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On Rational Monads and Free Iterative Theories

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Abstract

For every finitary endofunctor H of Set a rational algebraic theory (or a rational finitary monad) R is defined by means of solving all finitary flat systems of recursive equations over H. This generalizes the result of Elgot and his coauthors, describing a free iterative theory of a polynomial endofunctor H as the theory R of all rational infinite trees. We present a coalgebraic proof that R is a free iterative theory on H for every finitary endofunctor H, which is substantially simpler than the previous proof by Elgot et al., as well as our previous proof. This result holds for more general categories than Set.

1 Introduction

Calvin Elgot has introduced iterative theories as a model of (potentially infinite) computations — a model that, unlike other approaches based on complete partial orders or complete metric spaces, etc. does not require any additional structure. The motivating idea was to obtain an infinite computation as a unique solution of a finite system of equations. For example, in a machine

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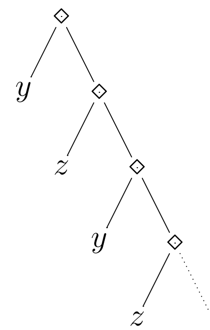
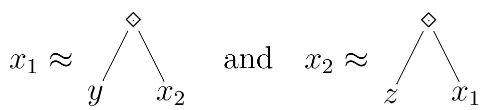
computing a binary operation $\diamond$ the following infinite computation

(1) 

can be described by recursive equations

(2) 

in variables x_1, x_2 (and parameters y, z). This has led Elgot to study Lawvere theories, or equivalently finitary[4] monads on Set, having the property that every finite system of recursive equations that is guarded (i.e., does not contain equations $x_i \approx x_j$) has a unique solution. He called such theories iterative.

A principal result of Elgot and his coauthors was a description of a free iterative monad on a given signature Σ: it is the theory formed by all *rational* Σ-labelled trees, i.e., trees which have only finitely many subtrees (up to tree isomorphism). The proof was quite involved, using refined algebraic techniques, and it occupied most of the series of papers [E], [BE] and [EBT]; see also the monograph [BÉ] for a compact (but not simpler) presentation.

It is the aim of our paper to show how coalgebraic methods lead to a much simpler proof of a more general result: we describe a *rational monad* R for every finitary endofunctor H of Set. Also for more general categories (e.g., posets, pointed sets, vector spaces, or unary algebras) we describe a rational monad on every finitary endofunctor preserving monomorphisms. And we prove that R is an iterative monad (which is equivalent to Elgot's iterative theory) and, in fact, a free iterative monad on H. We have to admit that this is already our second attempt of a coalgebraic proof. The first one, in [AMV$_1$], has been partially successful only: we did prove that, for H finitary, the rational monad is free on H, but the proof was very technical and long. Besides, for the generalization beyond Set we needed an unpleasant collection of side conditions which excluded all of the above examples except posets.

The present proof is much shorter. We rely on our previous work and provide detailed references to [AMV$_1$] – [AMV$_3$]; but all in all, if every detail were added to make the present paper self-contained, the size of the paper would hardly increase by a few pages (a dramatic improvement to [AMV$_1$]).

The main innovation of the current paper is a move from ideal monads to idealized ones. Ideal monads, introduced by C. Elgot, are monads (T, η, μ)

[4] A functor is called *finitary* if it preserves filtered colimits.

where $T = T' + Id$ is a coproduct with η as a coproduct injection and μ restricting to $T'T$. (A rare property.) Whereas an idealized monad is a monad (T, η, μ) together with a chosen right ideal $T' \longrightarrow T$ (in the usual sense of monoids); every monad has such an ideal, e.g., $T' = T$. By using idealized monads we are able (besides dropping some side conditions on the underlying category) to organize our proof in a much more compact and clear way than that in [AMV_1].

The crucial idea of the coalgebraic approach is to start with *completely iterative monads*, originally introduced in [EBT], that is, monads which allow for a unique solution of every guarded (not necessarily finite) system of recursive equations. As shown independently by Larry Moss [M] and our group in collaboration with Peter Aczel, see [AAV] and [AAMV], every finitary endofunctor H has the property that a final coalgebra, TY, exists for every functor $H(\_)+Y$, and TY is the object part of a naturally arising completely iterative monad T. In [AAMV] we proved that this is a free completely iterative monad on H. The proof is not trivial, but much simpler than any known proof for the rational monad. And, besides, this result about T holds in every category with finite coproducts such that coproduct injections are monomorphic.

The coalgebraic definition of the rational monad R is simple: it is the submonad of T obtained as the union of all solutions of finite guarded systems of recursive equations. The basic example has been mentioned above: let

$$H = H_\Sigma \quad \Big(X \mapsto \coprod_{\sigma \in \Sigma,\ \mathrm{ar}(\sigma)=n} X^n\Big)$$

be the polynomial functor w.r.t. the signature Σ, then RY is the (co)algebra of all rational Σ-labelled trees over Y. In the present paper we describe, e.g., the rational monad for the finitary-power-set functor $H = \mathcal{P}_{\mathit{fin}}$.

Following our paper [AMV_1], Ghani et al. define rational monads R in [GLM] for finitary endofunctors H of any locally finitely presentable category, and in this generality they prove that R is a monad. Moreover, if H preserves finitely presentable objects, R is proved to be "coalgebraic", a property stronger than "ideal". However, the main property of R of being iterative is not treated in that paper.

2 Free Completely Iterative Monad

2.1 Assumption. Throughout this section, whose aim is to recall the basic results of [M] and [AAMV], we assume that $\mathcal{A}$ is a category with finite coproducts, and that coproduct injections are monomorphisms.

2.2 For every finitary endofunctor H of $\mathcal{A}$ a final coalgebra, TY, of $H(\_)+Y$ exists (Y in $\mathcal{A}$). The coalgebra structure is, by Lambek's Lemma [L], an isomorphism

$$TY \xrightarrow{\cong} HTY + Y.$$

In other words, TY is a coproduct of HTY and Y; we denote by

$$\tau_Y : HTY \longrightarrow TY \qquad (\text{“}TY \text{ is an } H\text{-algebra”})$$

and

$$\eta_Y : Y \longrightarrow TY \qquad (\text{“}TY \text{ contains } Y\text{”})$$

the coproduct injections. Example: one binary operation symbol $\Sigma = \{\diamond\}$ corresponds to the polynomial functor $H_\Sigma Z = Z \times Z$ which is iteratable. Here TY is the (co)algebra of all finite and infinite binary trees over Y, i.e., with leaves labelled by elements of Y.

2.3 Substitution Theorem. (See [M] or [AAMV].) *For every morphism $s : X \longrightarrow TY$ there exists a unique extension to a homomorphism $\widehat{s} : TX \longrightarrow TY$ of H-algebras, i.e., such that $\widehat{s} \cdot \eta_X = s$.*

Corollary 2.4 *T is a monad w.r.t. $\eta : Id \longrightarrow T$ above, and $\mu : TT \longrightarrow T$ given by $\mu_X = \widehat{id_{TX}} : T(TX) \longrightarrow TX$. Moreover, $\widehat{s} = \mu_Y \cdot Ts$ for all $s : X \longrightarrow TY$.*

2.5 *Equation Morphisms.* Here we explain the concepts of a (guarded) equation morphism and its solution. We use elements in the explanation (i.e., $\mathcal{A} = \mathsf{Set}$), but the resulting concepts are meaningful in any category $\mathcal{A}$ with binary coproducts.

Given a set X of variables, we work with formal equations $x \approx e(x)$, one for each $x \in X$, whose right-hand sides live in $T(X + Y)$, for a set Y of parameters. Thus, *equation morphisms* are simply morphisms

$$e : X \longrightarrow T(X + Y)$$

Example: the equations (2) are identified with the function $e : \{x_1, x_2\} \longrightarrow T\{x_1, x_2, y, z\}$ where $e(x_i)$ is the right-hand side tree for x_i.

