Final Coalgebras And a Solution Theorem for Arbitrary Endofunctors

Jiří Adámek\textsuperscript{1,4} Stefan Milius\textsuperscript{2} Jiří Velebil\textsuperscript{3,4}

\textit{Institute of Theoretical Computer Science, Technical University, Braunschweig, Germany}

\begin{abstract}
Every endofunctor $F$ of $\textbf{Set}$ has an initial algebra and a final coalgebra, but they are classes in general. Consequently, the endofunctor $F^\infty$ of the category of classes that $F$ induces generates a completely iterative monad $T$. And solutions of arbitrary guarded systems of iterative equations w.r.t. $F$ exist, and can be found in naturally defined subsets of the classes $TY$.

More generally, starting from any category $\mathcal{K}$, we can form a free cocompletion $\mathcal{K}^\infty$ of $\mathcal{K}$ under small-filtered colimits (e.g., $\textbf{Set}^\infty$ is the category of classes), and we give sufficient conditions to obtain analogous results for arbitrary endofunctors of $\mathcal{K}$.

\textit{Key words:} initial algebra, final coalgebra, completely iterative monad
\end{abstract}

\section{Introduction}

In process algebra a system is often described in the form of equations

$$s = (s_1, a_1) \text{ or } (s_2, a_2) \text{ or } \ldots$$

where $s$, $s_1$, $s_2$, $\ldots$ are states (from a desired state set $S$) and $a_1$, $a_2$, $\ldots$ are actions (from a given set $\text{Act}$). Thus, the system is described by a labelled transition system

$$\sigma : S \rightarrow \mathcal{P}(S \times \text{Act})$$

assigning to every state $s$ the set $\sigma(s)$ of all the possible pairs on the right-hand side. Thus $\sigma$ represents a system of flat recursive equations, where “flat”

\textsuperscript{1} \text{Email: adamek@iti.cs.tu-bs.de}
\textsuperscript{2} \text{Email: milius@iti.cs.tu-bs.de}
\textsuperscript{3} \text{Email: velebil@iti.cs.tu-bs.de}
\textsuperscript{4} The support of the Grant Agency of the Czech Republic under the Grant No. 201/02/0148 is gratefully acknowledged.

\textit{©}2002 Published by Elsevier Science B. V.
refers to the fact that $\mathcal{P}$ appears just once, non-iterated, on the right-hand side. A “solution” of that system of equation is a description of the states of the system by the corresponding extensional trees, unique up to bisimilarity.

In a number of natural examples, non-flat equations play a rôle. For example the sequence

\[ x \equiv 1, 1, 1, \ldots \]

of natural numbers can be presented in the form of the equation

\[ x = (1, x). \]

Using the well-established set-theoretical notion for pairs, this means that

\[ x = \{\{1\}, \{1, x\}\} \]

This has, for $S = \{x\}$, the form of the (non-flat) iterative equation

\[ \sigma : S \to \mathcal{P}(S + \{1\}) \]

It is the aim of this paper to study equations of this kind, and to establish a general result on the existence and uniqueness of solutions.

In our previous work [AAV] and [AAMV] we have studied recursive equations for all “iteratable” endofunctors $H$ of $\text{Set}$, i.e., all endofunctors such that $H(\_)+X$ has a final coalgebra for every set $X$. This, of course, excluded important functors such as $H = \mathcal{P}$. The same restriction has been considered by Larry Moss [M]. In the present paper we show that the previous result, namely that every guarded system of recursive equations has a unique solution, can be proved for all endofunctors $H$ of $\text{Set}$. The trick is that we extend $H$ to an endofunctor of

\[ \text{Class} \]

the category of classes and class functions, obtaining an essentially unique functor $H^\infty : \text{Class} \to \text{Class}$ preserving small-filtered colimits (= large colimits which are $\lambda$-filtered for all small cardinals $\lambda$). Or, equivalently, to a set-based endofunctor $H^\infty$ in the terminology of Aczel and Mendler [AM]; recall that by their Final Coalgebra Theorem, $H^\infty(\_)+X$ has a final coalgebra, see also [HL]. Then $H^\infty$ is iterable, and we can thus use the previous results, just moving from sets to classes. But even better: no concrete system of iterative equations actually requires this move from $\text{Set}$ to $\text{Class}$! For example, the power-set functor $\mathcal{P}$ is iterable only when extended to $\mathcal{P}^\infty : \text{Class} \to \text{Class}$ (the functor assigning to every class the class of all subsets). A final coalgebra of $\mathcal{P}^\infty$ is the coalgebra $B/\sim$ where

\[ B \]

is the coalgebra of all extensional trees

and

\[ \sim \]

is the bisimilarity equivalence on $B$ (which we describe in Section 5 below).

Now $B$ is, of course, a proper class and so is $B/\sim$ (since a final coalgebra is a fixed point, by Lambek’s Lemma, but $\mathcal{P}$ has no fixed points in $\text{Set}$). However,
every system of equations (with a set of variables) has a unique solution that lives in a natural small subcoalgebra of $B$. This is so because every transition system is $\lambda$-branching for some cardinal number $\lambda$. Thus, it is an iterative equation morphism

$$\sigma : X \rightarrow \mathcal{P}_\lambda X$$

for the functor $\mathcal{P}_\lambda$ of all subsets of cardinality less than $\lambda$. And $\mathcal{P}_\lambda$ is iterable (in $\text{Set}$) with final coalgebra which is a natural subcoalgebra of that of $\mathcal{P}$. The morale of this is: for every transition system one has a unique solution in $B/\sim$, and the solution also lives in a small subcoalgebra (which one can ignore unless one objects to classes too much).

All this has nothing to do with $\mathcal{P}$. We prove that for every endofunctor $H : \text{Set} \rightarrow \text{Set}$ there is a natural iterable extension $H^\infty : \text{Class} \rightarrow \text{Class}$. And if $T^\infty Y$ denotes a final coalgebra of $H^\infty (\_ ) + Y$, then every guarded equation system of $H^\infty$ with parameters in $Y$ has a unique solution in $T^\infty Y$.

Now all this has nothing to do with $\text{Set}$ either! For every cocomplete category $\mathcal{K}$ we construct an extension $\mathcal{K}^\infty$ of $\mathcal{K}$ such that every endofunctor $H$ of $\mathcal{K}$ naturally extends to an iterable endofunctor $H^\infty$ of $\mathcal{K}^\infty$. Thus, guarded equation morphisms have unique solutions in $\mathcal{K}^\infty$.

The above case of non-labelled transition systems was one of the motivations for the introduction of non-well-founded set theory. Our paper could thus be considered as a continuation of the program of Michael Barr [B] of deleting non-well-foundedness from process algebra. There is no question that there is a certain loss of elegance in the process, but we feel that the loss is less heavy than expected. We return to this question in Section 5.

Set-Theoretical Assumptions

We have, essentially, just one, standard, assumption: that a universe of “small” sets has been chosen, so that we can form the category of all small sets. Now assuming that the universe itself is a (non-small) set in some higher universe, we can denote by

$$\aleph_\infty$$

the cardinality of that set. This enables us to identify

small sets with sets of cardinality less than $\aleph_\infty$

and

classes as sets with cardinality at most $\aleph_\infty$.

More precisely, for a set theorist, the universe of small sets can be the $\aleph_\infty$-th member $V(\aleph_\infty)$ of the cumulative hierarchy. However, we will take as

Set

3
the category of all sets of cardinality less than $\aleph\infty$ (equivalent to $V(\aleph\infty)$). And we take as

Class

the category of all sets of cardinality less than or equal to $\aleph\infty$.

We call a category $\mathcal{K}$ locally small if all objects form a class and every hom-set $\mathcal{K}(A, B)$ is small.

2 Solution Theorem for Iterable Functors

In the present section we recall results obtained independently by Larry Moss in [M] and our group [AAV], [AMMV]. Throughout this section, $\mathcal{K}$ denotes a category with binary coproducts.

