varnothing(u) = \delta_\varnothing(v) \).

**Definition 6.5.** For every finitary endofunctor \( H \) the **rational monad of \( H \)** is defined to be the rational monad \( \rho^* : H^+ \rightarrow R \) of the above reflection \( \gamma : H \rightarrow H^+ \).

**Corollary 6.6.** The rational monad of \( H \), together with the natural transformation \( \gamma : H \rightarrow H^+ \), is a free iterative monad on \( H \).

**Proof.** In fact, since \( \rho^* \) is an ideal natural transformation, so is \( \rho^* \gamma \). Let \( S \) be an iterative monad and \( \lambda : H \rightarrow S \) be an ideal transformation, \( \lambda = \sigma\lambda' \). Since the subfunction \( S' \) of \( S \) preserves monomorphisms (see Proposition 6.1), the natural transformation \( \lambda' : H \rightarrow S' \) extends uniquely to \( (\lambda')^+ : H^+ \rightarrow S' \), and we obtain an ideal natural transformation

\[
\lambda^+ \equiv H^+ \xrightarrow{(\lambda')^+} S' \xrightarrow{\sigma} S.
\]

By Proposition 5.7, this yields a unique ideal monad homomorphism \( \lambda^+_+ : R \rightarrow S \) extending \( \lambda^+ \), that is, such that

\[
\begin{array}{ccc}
H & \xrightarrow{\gamma} & H^+ \\
\downarrow{\lambda} & \nearrow{\lambda^+} & \downarrow{\lambda^+_+} \\
& & R
\end{array}
\]

commutes.

**7. Conclusion and future directions**

We have constructed a free iterative monad on every monos-preserving, finitary endofunctor \( H \) of a ‘set-like’ category. In particular, every finitary endofunctor of \( \text{Set} \) generates a free iterative monad (or free iterative Lawvere theory). The method of our construction was coalgebraic, making heavy use of the fact that final coalgebras \( TX \) of the functors \( H(\cdot) + X \) form a free completely iterative monad on \( H \).

The proof is surprisingly technical, and one question it naturally raises is whether a simpler proof can be found. When \( H \) is a polynomial functor, the existence of a free iterative theory was established by C.C. Elgot (Elgot 1975) and the fact that this
is the theory of rational trees was proved later in Elgot et al. (1978). Their method is fundamentally different from ours, and their proof was also quite involved.

The rational monad as presented here is based on solutions of finitary flat equations. This can be extended to solutions of finitary equations in much the same style as mentioned, for polynomial functors, in the Introduction (see Section 1.1), we have proved this is Adámek et al. (2002).

In the work of Bloom and Ésik (Bloom and Ésik 1993), equational properties of solutions are described and revealed to have a general pattern in various fields. See also a recent restatement of some of these results in Simpson and Plotkin (2000). It would be interesting to investigate these properties categorically. L. Moss (Moss 2001) has already taken first steps in that direction.

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References

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