2.6 A *solution* of e assigns to every variable x an element $e^\dagger(x)$ of TY, i.e., in general this is a morphism

$$e^\dagger : X \longrightarrow TY,$$

with the following property. If every variable x is substituted by $e^\dagger(x) \in TY$, and every parameter y by $\eta_Y(y) \in TY$, then this substitution makes the formal equations $x_i \approx e(x_i)$ actual identities. More precisely, we first consider the substitution morphism

$$s = [e^\dagger, \eta_Y] : X + Y \longrightarrow TY$$

and form the H-algebra homomorphism $\widehat{s} = \mu_Y \cdot Ts : T(X+Y) \longrightarrow TY$ of 2.3 above. Then the composite of $e : X \longrightarrow T(X + Y)$ and $\widehat{s}$ is $e^\dagger : X \longrightarrow TY$.

Thus, in an arbitrary category, a solution of $e : X \longrightarrow T(X+Y)$ is a morphism $e^\dagger : X \longrightarrow TY$ for which the following square

$$\begin{array}{ccc} X & \xrightarrow{\;e^\dagger\;} & TY \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \mu_Y} \\ T(X+Y) & \xrightarrow[T[e^\dagger,\eta_Y]]{} & TTY \end{array}$$

commutes. Example: for the above e expressing (2) the solution is $e^\dagger : \{x_1, x_2\} \longrightarrow T\{y, z\}$ where $e^\dagger(x_1)$ is the tree of (1), and $e^\dagger(x_2) = z \diamond e^\dagger(x_1)$.

Remark 2.7 Not all recursive equations have unique solutions — consider, e.g., $x \approx x$. This has led C. Elgot to consider only *ideal* equation morphisms, i.e., those where the right-hand sides are neither single parameters nor single variables. (That is, $e : X \longrightarrow T(X+Y)$ factors through $\tau_{X+Y} : HT(X+Y) \longrightarrow T(X+Y)$.) This is unnecessarily restrictive: we can allow single parameters on the right-hand sides, only single variables are excluded. Thus, expressing $T(X+Y) = HT(X+Y)+X+Y$ as a coproduct of $HT(X+Y)+Y$ and X, we will consider *guarded* equation morphisms, i.e., equation morphisms $e : X \longrightarrow T(X+Y)$ for which we have a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{\;e\;} & T(X+Y) \\ & \searrow & \uparrow{\scriptstyle [\tau_{X+Y},\, \eta_{X+Y}\mathsf{inr}]} \\ & & HT(X+Y)+Y \end{array}$$

2.8 *Parametric Corecursion* is the following theorem (see [M]): For every finitary endofunctor H each guarded equation has a unique solution.

In [AAMV] we have proved the same result, called Solution Theorem there, by showing how solutions are obtained by a restriction to *flat* equation morphisms. These are the morphisms

$$e : X \longrightarrow HX + Y.$$

Thus, one way of viewing e is simply as a coalgebra of $H(\_) + Y$. But we can also identify e with the equation morphism

$$X \xrightarrow{\;e\;} HX + Y \xrightarrow{\;m_{X,Y}\;} T(X+Y)$$

where $m_{X,Y}$ is the canonical monomorphism with the following components

$$HX \xrightarrow{\;H\eta_X\;} HTX \xrightarrow{\;\tau_X\;} TX \xrightarrow{\;T\mathsf{inl}\;} T(X+Y) \xleftarrow{\;T\mathsf{inr}\;} TY \xleftarrow{\;\eta_Y\;} Y$$

Each $m_{X,Y} \cdot e$ is guarded, and we denote by $e^\dagger$ its solution; we have proved in [AAMV] the following

Proposition 2.9 (Solution = Corecursion) *For every flat equation morphism $e : X \longrightarrow HX + Y$ the solution $e^\dagger$ is equal to the unique homomorphism from the coalgebra e to the final coalgebra TY.*

That is, the following square

$$\begin{array}{ccc} X & \xrightarrow{\;e\;} & T(X+Y) \\ {\scriptstyle e^\dagger}\downarrow & & \downarrow{\scriptstyle He^\dagger+Y} \\ TY & = & HTY+Y \end{array}$$

commutes.

Remark 2.10 The proof of Parametric Corecursion in [AAMV] consists of a (trivial) proof of 2.9, and a procedure of substituting an arbitrary guarded equation morphism by a flat one. For example, for the functor $H_\Sigma Z = Z \times Z$ above, the flat equation morphisms are $x_i \approx t_i$, where t_i is a flat tree (an element of $HX + Y$), i.e., either $t_i = x_j \diamond x_k$, or $t_i = y \in Y$. The flattening of (2) is the following system

$$x_1 \approx \begin{array}{c} \diamond \\ \swarrow \;\; \searrow \\ x_3 \qquad x_2 \end{array} \qquad x_2 \approx \begin{array}{c} \diamond \\ \swarrow \;\; \searrow \\ x_4 \qquad x_1 \end{array} \qquad x_3 \approx y \qquad x_4 \approx z$$

(which is guarded but not ideal — this is the reason why we have departed from Elgot's approach in this point).

A flat equation morphism $e : X \longrightarrow HX + Y$ with X finite is called *finitary.*

2.11 Ideal Monads. In case H is a polynomial functor, $H = H_\Sigma$, the completely iterative monad T (with TY the algebra of all infinite Σ-labelled trees on Y) has the property that T is a coproduct of HT and Id

$$T = HT + Id$$

with the right-hand coproduct injection $\eta : Id \longrightarrow T$: in fact, every infinite tree t in TY is either of the form $t = \sigma(t_1, \ldots, t_n)$ for some $\sigma \in \Sigma$ (thus, t lies in the σ-summand TY^n of $H_\Sigma TY$) or t is a single variable (i.e., $t = \eta_Y(y)$ for some $y \in Y$). This has led C. Elgot to the concept of an ideal algebraic theory. Translated to the language of monads (see [AAMV] for the proof) this yields the following concept: an *ideal monad* is a monad (S, η, μ) together with a subfunctor $\sigma : S' \longrightarrow S$ such that $S = S' + Id$ is a coproduct with the right-hand injection $\eta : Id \longrightarrow S$ (and the left-hand one called $\sigma : S' \longrightarrow S$), and μ restricts to a natural transformation $\mu' : S'S \longrightarrow S'$; that is, the following square

$$\begin{array}{ccc} S'S & \xrightarrow{\;\mu'\;} & S' \\ {\scriptstyle \sigma S}\downarrow & & \downarrow{\scriptstyle \sigma} \\ SS & \xrightarrow[\;\mu\;]{} & S \end{array}$$

commutes. For example, given a finitary endofunctor H, the above monad T (see 2.4) is ideal: here $T = HT + Id$, and $\mu' = H\mu : HTT \longrightarrow HT$.

The usual algebraic theories (groups, lattices etc.) are not ideal, and Elgot introduced ideal theories for technical reasons connected with the above notion of ideal equation. We prefer a more general approach based on the concept of a right ideal.

Definition.

(i) By a (right) *ideal* of a monad (S, η, μ) is understood a subfunctor $\sigma : S' \longrightarrow S$ such that the corresponding restriction $S'S \xrightarrow{\sigma S} SS \xrightarrow{\mu} S$ of μ factors through σ, i.e., the following square

$$\begin{array}{ccc} S'S & \xrightarrow{\mu'} & S' \\ {\scriptstyle \sigma S}\downarrow & & \downarrow{\scriptstyle \sigma} \\ SS & \xrightarrow[\mu]{} & S \end{array}$$

commutes (for a, necessarily unique, transformation μ').

(ii) A pair consisting of a monad and its ideal is called an *idealized monad.* And S is called an *ideal monad* if $S = S' + Id$ with injections σ and η.

(iii) Given idealized monads S_1 and S_2, a monad morphism $h : S_1 \longrightarrow S_2$ is called an *idealized-monad morphism*, if it preserves the chosen ideals (i.e., if there exists $h' : S_1' \longrightarrow S_2'$ with $h\sigma_1 = \sigma_2 h'$).

A natural transformation $H \longrightarrow S$, where S is an idealized monad, is called *ideal* provided that it factors through $\sigma : S' \longrightarrow S$.

Remark 2.12 One can show that the arrow $\mu' : S'S \longrightarrow S$ is just a right S-module and $\sigma : S' \longrightarrow S$ is then a monomorphism of right S-modules. (The terminology comes from the world of monoidal categories, since every monad is a monoid in the endofunctor category.)