Definition 2.1 A functor $H : \mathcal{K} \rightarrow \mathcal{K}$ is called iterable provided that for every object $X$ of $\mathcal{K}$ a final coalgebra

$$TX$$

of the functor $H(\_)+X$ exists.

Examples 2.2

(i) Every polynomial endofunctor $H_\Sigma$ of Set is iterable. Here $\Sigma$ is a (possibly infinitary) signature, i.e., a set of operation symbols $\sigma$ with prescribed arities $\text{ar}(\sigma)$, which are cardinal numbers. And $H_\Sigma$ assigns to every set $X$ the coproduct

$$\bigsqcup_{\sigma \in \Sigma} X^{\text{ar}(\sigma)}.$$  

Here $TX$ is the coalgebra of all (finite or infinite) $\Sigma$-labelled trees over $X$. That is, trees with leaves labelled by nullary operation symbols or variables from $X$, and inner nodes (of $n$ children) labelled by $n$-ary operation symbols.

(ii) More generally, every accessible (=bounded) endofunctor of Set is iterable.

(iii) The power-set functor $\mathcal{P} : \text{Set} \rightarrow \text{Set}$ is not iterable.

Notation 2.3 By Lambek’s Lemma, the structure arrow $TX \rightarrow HTX+X$ of the final coalgebra $TX$ is an isomorphism. That is, $TX$ is a coproduct of $HTX$ and $X$. We denote by

$$\tau_X : HTX \rightarrow TX \quad ("TX \text{ is an } H\text{-algebra")},$$

and

$$\eta_X : X \rightarrow TX \quad ("TX \text{ contains } X")$$

the coproduct injections.
Substitution Theorem 2.4 For every morphism \( s : X \rightarrow TY \) in \( \mathcal{K} \) there exists a unique extension to a homomorphism \( \hat{s} : TX \rightarrow TY \) of \( H \)-algebras. That is, a unique homomorphism with \( s = \hat{s} \eta_X \).

For a proof see either 2.4 in [M] or 2.11 in [AAV] (somewhat improved by 2.17 in [AAMV]).

**Corollary 2.5** The formation of \( TX \) (for all objects \( X \)) and \( \hat{s} \) (for all morphisms \( s : X \rightarrow TY \)) is a Kleisli triple. The corresponding monad \((T, \eta, \mu)\) has

\[
\mu_X = \text{id}_{TX} : TTX \rightarrow TX \quad \text{for all objects } X.
\]

This monad \( T \) is called the completely iterative monad generated by \( H \).

**Definition 2.6** By an (iterative) equation morphism with object \( X \) of variables and object \( Y \) of parameters is meant a morphism \( e : X \rightarrow T(X + Y) \).

**Example 2.7** Let \( \Sigma \) be the signature of two binary operations \( + \) and \( * \). The iterative system of equations

\[
\begin{align*}
    x_1 &= x_2 + y \\
    x_2 &= y \ast x_1
\end{align*}
\]

(1)

corresponds to the morphism

\[
e : \{x_1, x_2\} \longrightarrow T\{x_1, x_2, y\}
\]

defined by

\[
x_1 \mapsto \quad + \quad \quad x_2 \mapsto \quad y \quad \quad x_2 \mapsto \ast \quad y \quad \quad x_1 \mapsto \quad \ast
\]

This system has a unique solution \( x_1^\dagger, x_2^\dagger \), viz,

\[
\begin{array}{c}
    + \quad y \\
    * \\
    y \\
    y
\end{array}
\quad \quad \text{and} \quad \begin{array}{c}
    + \quad y \\
    * \\
    y \\
    x_1^\dagger
\end{array}
\]

The solution defines a morphism

\[
e^\dagger : X \longrightarrow TY
\]

with the following property: if we substitute in the right-hand sides of (1) each \( x_i \) by \( x_i^\dagger \) (and the parameter \( y \) by the corresponding tree \( \eta_T(y) \)), then \( e^\dagger \) is just \( e \) with that substitution.
The substitution morphism is
\[ s = [\eta, \eta_Y] : X + Y \to TY \]
and we extend it, using the Substitution Theorem, to
\[ \hat{s} : T(X + Y) \to TY. \]

Thus, solutions \( e^\dagger \) are morphisms defined by the property that the following triangle
\[
\begin{array}{ccc}
X & \xrightarrow{e^\dagger} & TY \\
\downarrow e & & \downarrow \hat{s} \\
T(X + Y) & & 
\end{array}
\]
commutes.

Now in every monad we have \( \hat{s} = \mu_Y Ts \), thus, we are led to the following

**Definition 2.8** By a *solution* of an equation morphism \( e : X \to T(X + Y) \) is meant a morphism \( e^\dagger : X \to TY \) such that the following square
\[
\begin{array}{ccc}
X & \xrightarrow{e^\dagger} & TY \\
\downarrow e & & \downarrow \mu_Y \\
T(X + Y) & \xrightarrow{T[e^\dagger, \eta_Y]} & TTY 
\end{array}
\]
commutes.

**Remark 2.9** Some trivial iteration equations, e.g., \( x = x \), have many solutions. But “almost” all systems of iterative equations turn out to have a unique solution. The cases we want to exclude are the equations \( x = x' \) where the right-hand side is a variable from \( X \). Now given an equation morphism \( e : X \to T(X + Y) \) recall that \( T(X + Y) \) is a coproduct of \( HT(X + Y) \) and \( X + Y \)—thus, it is a coproduct of
\[
\begin{array}{c}
X \\
\xleftarrow{\text{inl}} X + Y \\
\xrightarrow{\eta_{X + Y}} T(X + Y)
\end{array}
\]
and
\[
\begin{array}{c}
HT(X + Y) + Y \\
\xrightarrow{\text{inr}} T(X + Y)
\end{array}
\]
It is the first injection that we want to exclude. More precisely, we want \( e \) to factorize through the latter one:

**Definition 2.10** An equation morphism \( e : X \to T(X + Y) \) is called guarded provided that it factorizes through the coproduct injection \( HT(X + Y) + Y \to T(X + Y) \):
\[
\begin{array}{ccc}
X & \xrightarrow{e} & T(X + Y) \\
\downarrow & & \downarrow [\tau_{X + Y}, \eta_{X + Y} \\text{inr}] \\
HT(X + Y) + Y & & 
\end{array}
\]
Solution Theorem 2.11 Every guarded equation morphism has a unique solution.

For the proof see 2.11 in [M] or 3.3 in [AV] (much improved by 3.4–3.8 in [AAMV]).

Remark 2.12 In particular, every accessible endofunctor of Set (and, more generally, of any locally presentable category) is iteratable, see [AAMV].

3 All Functors Have Initial and Final (Co)Algebras

In the present section we prove that every endofunctor $F$ of Set has an initial $F$-algebra and a final $F$-coalgebra, but these can be classes. More precisely, we expand the category Set to the category Class of classes and class functions. Then every functor $F : \text{Set} \rightarrow \text{Set}$ has a unique extension to a small-accessible functor $F^\infty : \text{Class} \rightarrow \text{Class}$ (see 3.1 and 3.6 below for definitions), and both an initial $F^\infty$-algebra $I$ and a final $F^\infty$-coalgebra $T$ exist. Besides, $T$ is determined by finality w.r.t. all (small) $F$-algebras in Set.

All this is true for general categories $\mathcal{K}$ satisfying the following assumptions

1. $\mathcal{K}$ has small colimits (i.e., $\mathcal{K}$ is cocomplete)
2. $\mathcal{K}$ is (small) cowellpowered
   and
3. $\mathcal{K}$ is locally small (i.e., the objects of $\mathcal{K}$ form a class and the hom-sets $\mathcal{K}(A, B)$ are small sets for all objects $A, B$ of $\mathcal{K}$).