The most general definition of an idealized monad S is therefore as follows: a right S-module (S', μ') together with a right S-modules homomorphism $\sigma : S' \longrightarrow S$. All results can be proved at this level of generality. We postpone the details to a final version of the paper.

Remark 2.13 For every idealized monad S an *equation morphism*, i.e., a morphism $e : X \longrightarrow S(X + Y)$, is said to be *guarded* provided that e factors through $[\sigma_{X+Y}, \eta_{X+Y}\mathsf{inr}] : S'(X + Y) + Y \longrightarrow S(X + Y)$. And S is called *completely iterative* provided that every guarded equation morphism $e : X \longrightarrow S(X + Y)$ has a unique *solution*, i.e., a unique morphism $e^\dagger : X \longrightarrow SY$ such that the following square

$$\begin{array}{ccc} X & \xrightarrow{e^\dagger} & SY \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \mu_Y} \\ S(X+Y) & \xrightarrow[S[e^\dagger, \eta_Y]]{} & SSY \end{array}$$

commutes.

Thus, Parametric Corecursion states that the above monad T is completely iterative. The main result of [AAMV] is that T can be characterized as a free completely iterative monad on H.

Notation 2.14 For $T = HT + Id$ we put

(3) $$\tau^* \equiv H \xrightarrow{H\eta} HT \xrightarrow{\tau} T$$

It is clearly an ideal natural transformation.

The following result states that the functor $S \mapsto S'$ from the category of idealized monads on $\mathcal{A}$ to the category of endofunctors of $\mathcal{A}$ has a universal arrow at every *iteratable* functor H (i.e., such that for every object X a final $(H(\_) + X)$-coalgebra exists):

Theorem 2.15 (See [AAMV].) *The monad T is a free completely iterative monad on H.*

Explicitly: for every ideal natural transformation $\lambda : H \longrightarrow S$, where S is a completely iterative monad, there exists a unique idealized-monad morphism $\overline{\lambda} : T \longrightarrow S$ with $\lambda = \overline{\lambda} \cdot \tau^*$.

Let us remark that in [AAMV] we have only worked with ideal monads. But the equality $S = S' + Id$ has not been applied anywhere in the proof, i.e., we actually proved the above stronger result, related to idealized monads.

Remark 2.16 In the subsequent sections we work with *finitary equation morphisms* w.r.t. a monad S: these are morphisms

$$e : X \longrightarrow S(X + Y)$$

where X and Y are finitely presentable objects (i.e., objects whose hom-functors are finitary). The restriction on X to be finitely presentable is substantial (for Y this is just a technicality, whenever S is a finitary monad).

Definition 2.17 An idealized monad is called *iterative* provided that every finitary guarded equation morphism has a unique solution.

3 The Rational Monad

3.1 In the present section we define the rational monad of an arbitrary finitary endofunctor H of Set. In fact, the whole procedure works for much more general base categories $\mathcal{A}$ (and all endofunctors preserving monomorphisms) namely, all locally finitely presentable categories in the sense of Gabriel and Ulmer [GU] which have the following properties:

(a) a coproduct of two monomorphisms is a monomorphism,

(b) for every morphism $f : A \longrightarrow B$ and every object C the following square

$$\begin{array}{ccc} A & \xrightarrow{\mathsf{inl}} & A + C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f+C} \\ B & \xrightarrow[\mathsf{inl}]{} & B + C \end{array}$$

is a pullback

and

(c) every finitely generated object A (i.e., such that $\mathcal{A}(A, \_)$ preserves directed colimits of monomorphisms) is finitely presentable (i.e., $\mathcal{A}(A, \_)$ preserves filtered colimits).

(Observe that (a) and (b) hold in all extensive categories, which was our assumption in [AMV$_1$].)

Examples 3.2 Each of the following categories satisfies the above hypothesis:

$\mathsf{Set}_\bullet$ (pointed sets and base-point-preserving morphisms)

Pos (posets and order-preserving functions)

Vec (vector spaces and linear functions)

$\mathsf{Alg}(1)$ (algebras on one unary operation and homomorphism)

For reasons of a simpler presentation, we formulate our result just for endofunctors

$$H : \mathsf{Set} \longrightarrow \mathsf{Set}$$

preserving monomorphisms. The assumption of preserving monomorphisms can be, in case of Set, dropped, as explained in detail in Section 6 of [AMV$_1$], since for every set functor there there exists a monos-preserving set functor which defines the same rational monad. However, all our results hold in locally finitely presentable categories satisfying (a)–(c), as is easily seen by going through the proofs (and changing "set" to "object of $\mathcal{A}$" and "finite set" to "finitely presentable object" throughout). Let us observe that (b) implies that

coproduct injections are monomorphisms.

(Apply (b) to the unique morphism $f : A \longrightarrow 1$.)

3.3 Definition of RY. Let $H : \mathsf{Set} \longrightarrow \mathsf{Set}$ be a finitary functor preserving monomorphisms. Consequently, H generates a free completely iterative monad $T : \mathsf{Set} \longrightarrow \mathsf{Set}$, see Section 2.

For every finite set Y we denote by

$$\varepsilon_Y : RY \longrightarrow TY$$

the subset given by the union of the images, $\mathrm{im}(e^\dagger)$, of all solutions of finitary flat equation morphisms

$$e : X \longrightarrow HX + Y \quad (X \text{ finite}).$$

Shortly:

$$RY = \bigcup \mathrm{im}(e^\dagger)$$

Each $e^\dagger : X \longrightarrow TY$ thus restricts on the codomain to a morphism

$$e^\sharp : X \longrightarrow RY$$

satisfying

$$e^\dagger = \varepsilon_Y \cdot e^\sharp.$$

Lemma 3.4 (An alternative definition of RY.) *The object RY and the morphisms $e^\sharp$ form a colimit of a filtered diagram*

$$\mathrm{Eq}_Y : \mathsf{EQ}_Y \longrightarrow \mathsf{Set}$$

where EQ_Y is the category of all finitary flat equations (a full subcategory of the category of coalgebras of $H(\_) + Y$) and Eq_Y is the natural forgetful functor mapping $e : X \longrightarrow HX + Y$ to X.

Proof. See 4.6 of [AMV$_1$]. □

Remark 3.5 The proof of Lemma 3.4 needs conditions 3.1(a) and (c) and the fact that H preserves monomorphisms.

Corollary 3.6 *RY is a coalgebra over $H(\_) + Y$ and $\varepsilon_Y : RY \longrightarrow TY$ is a coalgebra homomorphism.*

In fact, colimits of coalgebras are formed on the level of Set.

3.7 Definition of R as a submonad of T. In order to define a finitary submonad $\varepsilon : R \longrightarrow T$ of a given monad T on Set, it is (necessary and) sufficient to specify

a subobject $\varepsilon_Y : RY \longrightarrow TY$ for every finite set Y

in such a way that (1) $\eta_Y : Y \longrightarrow TY$ factors through ε_Y and (2) for every morphism

$$s : X \longrightarrow RY \text{ with } X, Y \text{ finite}$$

the corresponding homomorphism $\widehat{\varepsilon_Y s} : TX \longrightarrow TY$ restricts to some (necessarily unique) $\widetilde{s} : RX \longrightarrow RY$, i.e., $\widetilde{s}$ exists for which the following square

$$\begin{array}{ccccc} RX & & \xrightarrow{\quad \widetilde{s} \quad} & & RY \\ {\scriptstyle \varepsilon_X}\downarrow & & & & \downarrow{\scriptstyle \varepsilon_Y} \\ TX & \xrightarrow[T(\varepsilon_Y s)]{} & TTY & \xrightarrow[\mu_Y]{} & TY \end{array}$$

commutes. See [AMV$_3$].

We define $\eta^R_Y = \mathsf{inr}^\sharp$, where $\mathsf{inr} : Y \longrightarrow HY + Y$ is the coproduct injection, forming a finitary flat equation morphism. It obvious that η_Y factors through ε_Y, since $\varepsilon_Y \cdot \mathsf{inr}^\sharp = \mathsf{inr}^\dagger = \eta_Y$.

To define $\widetilde{s}$, we construct a cocone

$$(e : Z \longrightarrow HZ + X \text{ in } \mathsf{EQ}_X) \mapsto (e^\ddagger : Z \longrightarrow RY)$$

of the diagram in Lemma 3.4 — then $\widetilde{s}$ is defined by $e^\ddagger = \widetilde{s} \cdot e^\sharp$ (for e in EQ_X). We first observe that since X is finite and $RY = \operatorname{colim} \mathrm{Eq}_Y$ is a filtered colimit, there exists a factorization

$$\begin{array}{ccc} X & \xrightarrow{\ s\ } & RY \\ & \searrow^{s'} & \uparrow{\scriptstyle f^\sharp} \\ & & V \end{array}$$

for some $f : V \longrightarrow HV + Y$ in EQ_Y. For every $e : Z \longrightarrow HZ + X$ we denote by $\overline{e} : Z + V \longrightarrow H(Z+V) + Y$ the flat equation morphism with the following components

$$Z \xrightarrow{\ e\ } HZ + X \xrightarrow{\ HZ+fs'\ } HZ + HV + Y \xrightarrow{\ [H\mathsf{inl}, H\mathsf{inr}]+Y\ } H(Z+V) + Y$$

and

$$V \xrightarrow{\ f\ } HV + Y \xrightarrow{\ H\mathsf{inr}+Y\ } H(Z+V) + Y$$

respectively. Then $e^\ddagger : Z \longrightarrow RY$ is defined as the left-hand component of

$$\overline{e}^\sharp : Z + V \longrightarrow RY.$$

The verification that $e^\ddagger$ form a cocone of Eq_X and that $\widetilde{s}$ makes the above square commutative is performed in [AMV$_1$], see Part (I) of the proof of 4.16.

Example 3.8 *Polynomial Functors.* For $HZ = Z \times Z$ as above, TY is the coalgebra of all finite and infinite binary trees on Y. The subcoalgebra RY consists of precisely all *rational* binary trees over Y, i.e., trees having (up to isomorphism) finitely many subtrees only. (For example, the tree t of (1) is rational: its only subtrees are t, y, z and $z \diamond t$.)

A flat equation $e : X \longrightarrow (X \times X) + Y$ can, as shown by Rutten in [R], be understood as a deterministic system with state set X with two inputs (say left and right) and with a "deadlock of type y" for every $y \in Y$. Here e assigns to every state x either a pair $(\mathsf{left}\, x, \mathsf{right}\, x)$ of next-step states, or the type y of deadlock that x is in. A solution $e^\dagger : X \longrightarrow TY$ assigns to every state x the tree unfolding $e^\dagger(x)$ of x: the nodes of the tree $e^\dagger(x)$ are all possible computation histories of x, which either end in a deadlock state (and are leaves labelled by $y \in Y$) or continue by the next possible left-hand and right-hand states. If the flat equation is finitary (i.e., we have finitely many states only) then $e^\dagger(x)$ is obviously rational. Conversely, every rational tree t is a tree unfolding of the (obvious) system whose states are all the subtrees of t.

More generally, for every polynomial functor $H_\Sigma : \mathsf{Set} \longrightarrow \mathsf{Set}$, TY is the coalgebra of all Σ-labelled trees over Y, and RY is the subcoalgebra of all rational trees.

Example 3.9 *Finitary Power-Set Functor $H = \mathcal{P}_{\mathit{fin}}$*, given by all finite subsets of the given set. Here TY has been described in [AMV$_2$], following the description of the final coalgebra ($T\emptyset$) by Barr [B], as follows. Recall that a labelled tree is called extensional if two different children of the same parent always define non-isomorphic subtrees. Let BY be the coalgebra of all finite-branching extensional (non-ordered) trees with leaves partially labelled in Y; the coalgebra structure $BY \longrightarrow \mathcal{P}_{fin}(BY) + Y$ is given by the inverse of tree-tupling. Then $TY = BY/{\sim_0}$ is the quotient modulo the bisimilarity congruence $\sim_0$ defined by

$$t \sim_0 s \text{ iff } t|_n = s|_n \text{ for all } n = 1, 2, 3, \ldots \ (t, s \in BY)$$

where $t|_n$ is the extensional quotient of the tree obtained by cutting t at depth n (and leaving all the new leaves unlabelled).

The description of RY is analogous to the previous example: it is the subcoalgebra of all ($\sim_0$-classes of) the trees which have, up to bisimilarity, only finitely many subtrees. Also the proof is analogous.

The case $Y = \emptyset$ is already interesting: a finitary flat equation morphism $e : X \longrightarrow \mathcal{P}_{fin}X$ is simply a finite graph. The solution $e^\dagger$ assigns to every vertex x the tree expansion $e^\dagger(x)$ of the graph at x — this is an (unlabelled) extensional tree which, obviously, has only finitely many subtrees modulo $\sim_0$. In the nonwellfounded set theory, see [A] or [BM], these are precisely the hereditarily finite nonwellfounded sets (defined as sets having a finite "picture", i.e., a finite graph whose expansion, in a chosen vertex, yields the "tree picture" of the set).

Remark 3.10

(i) The monad R is considered to be idealized w.r.t. the preimage R' of HT under $\varepsilon : R \longrightarrow T$ (i.e., the pullback of τ along ε):

$$\begin{array}{ccc} R' & \xrightarrow{\varepsilon'} & HT \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\tau} \\ R & \xrightarrow[\varepsilon]{} & T \end{array}$$

In fact, a preimage of an ideal is an ideal, as proved in [AMV$_3$].

Notice, that in a base category $\mathcal{A}$ with universal coproducts it holds that $R \cong R' + Id$. This follows immediately from the definition of R' and the fact that

$$\begin{array}{ccc} Id & \xrightarrow{id} & Id \\ {\scriptstyle\eta^R}\downarrow & & \downarrow{\scriptstyle\eta^T} \\ R & \xrightarrow[\varepsilon]{} & T \end{array}$$

is a pullback. Thus, if $\mathcal{A}$ has universal coproducts, then R is an ideal monad.

(ii) Observe that in the following diagram

$$\begin{array}{ccccc} & & \xrightarrow{\qquad\qquad\varepsilon'\qquad\qquad} & & \\ R' & \xrightarrow{i'} & HR & \xrightarrow{H\varepsilon} & HT \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle \mathsf{inl}} & & \downarrow{\scriptstyle \mathsf{inl}=\tau} \\ R & \xrightarrow{i} & HR+Id & \xrightarrow{H\varepsilon+Id} & HT+Id=T \end{array}$$

the right-hand square is a pullback by hypothesis and the equality

$$(H\varepsilon + Id)\cdot i \cdot \rho = \varepsilon \cdot \rho = \tau \cdot \varepsilon'$$

holds. Thus, there is a unique $i' : R' \longrightarrow HR$ such that the left-hand square commutes (in fact, it is even a pullback).

(iii) Every guarded equation morphism

$$\begin{array}{ccc} X & \xrightarrow{e} & R(X+Y) \\ & \searrow & \uparrow{\scriptstyle [\rho_{X+Y},\eta_{X+Y}\mathsf{inr}]} \\ & & R'(X+Y)+Y \end{array}$$

yields a guarded equation morphism $\varepsilon_{X+Y}e : X \longrightarrow T(X+Y)$ for T. This follows from the commutativity of the following square

$$\begin{array}{ccc} R(X+Y) & \xrightarrow{\varepsilon_{X+Y}} & T(X+Y) \\ {\scriptstyle [\rho_{X+Y},\eta_{X+Y}\mathsf{inr}]}\uparrow & & \uparrow{\scriptstyle [\tau_{X+Y},\eta_{X+Y}\mathsf{inr}]} \\ R'(X+Y)+Y & \xrightarrow[\varepsilon'_{X+Y}+Y]{} & HT(X+Y)+Y \end{array}$$

(iv) Recall from 2.16 and 2.17 the notions of finitary equation morphism and iterative monad. We are going to prove that every finitary guarded equation morphism $e : X \longrightarrow R(X+Y)$ has a unique solution which, moreover, is the restriction of the unique solution $(\varepsilon_{X+Y}e)^\dagger : X \longrightarrow TY$ w.r.t. T:

3.11 Rational Solution Theorem. *The rational monad is iterative.*

Proof. For every finitary guarded equation morphism $e : X \longrightarrow R(X+Y)$ we prove that e has a unique solution.