We form a free cocompletion

$$\mathcal{K}^\infty$$

of $\mathcal{K}$ w.r.t. small-filtered colimits (see 3.1). The cocompletion $\mathcal{K}^\infty$ can be described (analogously to the free cocompletion $\text{Ind}(\mathcal{K})$ w.r.t. filtered colimits of Grothendieck [AGV]) as a “suitable” category of all small-filtered diagrams in $\mathcal{K}$. The main example is $\text{Class} = \text{Set}^\infty$, see 3.7.

Then every endofunctor $F$ of $\mathcal{K}$ extends, uniquely up to natural isomorphism, to a small-accessible (see 3.1) endofunctor $F^\infty$ of $\mathcal{K}^\infty$, and $F^\infty$ has an initial algebra and a final coalgebra. There is a substantial difference between the two: for an initial $F^\infty$-algebra, $I$, we have a formula

$$I = \text{colim}_{i \in \text{Ord}} F^{(i)} 0$$

naturally expanding the well-known formula

$$I = \text{colim}_{n \in \omega} F^{(n)} 0$$

for $F$ $\omega$-cocontinuous.

That is, we iterate $F$ on an initial object, $0$, $\aleph_\omega$-many times (where, recall, $\aleph_\infty$ is the first large ordinal, thus, $\aleph_\infty$, as a well-ordered class, is precisely the same as the class Ord of all small ordinals), we obtain an initial $F^\infty$-algebra.
In contrast, the formula
\[ T = \lim_{n \in \omega} F^{(n)}1 \quad \text{for } F \text{ -continuous} \]
does not extend to \( T = \lim_{i \in \text{Ord}} F^{(i)}1 \). This has two reasons: the transfinite limit does not necessarily exist, and if it does, it need not be a terminal \( F \)-coalgebra. However, for \( \mathcal{K} = \text{Set} \) we use the ideas of James Worell [W] to show that by forming a limit
\[ F^{(\aleph_\infty)}1 = \lim_{i < \aleph_\infty} F^{(i)}1 \]
(albeit outside of \text{Class}), the next \( \aleph_\infty \) steps
\[ F^{(\aleph_\infty + 1)}1 = F(F^{(\aleph_\infty)}1), \ldots, F^{(\aleph_\infty + i)}1 = F(F^{(\aleph_\infty + i - 1)}1), \ldots \]
yield a transfinite chain of subsets
\[ F^{(\aleph_\infty)}1 \supseteq F^{(\aleph_\infty + 1)}1 \supseteq \ldots \supseteq F^{(\aleph_\infty + i)}1 \supseteq \ldots \]
such that the correct formula for a final \( F \)-coalgebra is
\[ T = \lim_{i < \aleph_\infty + \aleph_\infty} F^{(i)}1 = \bigcap_{i \in \text{Ord}} F^{(\aleph_\infty + i)}1. \]

3.1 Free Cocompletion Under Small-Filtered Colimits

Recall the concept of a \( \lambda \)-filtered category, for a given infinite cardinal \( \lambda \): it is a category \( \mathcal{D} \) such that every (non-full) subcategory on less than \( \lambda \) morphisms has a cocone in \( \mathcal{D} \). Colimits of diagrams with \( \lambda \)-filtered domains are called \( \lambda \)-filtered colimits. Basic example: a colimit of a \( \lambda \)-chain. And functors preserving \( \lambda \)-filtered colimits are called \( \lambda \)-accessible.

**Definition 3.1** A category \( \mathcal{D} \) is called small-filtered if it has a class of morphisms, and every small subcategory of \( \mathcal{D} \) has a cocone in \( \mathcal{D} \); that is, \( \mathcal{D} \) is \( \lambda \)-filtered for all small cardinals \( \lambda \).

Colimits of diagrams with small-filtered domains are called small-filtered colimits.

A functor preserving small-filtered colimits is called small-accessible.

**Example 3.2** The well-ordered category \( \text{Ord} \) of all small ordinals is small-filtered. Thus, a small-accessible functor preserves colimits of transfinite chains.

As a concrete example of a small-accessible functor, consider the usual extension of the power-set functor \( \mathcal{P} : \text{Set} \rightarrow \text{Set} \) to the power-set functor
\[ \mathcal{P}^\infty : \text{Class} \rightarrow \text{Class} \]
assigning to every class \( X \) the class \( \mathcal{P}^\infty X \) of all subsets of \( X \).

**Remark 3.3** Peter Aczel and Nax Mendler [AM] call an endofunctor \( F \) of Class set-based provided that for every element of \( F \), \( x \in FX \), there exists a small subset \( m : Y \rightarrow X \) of the class \( X \) such that \( x \) lies in the image of \( Fm : FY \rightarrow FX \). This is equivalent to \( F \) being small-accessible, see the argument in [AP] for “bounded-accessible”.
Notation 3.4 Let \( \mathcal{K} \) be any category. We denote by
\[
E : \mathcal{K} \longrightarrow \mathcal{K}^\infty
\]
a free cocompletion of \( \mathcal{K} \) under small-filtered colimits.

Explicitly: \( \mathcal{K}^\infty \) is a category having small-filtered colimits and \( E \) is a full embedding with the following universal property:

for every functor \( F : \mathcal{K} \longrightarrow \mathcal{L} \) where \( \mathcal{L} \) has small-filtered colimits there exists a small-accessible extension \( F' : \mathcal{K}^\infty \longrightarrow \mathcal{L} \) of \( F \), unique up to a natural isomorphism.

Remark 3.5

(a) Every object \( K \) of \( \mathcal{K} \) is small-presentable in \( \mathcal{K}^\infty \). This means that for every morphism
\[
f : K \longrightarrow \colim_{i \in I} X_i
\]
from \( K \) into a small-filtered colimit in \( \mathcal{K}^\infty \) (with a colimit cocone \( c_j : X_j \longrightarrow \colim_{i \in I} X_i \)) we have that

(i) \( f \) factorizes through some \( c_j \):

\[
\begin{array}{ccc}
K & \xrightarrow{f} & \colim_{i \in I} X_i \\
\downarrow{g} & & \downarrow{c_j} \\
X_j & \end{array}
\]

(ii) the factorization is essentially unique, i.e., given \( g' : K \longrightarrow X_j \) with \( f = c_j \cdot g' \) then there exists a morphism \( x_{jk} : X_j \longrightarrow X_k \) of the given diagram with
\[
x_{jk} \cdot g = x_{jk} \cdot g'.
\]

Conversely, every small-presentable object \( K \) of \( \mathcal{K}^\infty \) is a retract of an object of \( \mathcal{K} \). Thus, whenever idempotents split in \( \mathcal{K} \), then small-presentable objects of \( \mathcal{K}^\infty \) are precisely those isomorphic to objects of \( \mathcal{K} \).

(b) The universal property of \( \mathcal{K}^\infty \) mentioned above can be restated as follows: the functor category \( [\mathcal{K}, \mathcal{L}] \) is equivalent to the full subcategory \( [\mathcal{K}^\infty, \mathcal{L}]_{\text{succ}} \) of \( [\mathcal{K}^\infty, \mathcal{L}] \) formed by all small-accessible functors under the equivalence functor
\[
(\_ \cdot E) : [\mathcal{K}^\infty, \mathcal{L}]_{\text{succ}} \longrightarrow [\mathcal{K}, \mathcal{L}]
\]

This explains the following extension of the above notation.

Notation 3.6 Let \( \mathcal{K} \) be a locally small category. For every functor \( F : \mathcal{K} \longrightarrow \mathcal{K} \) we denote by
\[
F^\infty : \mathcal{K}^\infty \longrightarrow \mathcal{K}^\infty
\]
the (essentially unique) extension of \( F \cdot E : \mathcal{K} \longrightarrow \mathcal{K}^\infty \) to a small accessible endofunctor. For every natural transformation
\[
f : F \longrightarrow G \quad \text{in} \quad [\mathcal{K}, \mathcal{K}]
\]
we denote by

\[ f^\infty : F^\infty \to G^\infty \] in \([\mathcal{K}^\infty, \mathcal{K}^\infty]\)

the unique natural transformation extending \(E \cdot f\), i.e., such that

\[ f^\infty \cdot E = E \cdot f. \]

**Examples 3.7**

(i) \(\text{Set}^\infty = \text{Class}\). In fact firstly, \(\text{Class}\) has small-filtered colimits, in fact, all class-indexed colimits. (This is obvious: a coproduct of a class of classes is a class, since \((\aleph_\infty)^2 = \aleph_\infty\), and coequalizers also clearly exist.)