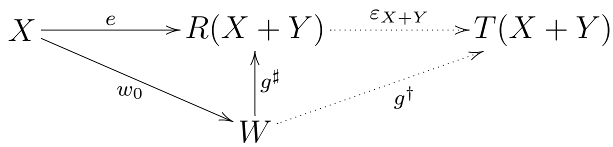
(I) Existence. Since e is guarded, we have a factorization

$$\begin{array}{ccc} X & \xrightarrow{e} & R(X+Y) \\ & \searrow{\scriptstyle e_0} & \uparrow{\scriptstyle [\rho_{X+Y},\eta_{X+Y}\mathsf{inr}]} \\ & & R'(X+Y)+Y \end{array}$$

Moreover, since X is finite and $R(X+Y)$ is a filtered colimit of Eq_{X+Y} there exists an object

$$g : W \longrightarrow HW + (X+Y)$$

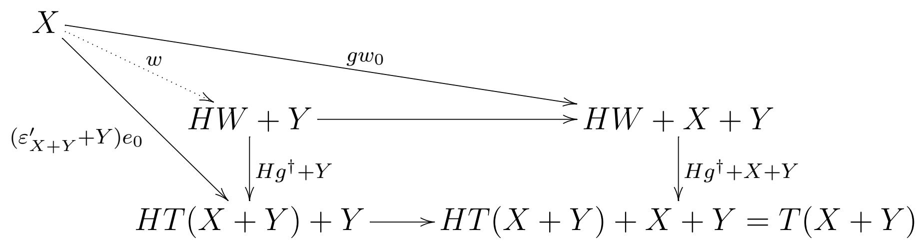
of EQ_{X+Y} such that e factors through the corresponding colimit map $g^\sharp$:



Due to $\varepsilon_{X+Y} g^\sharp = g^\dagger = (Hg^\dagger + X + Y)g$ (see 2.9), this yields

$$\begin{aligned}(Hg^\dagger + X + Y)gw_0 &= \varepsilon_{X+Y} e \\ &= \varepsilon_{X+Y}[\rho_{X+Y}, \eta_{X+Y}\mathsf{inr}]e_0 \\ &= [\tau_{X+Y}, \eta_{X+Y}\mathsf{inr}](\varepsilon'_{X+Y} + Y)e_0.\end{aligned}$$

Consequently, the following diagram (without the arrow w)

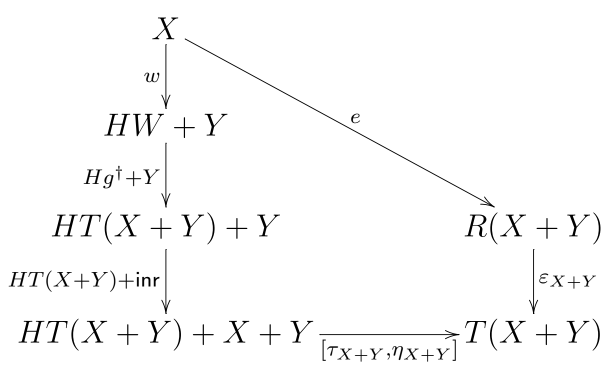


with coproduct injections as horizontal arrows, commutes. The square in that diagram is a pullback (due to Condition 3.1(b)), thus, we obtain $w : X \longrightarrow HW + Y$ as indicated.

We define a new finitary flat equation morphism, where $\mathsf{inm} : X \longrightarrow HW + X + Y$ denotes the middle injection, as follows

$$h \equiv W + X \xrightarrow{[g,\mathsf{inm}]} HW + X + Y \xrightarrow{[\mathsf{inl},w,\mathsf{inr}]} HW + Y \xrightarrow{H\mathsf{inl}+Y} H(W + X) + Y$$

and prove that the right-hand component of $h^\sharp : W + X \longrightarrow RY$ is a solution for e. In fact, this is proved in all detail in [AMV$_1$]: see the proof of Theorem 4.23 starting from the definition of h there. The only modification needed concerns the last diagram, called (29), of that argument, whose left-hand side, with numbers (ii) to (iv), is to be substituted by the following diagram:



This commutes by definition of w.

(II) Uniqueness. Since solutions for T are unique, see 2.8, and ε_Y is a monomorphism, it is sufficient to prove that the solution $(\varepsilon_{X+Y} e)^\dagger : X \longrightarrow TY$ w.r.t.

T is related to (any) solution $e^\ddagger$ of e w.r.t. R by

$$\varepsilon_Y e^\ddagger = (\varepsilon_{X+Y} e)^\dagger.$$

The last equation follows from the fact that $\varepsilon_Y e^\ddagger$ is a solution of $\varepsilon_{X+Y} e$, see Diagram (25) in the proof of Theorem 4.23 of [AMV$_1$]. □

Example 3.12 The endofunctor $H : Z \mapsto \mathbb{N} \times Z$ of Set is finitary. The final coalgebra TY of $\mathbb{N} \times (\_) + Y$ is given by

$$TY = \mathbb{N}^\infty + \mathbb{N}^* \times Y$$

(where $\mathbb{N}^\infty$ is the set of all infinite sequences on $\mathbb{N}$). For every flat equation morphism $e : X \longrightarrow \mathbb{N} \times X + Y$ the solution $e^\dagger$ assigns to a variable x_0 the sequence $(a_1, a_2, \ldots, a_n, y) \in \mathbb{N}^* \times Y$ if there are variables $x_1, \ldots, x_n$ with $e(x_0) = (a_1, x_1)$, $e(x_1) = (a_2, x_2)$, $\ldots$, $e(x_n) = y$. If no such y exists, then $e^\dagger(x_0) = (a_1, a_2, \ldots)$, where $e(x_0) = (a_1, x_1)$, $e(x_1) = (a_2, x_2)$, $\ldots$. The latter sequence is periodic whenever X is finite. It follows easily that the rational monad is given by

$$RY = \mathbb{N}^p + \mathbb{N}^* \times Y$$

where $\mathbb{N}^p \subseteq \mathbb{N}^\infty$ is the set of all periodic sequences.

Example 3.13 Let us perform the analogous example in the category

$$\mathsf{Alg}(1)$$

of algebras with one unary operation. Thus, let $H : \mathsf{Alg}(1) \longrightarrow \mathsf{Alg}(1)$ be defined by $Z \mapsto \mathbb{N} \times Z$, where $\mathbb{N}$ is the set of natural numbers with the operation of successor. Again H is finitary and it preserves monomorphisms. The completely iterative monad is given, as above, by

$$TY = \mathbb{N}^\infty + \mathbb{N}^* \times Y.$$

Here the unary operation α_{TY} of TY is given by that of Y (and by succ of $\mathbb{N}$) as follows:

$$\alpha_{TY}(a_n)_{n<\omega} = (\text{succ } a_n)_{n<\omega}$$

and

$$\alpha_{TY}((a_0, \ldots, a_{n-1}), y) = ((\text{succ } a_0, \ldots, \text{succ } a_{n-1}), \alpha_Y(y)).$$

However, some nonperiodic sequences of $\mathbb{N}^\infty$ appear in the rational monad RY: for the object

$$X = \{x_0, x_1, \ldots\} \quad \text{with } \alpha_X(x_i) = x_{i+1}$$

of variables (which is finitely presentable) consider the equation morphism $e : X \longrightarrow HX$ given by

$$e(x_i) = (k + i, x_{r+i}) \quad \text{for } i = 0, 1, 2, \ldots \quad (r, k < \omega).$$

The solution $e^\dagger : X \longrightarrow T\emptyset = \mathbb{N}^\infty$ turns x_0 into the nonperiodic sequence $(k, k + r, k + 2r, \ldots)$.

4 The Freeness of the Rational Monad

4.1 Throughout this section H denotes a finitary endofunctor of Set (or, more generally, of a locally finitely presentable category satisfying (a)–(c) in 3.1) preserving monomorphisms. We repeat that the last condition can be dropped in case of Set (see the last section of [AMV$_1$]).

Observation 4.2 Every morphism $p : Z \longrightarrow HY$ with Y and Z finite defines a finitary flat equation morphism

(4)
$$e_p \equiv Z + Y \xrightarrow{p+Y} HY + Y \xrightarrow{H\mathsf{inr}+Y} H(Z+Y) + Y$$

whose solution is

(5)
$$e_p{}^{\dagger} \equiv Z + Y \xrightarrow{[\tau_Y H\eta_Y p, \eta_Y]} TY$$

This follows from 2.9 because the following square

$$\begin{array}{ccccc} Z+Y & \xrightarrow{p+Y} & HY+Y & \xrightarrow{H\mathsf{inr}+Y} & H(Z+Y)+Y \\ \big\downarrow{\scriptstyle [\tau_Y H\eta_Y p, \eta_Y]} & & & & \big\downarrow{\scriptstyle H[\tau_Y H\eta_Y p, \eta_Y]+Y} \\ TY & & \xrightarrow[{[\tau_Y, \eta_Y]^{-1}}]{} & & HTY+Y \end{array}$$

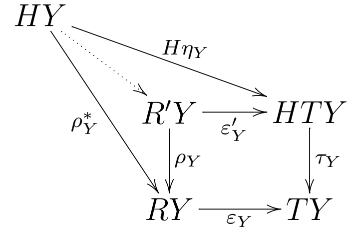
commutes. (For the right-hand component, with domain Y, this is obvious. For the left-hand one this follows from $H[\tau_Y H\eta_Y p, \eta_Y] \cdot H\mathsf{inr} = H\eta_Y$.)

4.3 Definition of $\rho^* : H \longrightarrow R$. We define a natural transformation ρ^* by specifying the components $\rho^*_Y : HY \longrightarrow RY$ for all finite Y — since both H and R are finitary, this is sufficient. For every $p : Z \longrightarrow HY$ in the comma-category Set_{fin}/HY, where Set_{fin} denotes the category of finite sets, the left-hand components of $e^\sharp_p : Z + Y \longrightarrow RY$ (see (5)) are easily seen to form a cocone for the canonical diagram $\mathsf{Set}_{fin}/HY \longrightarrow \mathsf{Set}$, thus, we can define ρ^*_Y by commutativity of the following squares

(6)
$$\begin{array}{ccc} Z & \xrightarrow{\mathsf{inl}} & Z+Y \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle e^\sharp_p} \\ HY & \xrightarrow[\rho^*_Y]{} & RY \end{array}$$

for all p in Set_{fin}/HY.

Observation 4.4 ρ^* is ideal, i.e., it factors through $\rho : R' \longrightarrow R$. In fact, from (5) it follows that $\tau_Y \cdot H\eta_Y = \varepsilon_Y \cdot \rho^*_Y$:



because given any p in Set_{fin}/HY we have

$$(\tau_Y \cdot H\eta_Y) \cdot p = e_p{}^\dagger \cdot \mathsf{inl} = \varepsilon_Y \cdot e_p^\sharp \cdot \mathsf{inl} = (\varepsilon_Y \cdot \rho^*_Y) \cdot p.$$

Theorem 4.5 *The rational monad is a free iterative monad on H. That is, for every iterative monad S and every ideal transformation $\lambda : H \longrightarrow S$ there exists a unique idealized-monad morphism $\overline{\lambda} : R \longrightarrow S$ with $\lambda = \overline{\lambda} \cdot \rho^*$.*

Proof. (I) Definition of $\overline{\lambda}_Y$. First, for every object

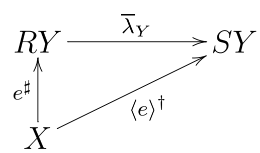
$$e : X \longrightarrow HX + Y \quad \text{in } \mathsf{EQ}_Y$$

define a (guarded) equation morphism $\langle e \rangle$ w.r.t. S as follows:

$$\langle e \rangle \equiv X \xrightarrow{e} HX + Y \xrightarrow{\lambda_X + \eta^S_Y} SX + SY \xrightarrow{[S\mathsf{inl}, S\mathsf{inr}]} S(X+Y)$$

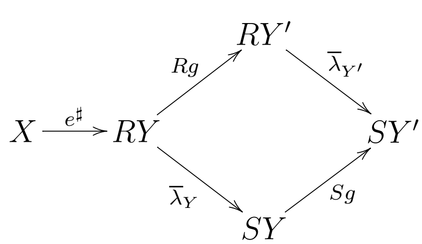
Since λ is natural, it is easy to see that all $\langle e \rangle^\dagger$ form a cocone for $\mathrm{Eq}_Y : \mathsf{EQ}_Y \longrightarrow \mathsf{Set}$, thus, we can define $\overline{\lambda}_Y$ to be the unique morphism making the diagrams

(7)



commutative for all $e : X \longrightarrow HX + Y$ in EQ_Y.

(II) $\overline{\lambda}_Y$ is natural in Y. In fact, consider any $g : Y \longrightarrow Y'$ in Set and any $e : X \longrightarrow HX + Y$ in EQ_Y. We want to show that both legs of the following diagram



are equal. The lower passage gives

$$Sg \cdot \overline{\lambda}_Y \cdot e^\sharp = Sg \cdot \langle e \rangle^\dagger$$

and the upper one yields

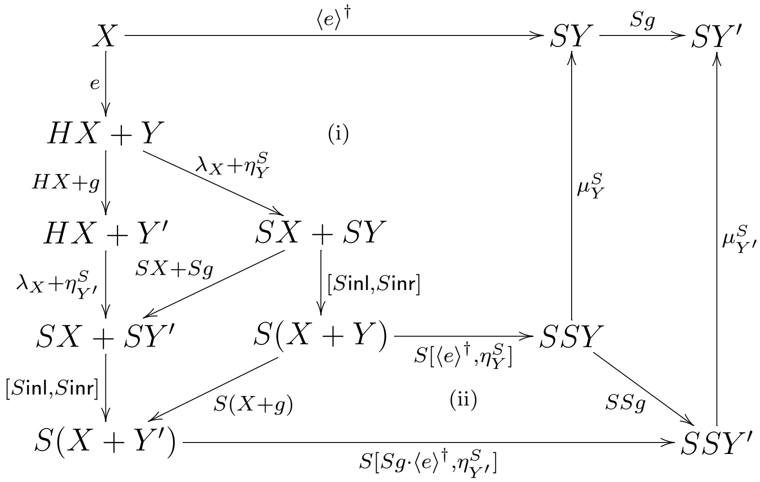
$$\overline{\lambda}_{Y'} \cdot Rg \cdot e^\sharp = \overline{\lambda}_{Y'} \cdot (HX + g)e)^\sharp = \langle (HX + g)e \rangle^\dagger.$$

It therefore suffices to show

(8) $$Sg \cdot e^{\dagger} = \langle (HX + g)e \rangle^{\dagger}.$$

In other words, we need to show that the outward square of (9) commutes.

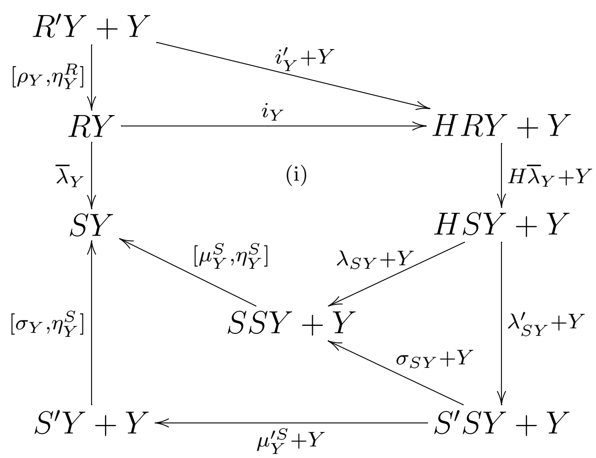
(9)



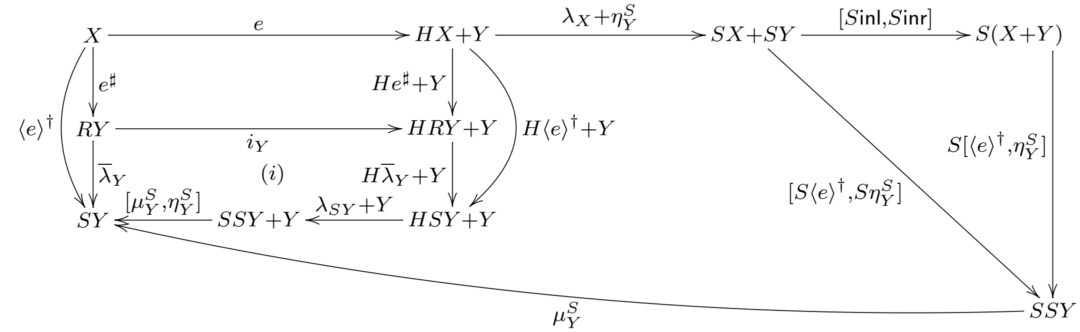
The region (i) commutes by definition of $(\_)^{\dagger}$, and (ii) clearly commutes when S is removed. The rest of (9) commutes trivially.