Next, let \(F : \text{Set} \to \mathcal{L}\) be a functor, where \(\mathcal{L}\) has small-filtered colimits. For every class \(X\) form the small-filtered diagram \(D_X : D_X \to \text{Set}\) of all small subsets \(A\) of \(X\) and all inclusion functions, and choose a colimit \(F' X\) of \(F \cdot D_X\) with a colimit cocone

\[ c_{A,X} : F A \to F' X \quad (A \text{ in } D_X) \]

In any case \(X\) is small, it is the largest element of \(D_X\) and we choose \(F' X = FX\) and \(c_{A,X} = F(A \to X)\).

For every morphism \(f : X \to Y\) in \(\text{Class}\) denote by \(F' f : F' X \to F' Y\) the unique morphism of \(\mathcal{L}\) such that for every set \(A \subseteq X\) with image \(B = f[A]\) in \(Y\) and domain-codomain restriction \(f_0 : A \to B\) the following square

\[
\begin{array}{ccc}
FA & \xrightarrow{c_{A,X}} & F'X \\
Ff_0 \downarrow & & \downarrow F'f \\
FB & \xrightarrow{c_{B,Y}} & F'Y
\end{array}
\]

commutes. It is easy to verify that this defines a functor \(F' : \text{Class} \to \mathcal{L}\) which preserves small-filtered colimits. Obviously, \(F'\) extends \(F\), and is unique up to a natural isomorphism. Thus, \(\text{Class}\) is a free cocompletion of \(\text{Set}\) under small-filtered colimits.

(ii) An analogous description can be provided for the cocompletions \(\mathcal{K}^\infty\) of other “everyday-life” categories. E.g., if \(\mathcal{K} = \text{Pos}\) is the category of small posets and order-preserving maps, then

\(\text{Pos}^\infty\)

is the category of all partially ordered classes and order-preserving maps. The argument is analogous to \(\text{Class}\) above. Or for \(\mathcal{K} = \text{Cpo}\), the category of all small posets with directed joins and continuous (= directed-joins-preserving) maps we have

\(\text{Cpo}^\infty\)

the category of partially ordered classes having joins of directed subsets, and functions preserving such joins.

(iii) Let \(\text{Ord}^+\) be the well-ordered category of (a) all small ordinals and (b) a largest object, \(\mathbb{T}\). Then \((\text{Ord}^+)^\infty\) is the extension of \(\text{Ord}^+\) by a new
element, \( u \), satisfying
\[
i < u < \top \quad \text{for all } i \in \text{Ord.}
\]

**Lemma 3.8** Every locally small, cowellpowered category \( \mathcal{K} \) with small colimits is closed under small colimits in \( \mathcal{K}^\infty \), and \( \mathcal{K}^\infty \) has class-indexed colimits (i.e., colimits with at most \( \aleph_\infty \) morphisms in the diagram scheme) and arbitrary multiple pushouts of epimorphisms.

**Proof.** The first statement is trivial, since objects of \( \mathcal{K} \) are small-presentable in \( \mathcal{K}^\infty \) (see Remark 3.5(a) and recall that in small-complete categories idempotents split). The second statement requests just showing that \( \mathcal{K}^\infty \) has small colimits: since it has small-filtered colimits, it has, then, class-indexed colimits (given a class-indexed diagram \( D \), consider the small-filtered colimit of the diagram of colimits of all small subdiagrams of \( D \); this is a colimit of \( D \)).

The existence of small coproducts in \( \mathcal{K}^\infty \) is evident since objects of \( \mathcal{K}^\infty \) are small-filtered colimits of objects of \( \mathcal{K} \); given a small collection of small-filtered diagrams \( D_i : D_i \to \mathcal{K}^\infty \) (\( i \in I \)), form the small-filtered diagram
\[
\prod D_i \to \prod D_i \to \mathcal{K}^I \to \mathcal{K},
\]
where the second part is taking coproducts in \( \mathcal{K} \). Its colimit is the coproduct of \( \text{colim} D_i \) in \( \mathcal{K}^\infty \). Analogously with coequalizers: given a parallel pair \( f, g : \text{colim} D \to \text{colim} D' \) in \( \mathcal{K}^\infty \), where \( D, D' \) are small-filtered in \( \mathcal{K} \), we can find natural transformations \( f_i, g_i : D_i \to D'_i \) in \( \mathcal{K} \) with \( f = \text{colim} f_i \) and \( g = \text{colim} g_i \). By forming coequalizers \( c_i : D'_i \to D'^i \) in \( \mathcal{K} \) we obtain a small-filtered diagram \( D'^i \) in \( \mathcal{K} \) and a natural transformation \( c_i : D' \to D'^i \).

It is easy to see that \( \text{colim} c_i \) is a coequalizer of \( f \) and \( g \).

The existence of multiple pushouts of epimorphisms is proved analogously to the proof that locally presentable categories are cowellpowered, see Theorem 2.14 of [GU]. \( \square \)

**Remark 3.9** Every \( F \)-coalgebra is also an \( F^\infty \)-coalgebra (since \( FA = F^\infty A \) for all \( A \in \mathcal{K} \)). And every \( F^\infty \)-coalgebra is a small-filtered colimit of \( F \)-coalgebras. This has been proved in [AP1] (see Theorem IV.2 applied to \( \lambda = \aleph_\infty \)).

### 3.2 Initial Algebras and Final Coalgebras

**Remark 3.10** Let \( \mathcal{K} \) be a locally small, cowellpowered category with small colimits. By Lemma 3.8, for every endofunctor \( F \) and every \( F^\infty \)-coalgebra \( A \) there exists a greatest congruence on \( A \), i.e., a homomorphism \( e : A \to A^e \) of \( F^\infty \)-coalgebras carried by an epimorphism of \( \mathcal{K} \) such that every other epimorphic homomorphism \( f : A \to B \) has a factorization \( f^* : B \to A^e \) with \( f^* f = e \). (Viz., \( e \) is a multiple pushout of all \( f^* 's \.).

**Theorem 3.11** Let \( \mathcal{K} \) be a locally small, cowellpowered category with small colimits. For every endofunctor \( F \) of \( \mathcal{K} \) an initial \( F^\infty \)-algebra, \( I \), exists, in
\[ I = \varinjlim_{i \in \text{Ord}} F^{(i)}0, \]

where 0 is initial in \( \mathcal{K} \), and \( \text{Ord} \) is the chain of all small ordinals. And a final \( F^{\infty} \)-coalgebra, \( T \), exists, in fact

\[
T = \left( \prod_{A \in \text{Coalg} F} A \right)^*,
\]

is a quotient of the coproduct of all \( F \)-coalgebras modulo the greatest congruence.

**Remark.** The statement on the existence of \( T \) is a generalization of the Final Coalgebra Theorem of [AM], see also the paper [B] of Barr.

**Proof.** (1) Following [Ad] define an Ord-chain \( F^{(i)}0 \) \((i \in \text{Ord})\) with connecting morphisms \( w_{ij} : F^{(i)}0 \to F^{(j)}0 \) \((i, j \in \text{Ord}, i \leq j)\) in \( \mathcal{K} \) by the following transfinite induction over \( \text{Ord} \):

\[
F^{(0)}0 = 0, \quad F^{(1)}0 = F0, \quad \text{and} \quad w_{01} : 0 \to F0 \text{ is uniquely determined.}
\]

For the isolated step, given \( F^{(i)}0 \) and \( w_{ij} \) put

\[
F^{(i+1)}0 = F(F^{(i)}0) \quad \text{and} \quad w_{i+1,j+1} = Fw_{ij}.
\]

For the limit step, assume that \( j \) is a small limit ordinal such that the chain \( (F^{(i)}0)_{i < j} \) has already been defined. Put

\[
F^{(j)}0 = \varinjlim_{i < j} F^{(i)}0
\]

with a colimit cocone

\[
w_{ij} : F^{(i)}0 \to F^{(j)}0 \quad (i < j).
\]

The requirement that we define a chain makes \( w_{ij,j+1} : F^{(j)}0 \to F(F^{(j)}0) \) uniquely determined:

\[
w_{ij,j+1} \cdot w_{i+1,j+1} = w_{i+1,j+1} = Fw_{ij} \quad (\text{for all } i < j).
\]

Denote by \( I \) a colimit of this (small-filtered) chain in \( \mathcal{K}^{\infty} \). Then \( F^{\infty} \) preserves that colimit, yielding a canonical isomorphism

\[
F^{\infty}I \cong \varinjlim_{i \in \text{Ord}} F^{(i+1)}0 = \varinjlim_{i \in \text{Ord}} F^{(i)}0 = I.
\]

This is an initial \( F^{\infty} \)-algebra, as proved in [Ad].

(2) The collection of all \( F \)-coalgebras \( A = (X_A, \xi_A : X_A \to FX_A) \) is a class because it is a class-indexed union of small sets \( \mathcal{K}(X, FX) \). The category \( \mathcal{K}^{\infty} \) has class-indexed coproducts, by Lemma 3.8, thus, the coproduct

\[
B = \coprod_{A \in \text{Coalg} F} A
\]

exists as an \( F^{\infty} \)-coalgebra. In fact, the forgetful functor \( \text{Coalg} F^{\infty} \to \mathcal{K}^{\infty} \)
creates colimits, thus, $B$ is the unique $F^\infty$-coalgebra on the coproduct

$$\coprod_{A \in \text{Coalg } F} X_A$$

in $\mathcal{K}^\infty$ forming a coproduct in $\text{Coalg } F^\infty$. It follows from Remark 3.9 that $B$ is weakly final; thus, so is $B^a$. Consequently, $B^a$ is final: suppose that $p,q : C \to B^a$ are $F^\infty$-coalgebra homomorphisms. We can form their coequalizer and find that, since $B^a$ has no non-trivial quotients, we have $p = q$. □

**Remark 3.12** For set functors James Worrell [W] has provided a different, much more natural construction of a final coalgebra $T$:

$$T = \lim_{i \in \text{Ord}} F^{(\aleph_\infty+i)}1 = F^{(\aleph_\infty+\aleph_\infty)}1.$$ 

More precisely, given $F : \text{Set} \to \text{Set}$, we can form a cochain indexed by $\text{Ord}$ (or, which is the same, indexed by the first non-small ordinal $\aleph_\infty$), $F^{(i)}1$ ($i \in \text{Ord}$), by dualizing the chain of the proof of Theorem 3.11:

$$F^{(0)}1 = 1, \quad F^{(1)} = F1 \quad \text{and} \quad w_{10} : F1 \to 1 \text{ is unique;}$$

for the isolated steps we put

$$F^{(i+1)}1 = F(F^{(i)}1) \quad \text{and} \quad w_{ij+1} = Fw_{ij}$$

and on limit steps, where $j$ is a limit ordinal, put

$$F^{(j)}1 = \lim_{i < j} F^{(i)}1 \quad \text{with limit cone } w_{ji} \ (i < j).$$

Notice that by forming the class-indexed limit

$$F^{(\aleph_\infty)}1 = \lim_{i \in \text{Ord}} F^{(i)}1 = \lim_{i < \aleph_\infty} F^{(i)}1$$

we can leave not only $\text{Set}$, but also $\text{Class}$: there is no guarantee that $F^{(\aleph_\infty)}1$ is a class! And, whenever it is not a class, then we have not found our final coalgebra yet (since, by Theorem 3.11, $T$ is a class). Fortunately, another $\text{Ord}$-indexed cochain repairs the damage.

Let us denote by $\text{Set}^\alpha$ the category of all sets of cardinality at most $2^{\aleph_\infty}$; since $\text{card}(F^{(i)}1) < \aleph_\infty$ for all $i < \aleph_\infty$, it follows that $\text{card}(F^{(\aleph_\infty)}1) \leq \aleph_\infty < \aleph_\infty = 2^{\aleph_\infty}$ and our limit thus lives in $\text{Set}^\alpha$. We have an essentially unique $2^{\aleph_\infty}$-accessible extension

$$F^\alpha : \text{Set}^\alpha \to \text{Set}^\alpha$$

of $F$. And this allows us to define an $\text{Ord}$-indexed cochain

$$F^{(\aleph_\infty+i)}1 \quad (i \in \text{Ord})$$

in $\text{Set}^\alpha$ by a transfinite induction which precisely follows the previous one, except that $F$ is now substituted by $F^\alpha$:

$$F^{(\aleph_\infty)}1 \text{ has been defined already,} \quad F^{(\aleph_\infty+1)}1 = F^\alpha(F^{(\aleph_\infty)}1)$$

and

$$w_{\aleph_\infty+1,\aleph_\infty} : F^\alpha(F^{(\aleph_\infty)}1) \to F^{(\aleph_\infty)}1$$
is uniquely determined by the commutativity of the following triangles

$$F^0(F^{(N_\infty)}_1) \quad \xrightarrow{w_{i+i+1}} \quad F^{(N_\infty+i+1)}_1$$

for all $i \in \text{Ord}$. The isolated step is, as above,

$$F^{(N_\infty+i+1)}_1 = F^0(F^{(N_\infty+i)})$$

$$w_{i+1,j+1} = F^0 w_{ij}.$$ 

And limit steps are given by the formation of limits. We denote by

$$F^{(N_\infty+N_\infty)}_1 = \lim_{i \in \text{Ord}} F^{N_\infty+i}$$

a limit of this cochain in $\text{Set}^\alpha$ with limit cone $\overline{w_i} : F^{(N_\infty+N_\infty)}_1 \to F^{(N_\infty+i)}_1$. This is an $F^\alpha$-coalgebra w.r.t. the unique

$$\tau : F^\alpha(F^{(N_\infty+N_\infty)}_1) \to F^{(N_\infty+N_\infty)}_1$$

with

$$\overline{w_{i+1}} \tau = F^\alpha \overline{w_i} : F^\alpha(F^{(N_\infty+N_\infty)}_1) \to F^{(N_\infty+N_\infty+i+1)}_1$$

for all ordinals $i \in \text{Ord}$.

It has been proved by J. Worrell that this $F^\alpha$-coalgebra is final. And, unlike $F^{(N_\infty)}_1$, we are now sure that

$$F^{(N_\infty+N_\infty)}_1$$

is a class. In fact, the argument that a final $F^\alpha$-coalgebra is a class is the same as that presented in Theorem 3.11: all $F$-coalgebras form a generator of $\text{Coalg} F^\alpha$, thus, a final $F^\alpha$-coalgebra is a quotient of the class-coalgebra $\prod_{A \in \text{Coalg} F} A$.

**Remark 3.13**

(i) In [W] J. Worrell has shown that the connecting maps starting after $N_\infty$:

$$F^{(N_\infty)}_1 \xrightarrow{F^{(N_\infty+1)}} \cdots \cdots \cdots F^{(N_\infty+i)}_1 \cdots$$

are all monomorphisms, i.e., $F^{(N_\infty+i)}_1$ is a subobject of $F^{(N_\infty)}_1$, and a final $F^\infty$-coalgebra is thus an intersection

$$T = \bigcap_{i \in \text{Ord}} F^{(N_\infty+i)}_1$$

of these subobjects.