(III) To show that $\overline{\lambda}$ is ideal, it is sufficient to prove that diagram (10) commutes.

(10)

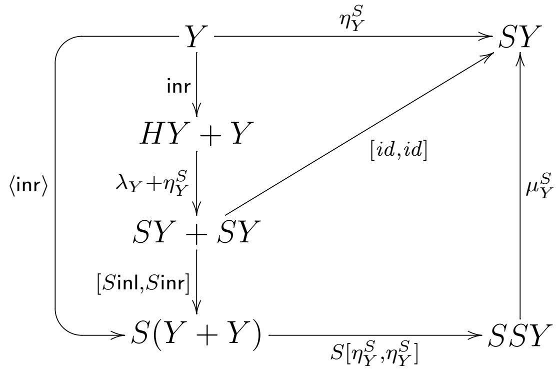


In fact, this is obvious except for the region (i), which commutes, since the following diagram does:



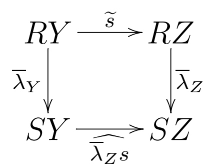
(IV) $\overline{\lambda}$ is a monad morphism. In fact, we first prove that $\overline{\lambda}_Y \cdot \eta^R_Y = \eta^S_Y$. Since η^R_Y is, by Definition 3.7, $\mathsf{inr}^\sharp$, the equation follows from the commutativity of (11).

(11)



Next we show that the following square

(12)



where $\widehat{\overline{\lambda}_Z s} = \mu^S_Z \cdot S(\overline{\lambda}_Z s)$, commutes for every $s : Y \longrightarrow RZ$, where Y and Z are finite sets. Since Y is finite, s factors through a colimit morphism of $RZ = \operatorname{colim} \mathsf{EQ}_Z$, see Lemma 3.4, i.e., we have a morphism

$$f : V \longrightarrow HV + Z \quad (V \text{ finite})$$

such that

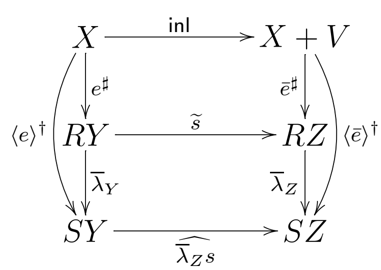
$$s = f^\sharp s' \quad \text{for some } s' : Y \longrightarrow V.$$

Recall from 3.7 that $\widetilde{s}$ is defined by

(13) $$\widetilde{s}\, e^\sharp = \bar{e}^\sharp \mathsf{inl} \quad \text{for all } e : Z \longrightarrow HZ + X \text{ in } \mathsf{EQ}_X.$$

Now (12) commutes because when precomposed with any of the colimit maps $e^\sharp$ it yields the following diagram

(14)



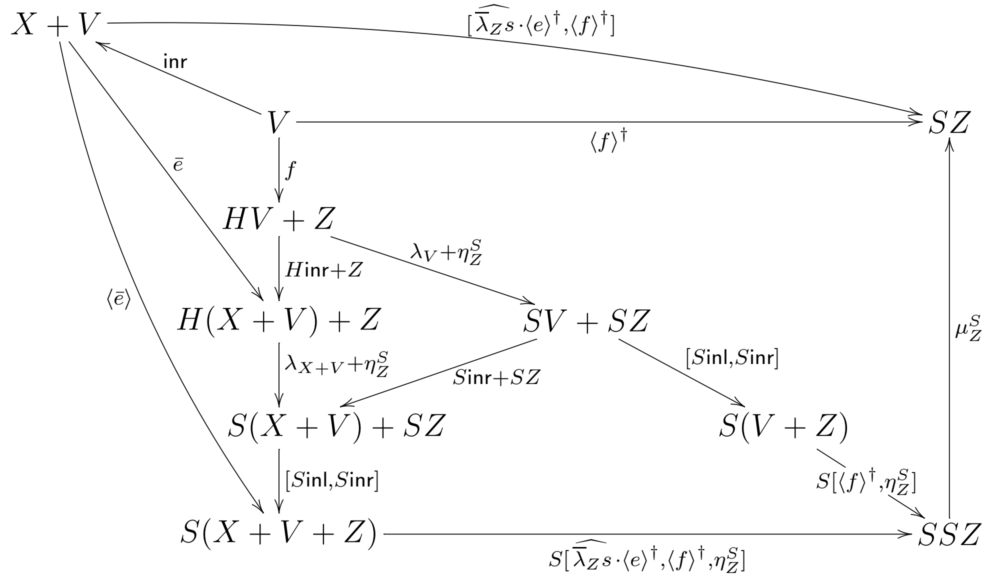
whose commutativity follows from (13) and the following equality

(15) $$\langle \bar{e} \rangle^\dagger = [\widehat{\overline{\lambda}_Z s} \cdot \langle e \rangle^\dagger, \langle f \rangle^\dagger] : X + V \longrightarrow SZ$$

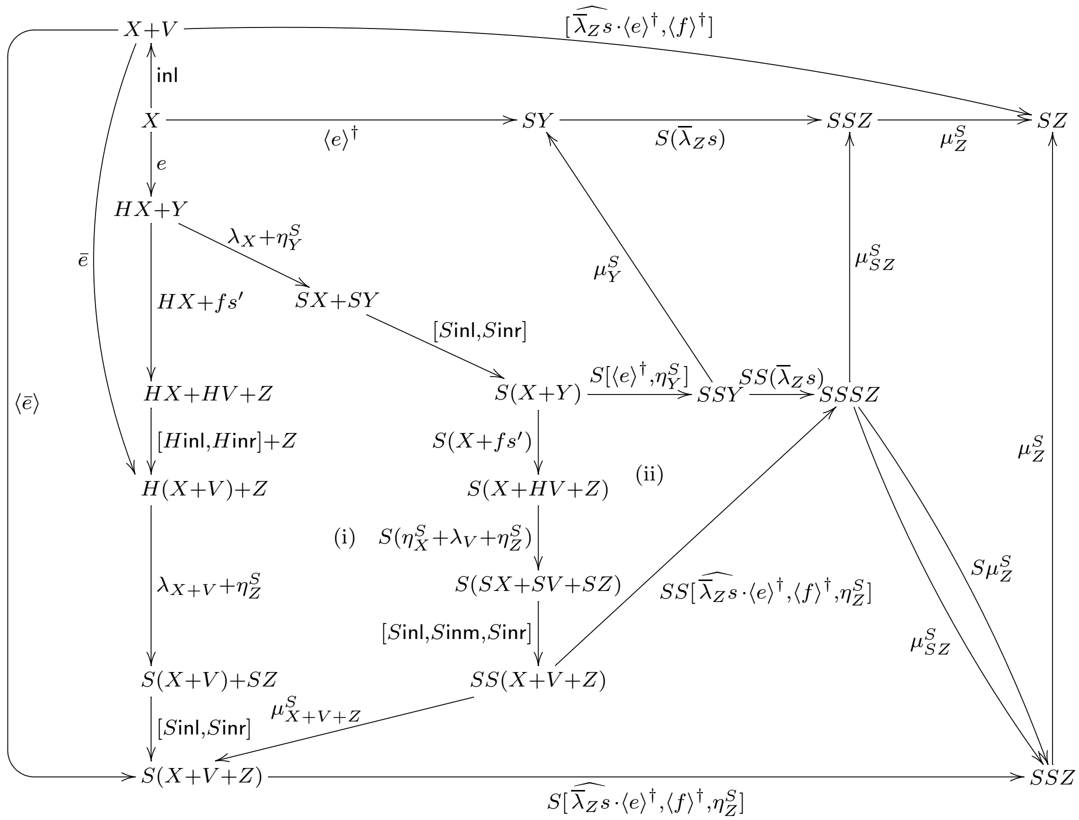
which we prove now.