(ii) All the above results hold not only for functors $F^\infty$, but for all small-accessible endofunctors of $\text{Class}$.

**Definition 3.14** A category $\mathcal{C}$ is called smooth provided that it has no non-trivial small-filtered colimits of monomorphisms.

That is, given a small-filtered diagram $D : \mathcal{D} \to \mathcal{C}$ of monomorphisms with a colimit $c_d : Dd \to K$ ($d$ in $\mathcal{D}$) then some of the colimit morphisms $c_d$ is an isomorphism.
Examples 3.15

(i) Set is smooth. In fact, given a small-filtered diagram $D$ of monomorphisms whose colimit (= union) is a set, then this set is simply $Dd$ for some object $d$.

(ii) All “everyday-life” categories are smooth, e.g., $\text{Pos}$, $\text{Cpo}$, etc. The argument is similar to that for Set.

(iii) Every locally presentable category is smooth. Given a small-filtered colimit $c_d : Dd \rightarrow K$ of monomorphisms, then, since $K$ is a $\lambda$-presentable object for some $\lambda$, the morphism $id_K : K \rightarrow K$ factorizes through some $c_d$. Thus, $c_d$ is both a monomorphism and a split epimorphism.

(iv) Categories $\mathcal{K}^\infty$ are typically not smooth, e.g., Class, $\text{Pos}^\infty$ or $\text{Cpo}^\infty$ are certainly not smooth.

Lemma 3.16 For every smooth category $\mathcal{K}$ the functor $(\_)^\infty$ from $[\mathcal{K}, \mathcal{K}]$ to $[\mathcal{K}^\infty, \mathcal{K}^\infty]$ preserves all existing small-filtered colimits of monomorphisms.

Proof. Let $(f_i : F_i \rightarrow F)_{i \in I}$ be a small-filtered colimit of monomorphisms in $[\mathcal{K}, \mathcal{K}]$. This means, of course, that for every object $K$ of $\mathcal{K}$ we have a trivial colimit $((f_i)_K : F_i K \rightarrow FK)_{i \in I}$, since $\mathcal{K}$ is smooth and since colimits in $[\mathcal{K}, \mathcal{K}]$ are, whenever they exist, formed pointwise. We are to prove that $(f_i^\infty : F_i^\infty \rightarrow F^\infty)_{i \in I}$ is a colimit in $[\mathcal{K}^\infty, \mathcal{K}^\infty]$. We know that, since $I$ is a small-filtered category, a colimit $G = \text{colim}_{i \in I} F_i^\infty$ exists in $[\mathcal{K}^\infty, \mathcal{K}^\infty]$ (with colimit counit $g_i : F_i^\infty \rightarrow G$). To prove that $G \cong F^\infty$, observe that $G$ preserves small-filtered colimits (since each $F_i^\infty$ does), thus, it is sufficient to show that $G$ extends $F$. In fact, for every object $K$ of $\mathcal{K}$ we have $i \in I$ such that $(f_i)_K : F_i K \rightarrow FK$ is an isomorphism. Then $(g_i)_K$ is an isomorphism, making $G K$ essentially equal to $F_i^\infty K = F_i K = FK$. \hfill \Box

Theorem 3.17 For every smooth category $\mathcal{K}$ the functor
\[ F \mapsto T_F \]
assigning a final coalgebra to every endofunctor of $\mathcal{K}$ preserves existing small-filtered colimits of monomorphisms.

Remark 3.18 What we mean is, of course, the following functor
\[ \Phi : [\mathcal{K}, \mathcal{K}] \rightarrow \mathcal{K}^\infty \]
assigning to every $F$ the object $T_F$ of a final $F^\infty$-coalgebra $(T_F, \tau_F)$ and to every natural transformation $f : F \rightarrow G$ the unique homomorphism $\Phi f : T_F \rightarrow T_G$ of $G^\infty$-coalgebras:
\[
\begin{array}{c}
T_F \xrightarrow{\tau_F} F^\infty T_F \xrightarrow{(f)_{T_F}} G^\infty T_F \\
\Phi f \downarrow \quad \quad \quad \downarrow \Phi f \\
T_G \xrightarrow{\tau_G} G^\infty T_G
\end{array}
\]

Proof. Let
\[(f_i : F_i \rightarrow F)_{i \in I}\]
be a small-filtered colimit of monomorphisms in $[\mathcal{K}, \mathcal{K}]$. We obtain the corresponding diagram of objects $T_R (i \in I)$, more precisely, we apply $\Phi$ to the given diagram. This diagram is small-filtered in $\mathcal{K}^\infty$, thus, it has a colimit

$$(t_i : T_R \rightarrow T)_{i \in I}$$

in $\mathcal{K}^\infty$. There is a unique $F^\infty$-coalgebra structure

$$\tau : T \rightarrow F^\infty T$$

making each $t_i$ a homomorphism of $F^\infty$-coalgebras:

$$T_R \xrightarrow{\tau_R} F^\infty T_R \xrightarrow{(f_i^\infty)_{T_R}} F^\infty T_R \xrightarrow{\tau} F^\infty T$$

To prove that $(T, \tau)$ is a final $F$-coalgebra, we only have to consider an $F$-coalgebra

$$\beta : B \rightarrow FB$$

see Remark 3.9. In order to prove the existence and uniqueness of a homomorphism $B \rightarrow T$, we first observe that since $F^\infty$ preserves small-filtered colimits, we have

$$F^\infty T = \text{colim}_{i \in I} F^\infty T_R$$

with the colimit cocone $F^\infty t_i$ $(i \in I)$. By Lemma 3.16

$$(f_i^\infty : F^\infty_i \rightarrow F^\infty)_{i \in I}$$

is a small-filtered colimit in $[\mathcal{K}^\infty, \mathcal{K}^\infty]$.

Consequently, we also have

$$F^\infty T = \text{colim}_{i \in I} F^\infty_i T_R$$

with the colimit cocone

$$F^\infty_i T_R \xrightarrow{i^\infty} F^\infty T_R \xrightarrow{(f_i^\infty)_{T_R}} F^\infty T \quad (i \in I).$$

**Existence of a homomorphism** $B \rightarrow T$. Since $B$ is small-presentable, see Remark 3.5(a), the morphism

$$\beta : B \rightarrow \text{colim}_{i \in I} F_i B$$

factorizes through some $(f_i)_B$:

$$\begin{array}{ccc}
B & \xrightarrow{\beta} & FB \\
& \searrow^{\beta'} & \downarrow^{(f_i)_B} \\
& & F_i B
\end{array}$$

The unique homomorphism $h : B \rightarrow T_R$ of $F^\infty_i$-coalgebras defines a homo-
morphism $\overline{h} = t_i \cdot h : B \to T$ of $F^\infty$-coalgebras:

$$
\begin{array}{c}
B & \xrightarrow{\beta} & F^\infty_i B \\
\downarrow h & & \downarrow (f^\infty_i)_b \\
T_{F_i} & \xrightarrow{\tau_{F_i}} & F^\infty_i T_{F_i}
\end{array}
$$

Uniqueness of a homomorphism $B \to T$. The uniqueness of $\overline{h}$ follows, again, from small presentability, see (ii) in 3.5(a): given a homomorphism $k : B \to T_F$ of $F^\infty$-coalgebras, then there is a factorization $k = t_i \cdot k'$ for some $i \in I$, and without loss of generality we can assume $i = i$ (since $I$ is small-filtered):

$$
\begin{array}{c}
B & \xrightarrow{\beta} & F^\infty_i B \\
\downarrow h & & \downarrow (f^\infty_i)_b \\
T_{F_i} & \xrightarrow{\tau_{F_i}} & F^\infty_i T_{F_i}
\end{array}
$$

If $k'$ is a homomorphism of $F^\infty_i$-coalgebras, then the proof is finished: we have $k' = h$, thus, $k = t_i \cdot h = \overline{h}$. If not, we use the fact that $F^\infty T$ is a small-filtered colimit of $F^\infty_i T_{F_i}$. Now the two morphisms $(\tau_{F_i} \cdot k')$ and $(F^\infty_i k' \cdot \beta')$ are merged by the colimit map $F^\infty i \cdot (f^\infty_i)_{T_{F_i}}$ of the colimit (2):

$$
\begin{align*}
F^\infty t_i \cdot (f^\infty_i)_{T_{F_i}} \cdot (F^\infty_i k' \cdot \beta') &= F^\infty t_i \cdot F^\infty k' \cdot (f_i)_B \cdot \beta' \\
&= F^\infty k \cdot \beta \\
&= \tau \cdot k \\
&= \tau \cdot t_i \cdot k' \\
&= F^\infty t_i \cdot (f^\infty_i)_{T_{F_i}} \cdot (\tau_{F_i} \cdot k') \text{definition of } \tau
\end{align*}
$$

Indeed, the first equation uses naturality of $f_i$, the second one the definitions of $k'$ and $\beta'$, the third one holds since $k$ is a homomorphism, and the 4th and 5th follow from the definitions of $k$ and $\tau$, respectively. Since $B$ is small-presentable, there is a connecting morphism

$$
x_{ij} : F_i \to F_j
$$

of the original diagram such that the corresponding connecting morphism

$$
F^\infty_i T_{F_i} \xrightarrow{(x^\infty_{ij})_{T_{F_i}}} F^\infty j T_{F_i} \xrightarrow{F^\infty j \Phi x_{ij}} F^\infty j T_{F_j}
$$

also merges the pair $\tau_{F_i} \cdot k'$ and $F^\infty_i k' \cdot \beta'$.

It follows that $\Phi x_{ij} \cdot k^j : B \to T_{F_j}$ is a homomorphism of $F^\infty_j$-coalgebras—
in fact, the following diagram

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{ B \ar[r]^-{\beta} \ar[d]^-{k'} & F_{i}^{\infty} \ar[d]^-{F_{i}^{\infty} k'} & F_{i}^{\infty} B \\
 T_{F_{i}} \ar[r]^-{\tau_{F_{i}}} & F_{j}^{\infty} T_{F_{i}} & F_{j}^{\infty} T_{F_{i}} \\
 \Phi_{x_{ij}} \ar[d]^-{\Phi_{x_{ij}} \tau_{F_{i}}} & F_{j}^{\infty} \Phi_{x_{ij}} \\
 T_{F_{j}} \ar[r]^-{\tau_{F_{j}}} & F_{j}^{\infty} T_{F_{j}} }
\end{array}
\end{array}
\]

commutes. Consequently, \( \Phi_{x_{ij}} \cdot k' = \Phi_{x_{ij}} \cdot h \) (since the right-hand side is also a homomorphism). Therefore

\[
k = t_{i} \cdot k' = t_{j} \cdot \Phi_{x_{ij}} \cdot k' = t_{j} \cdot \Phi_{x_{ij}} \cdot h = t_{i} \cdot h = \overline{h}.
\]

\[\Box\]

4 A General Solution Theorem

We apply here the results of Section 3 to show that for every endofunctor \( H \) of \( \text{Set} \) we have a solution theorem concerning guarded sets of iterative equations. This is so because the class extension \( H^{\infty} : \text{Class} \rightarrow \text{Class} \) is iterable, thus, we have the completely iterative monad \( T^{i} \) of \( H^{\infty} \), see 2.5. (If \( H \) is iterable and defines thus a completely iterative monad \( T : \text{Set} \rightarrow \text{Set} \), then \( T^{i} \) is nothing else than the extension \( T^{\infty} : \text{Class} \rightarrow \text{Class} \).) But we can say more: for every infinite cardinal number \( \lambda \) we can form the \( \lambda \)-accessible coreflection, \( H_{\lambda} \), of \( H \): to every set \( X \) it assigns the union of images of \( Hi \) for all inclusions \( i : Y \rightarrow X \) of subsets \( Y \) of cardinality less than \( \lambda \). The functor \( H_{\lambda} \) is iterable in \( \text{Set} \), see [AAMV], and we denote by \( T_{\lambda} \) the corresponding completely iterative monad on \( \text{Set} \).

We are going to prove that for every set \( X \) the class \( T^{i} X \) is a canonical colimit of the sets \( T_{\lambda} X \), where \( \lambda \) is a small cardinal number. Consequently, every iterative system of equations

\[
e : X \longrightarrow T^{i}(X + Y) \quad (X, Y \text{ in } \text{Set})
\]

for \( H \) actually has the form of a morphism

\[
\overline{e} : X \longrightarrow T_{\lambda}(X + Y) \quad \text{for some small cardinal } \lambda
\]

followed by the colimit map \( T_{\lambda}(X + Y) \longrightarrow T^{i}(X + Y) \). We then solve \( \overline{e} \) with respect to \( T_{\lambda} \) and obtain

\[
\overline{\mathfrak{r}} : X \longrightarrow T_{\lambda} Y
\]

which, composed with the colimit map \( T_{\lambda} Y \longrightarrow T^{i} Y \), is the (unique) solution of \( e \).

Notation 4.1 For every endofunctor

\[
H : \text{Set} \rightarrow \text{Set}
\]
denote by

\[ H_\lambda : \mathbf{Set} \rightarrow \mathbf{Set} \quad (\lambda \text{ any small cardinal}) \]

the subfunctor given by

\[ H_\lambda X = \bigcup f[H[Y]] \]

where the union ranges over all \( f : Y \rightarrow X \) with \( \operatorname{card}(Y) < \lambda \). Let \( h_\lambda : H_\lambda \rightarrow H \) be the inclusion.

We denote by \( T_\lambda : \mathbf{Set} \rightarrow \mathbf{Set} \) the free completely iterative monad of \( H_\lambda \) and by \( T^i \) the free completely iterative monad of \( H^\infty : \mathbf{Class} \rightarrow \mathbf{Class} \).

**Lemma 4.2** \( H = \operatorname{colim} H_\lambda \) is a small-filtered colimit of monomorphisms in \([\mathbf{Set}, \mathbf{Set}]\).

**Example 4.3** If \( H = \mathcal{P} \) is the power-set functor then for every \( \lambda \geq \omega \) we get the functor \( \mathcal{P}_\lambda \) of all subsets of cardinalities less than \( \lambda \).

We now extend the definition of (guarded) equation morphism and solution to arbitrary endofunctors of \( \mathbf{Set} \).

**Definition 4.4** Let \( H \) be an endofunctor of \( \mathbf{Set} \).

(i) By an equation morphism for \( H \) we understand a morphism

\[ e : X \rightarrow T^i(X + Y) \quad \text{for } X, Y \text{ in } \mathbf{Set} \]

It is called guarded if it factorizes through \( [\eta_{X+Y \text{int}}^i, \tau_{X+Y \text{int}}^i] \):

\[ \begin{array}{ccc}
X & \xrightarrow{e} & T^i(X + Y) \\
\downarrow & & \downarrow \\
& & H^\infty T^i(X + Y) + Y
\end{array} \]

(ii) By a solution of \( e \) we understand a morphism

\[ e^1 : X \rightarrow T^i Y \]

such that the following square

\[ \begin{array}{ccc}
X & \xrightarrow{e^1} & T^i Y \\
\downarrow & & \downarrow \\
T^i(X + Y) & \xrightarrow{T^i e + \eta_{X+Y}^i} & T^i T^i Y
\end{array} \]

commutes.

**Lemma 4.5** For every accessible functor \( H : \mathbf{Set} \rightarrow \mathbf{Set} \) with a free completely iterative monad \( T \) the functor \( H^\infty : \mathbf{Class} \rightarrow \mathbf{Class} \) has a free completely iterative monad with underlying functor \( T^\infty \).