The right-hand components of (15) are equal due to the commutativity of (16).

(16)

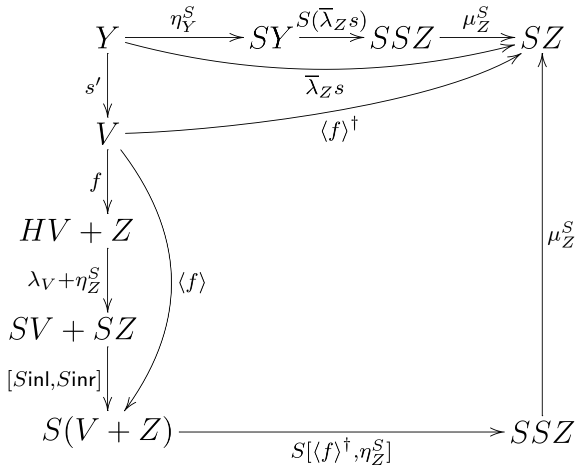


For the left-hand components of (15), consider the following diagram:



To see that (i) commutes consider the components of $HX+Y$ separately and use naturality of λ and η as well as the monad laws for S. All other regions, except for (ii), commute for obvious reasons. We do not claim that (ii) commutes: it is sufficient to show that $S\mu^S_Z$ merges both sides of (ii). In fact, consider the components after removing S: the right-hand components commute due to the diagram (17).

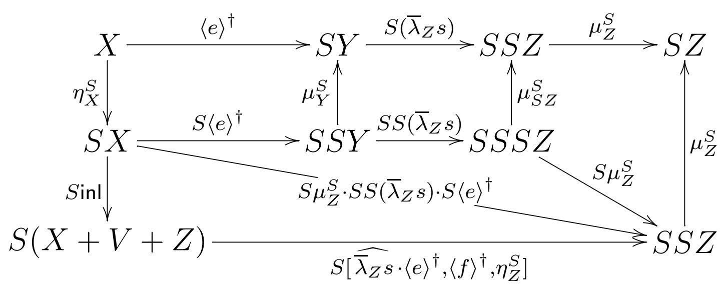
(17)



The triangle in the upper part of (17) commutes, since $\overline{\lambda}_Z \cdot s = \overline{\lambda}_Z \cdot f^\sharp \cdot s' = \langle f\rangle^\dagger \cdot s'$, thus, (17) commutes.

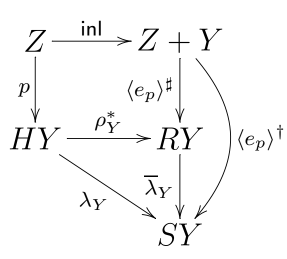
For the left-hand components of (ii) composed with $S\mu^S_Z$ consider the commutative diagram (18).

(18)



This completes the proof that $\overline{\lambda}$ is a monad morphism.

(V) $\overline{\lambda}$ extends λ. By definition of ρ^*_Y the square in the following diagram

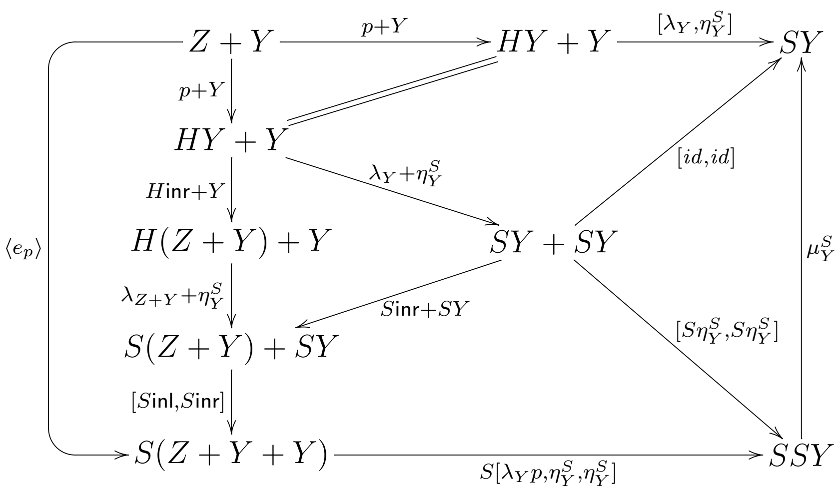


commutes. To prove that $\overline{\lambda}_Y \cdot \rho^*_Y = \lambda_Y$ it therefore suffices to show that

$$\langle e_p\rangle^\dagger = [\lambda_Y \cdot p, \eta^S_Y]$$

This is verified by the following commutative diagram.

(19)

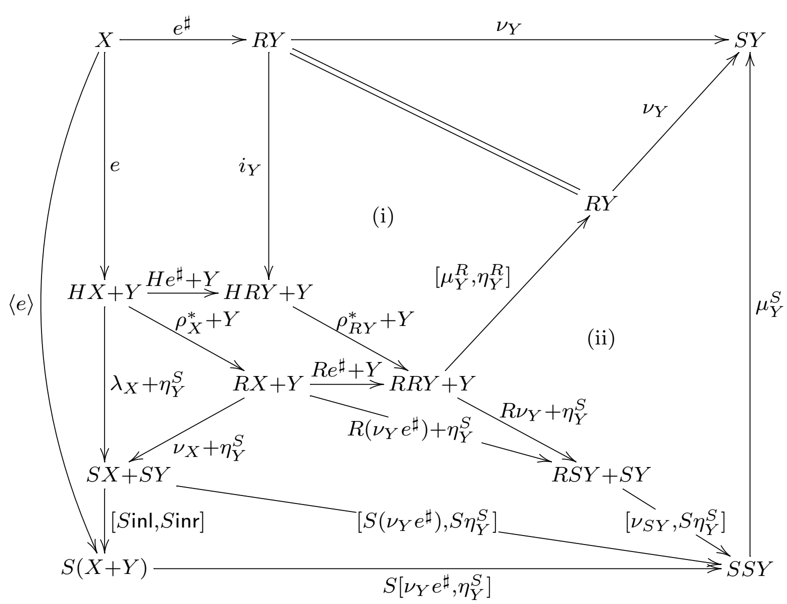


(VI) Uniqueness of $\overline{\lambda}$. Suppose that $\nu : R \longrightarrow S$ is an idealized-monad morphism extending λ. It suffices to show, due to (7), that for every finite set Y and every $e : X \longrightarrow HX + Y$ in EQ_Y we have

$$\nu_Y e^\sharp = \langle e \rangle^\dagger \tag{20}$$

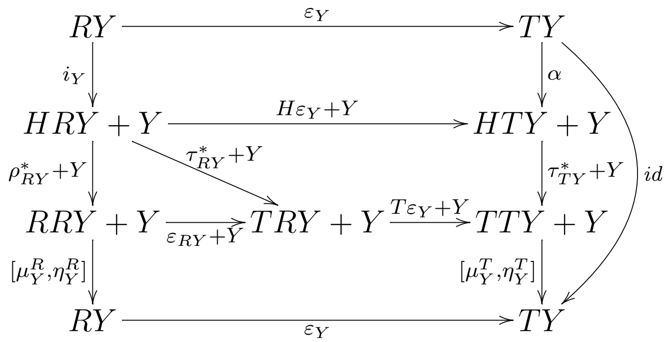
To conclude the proof, let us prove that the diagram (21) commutes.

(21)



For (i) use the fact that ε_Y is a monomorphism. Thus, it is sufficient to prove that the following diagram (22) commutes (where $\alpha = [\tau_Y, \eta_Y^T]^{-1}$ is the final coalgebra structure on TY and $\tau^* = \tau \cdot H\eta$, see (3)):

(22)



The lower rectangle of (22) commutes, since ε is a monad morphism (see Paragraph 3.7). The upper rectangle commutes by Corollary 3.6. In the middle part of (22) the triangle on the left commutes by Observation 4.4 and the trapezoid on the right is naturality of τ^*. Finally, the curved region on the right commutes due to definition of α and τ^*_Y.

For (ii) in (21) use the fact that ν is a monad morphism, all other parts of (21) obviously commute. □

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