**Proof.** We prove that \( T^\infty X \) is a final coalgebra for \( H^\infty(\_ + X) \): (a) If \( X \) is a small set, this is trivial:

\[ H^\infty(T^\infty X) + X = HTX + X = TX = T^\infty X. \]

Now use Remark 3.9.
(b) If $X$ is a class, express it as a small-filtered union of all of its subsets, and use the fact that $H^\infty$ and $T_i^\infty$ preserve small-filtered colimits

$$H^\infty(T^\infty X) + X = \operatorname{colim}_i \left( H^\infty(T^\infty X_i) + X_i \right) = \operatorname{colim} T^\infty X_i = T^\infty X$$

and use Remark 3.9 again.

\[ \square \]

**Remark 4.6** In [AAMV] we have proved that the formation of free completely iterative monads over accessible endofunctors is (as the name suggests) a universal construction. Therefore, the natural transformation $h^\infty_\lambda : H^\infty_\lambda \longrightarrow H^\infty$ (inclusion) extends to a unique ideal monad morphism $t^\infty_\lambda : T^\infty_\lambda \longrightarrow T_i^\lambda$.

“Ideal” means that

$$t^\infty_\lambda = h^\infty_\lambda \ast t^\infty_\lambda + id : H^\infty_\lambda T^\infty_\lambda + Id \longrightarrow H^\infty T_i^\lambda + Id$$

(here, $\ast$ denotes the horizontal composition of natural transformations).

Moreover, the obvious small-filtered diagram formed by all $T^\infty_\lambda$ ($\lambda$ a small cardinal) has a colimit cocone

$$t^\infty_\lambda : T^\infty_\lambda \longrightarrow T_i^\lambda$$

because left adjoints preserve colimits.

**General Solution Theorem 4.7** For every endofunctor $H$ of $\text{Set}$, every guarded equation morphism has a unique solution.

Moreover, the solution can be found as follows: we find a factorization

$$X \xrightarrow{e} T_i^\lambda (X + Y)$$

$$\downarrow \pi$$

$$T_\lambda (X + Y)$$

for some small cardinal number $\lambda$ and some guarded equation morphism $\pi$, and by solving $\pi$ w.r.t. $H_\lambda$ we solve $e$ w.r.t. $H^\infty$ since the following triangle

$$X \xrightarrow{e_i} T_i^\lambda Y$$

$$\downarrow \pi_i$$

$$T_\lambda Y$$

commutes.

**Remark.** The above theorem states that solutions of all guarded equations w.r.t. $H$ are found in the small coalgebras $T_\lambda Y$ for various cardinal numbers $\lambda$.

**Proof.** Suppose that a guarded equation morphism $e : X \longrightarrow T_i^\lambda (X + Y)$ is given and consider the factorization

$$X \xrightarrow{e} T_i^\lambda (X + Y)$$

$$\downarrow e_i$$

$$H^\infty T_i^\lambda (X + Y) + Y$$

20
Since $X$ is a small set, $e'$ factorizes through some $(h^\infty_X * t^\infty_X)_X + Y + i d_Y$:

$$
\begin{array}{c}
X \xrightarrow{e'} H^\infty T^i(X + Y) + Y \\
\downarrow \downarrow \\
H^\infty_T X(\lambda) + Y + i d_Y
\end{array}
$$

Observe that the following square

$$
\begin{array}{c}
H^\infty T^i(X + Y) + Y \xrightarrow{[\tau^i_{X + Y}, \eta^i_{X + Y}] \text{inv}} T^i(X + Y) \\
\downarrow \downarrow \\
H^\infty_T X(\lambda) + Y + Y \xrightarrow{[\tau^i_{X + Y}, \eta^i_{X + Y}] \text{inv}} T^i_X(\lambda) + Y
\end{array}
$$

commutes. Thus, by putting

$$
\bar{e} = [(\tau^i_{X + Y}, \eta^i_{X + Y})] \cdot \bar{e}'
$$

we define a guarded equation morphism such that the following triangle

$$
\begin{array}{c}
X \xrightarrow{e} T^i(X + Y) \\
\downarrow \downarrow \\
T^i_X Y = T^i_Y
\end{array}
$$

commutes. Since $t^\infty_X$ is an ideal monad morphism, it preserves solutions (see 4.11 of [AAMV]), i.e., the following triangle

$$
\begin{array}{c}
X \xrightarrow{e} T^i Y \\
\downarrow \downarrow \\
T^i_X Y = T^i_Y
\end{array}
$$

commutes. \hfill \Box

**Remark 4.8** A special case of guarded equation morphisms are the flat ones, i.e., equation morphisms of the form

$$
e : X \longrightarrow H X + Y \quad (X, Y \text{ in Set}).
$$

We have a natural connecting morphism

$$
\rho_{X,Y} : H X + Y \longrightarrow T^i(X + Y)
$$

whose left-hand component is

$$
H X = H^\infty X \xrightarrow{H^\infty \eta^i_X} H^\infty T^i X \xrightarrow{H^\infty T^i \text{inv}} H^\infty T^i(X + Y) \xrightarrow{\tau^i_{X + Y}} T^i(X + Y)
$$

and the right-hand one is

$$
Y \xrightarrow{\text{inv}} X + Y \xrightarrow{\eta^i_{X + Y}} T^i(X + Y)
$$

Thus, every flat equation morphism $e : X \longrightarrow H X + Y$ yields an equation morphism $\rho_{X,Y} e : X \longrightarrow T^i(X + Y)$ which is easily seen to be guarded. We denote by

$$
e^i : X \longrightarrow T Y
$$
the unique solution of $\rho_{X,Y} e$, for short.

In case of flat equation morphisms we have shown in [AAMV] that

solution = corecursion.

That is, $e^1 : X \rightarrow T^2 Y$ is the unique homomorphism from the coalgebra $e : X \rightarrow HX + Y$ to the final coalgebra $T^2 Y$ of $H^\infty(\_ + Y)$.

5 Example: Power-Set Functor

We apply the above results to non-labelled transition systems, i.e., to coalgebras of the power-set functor $\mathcal{P} : \text{Set} \rightarrow \text{Set}$. It has been noticed by several authors [AM], [B], [JPTWW], [RT], [W] that $\mathcal{P}^\infty$ has a very natural weakly final coalgebra $B$ (i.e., such that every $\mathcal{P}$-coalgebra $A$ has at least one homomorphism from $A$ to $B$): the coalgebra of all small extensional trees. Recall that a (rooted, non-ordered) tree is called extensional provided that any two distinct nodes with a common parent define non-isomorphic subtrees. Throughout this section trees are always taken up to (graph) isomorphism. Thus, shortly, a tree is extensional if and only if distinct siblings define distinct subtrees. We call a tree small if it has only a small set of children (= maximal proper subtrees).

5.1 Coalgebra $B$

It has as elements all small extensional trees, and the coalgebra structure

$\beta : B \rightarrow \mathcal{P}^\infty B$

is the inverse of tree tupling, i.e., $\beta$ assigns to every tree $t$ the set of all children of $t$.

5.2 Final Coalgebra $B/\sim$

We know from Theorem 3.11 that a final coalgebra exists. Recall here that $\mathcal{P}^\infty$ preserves weak pullbacks. Hence, the greatest congruence coincides with the greatest bisimulation on any $\mathcal{P}^\infty$-coalgebra, see e.g. [R]. Since $B$ is weakly final, it follows that a final coalgebra is a quotient of $B$ modulo the bisimilarity equivalence $\sim$ (i.e., the largest bisimulation on $B$). We are going to describe this equivalence $\sim$. We start by describing one interesting class.

Example 5.1 An extensional tree $t$ is bisimilar to the following tree

```
    Ω
   / \  \
  /   \  \
 /     \  \
```

22
if and only if all paths in $t$ are infinite. Thus, for example, the following tree

![Tree Diagram]

is bisimilar to $\Omega$. This illustrates that the bisimilarity equivalence is non-trivial. We prove $\Omega \sim \Omega'$ below.

**Remark 5.2** For the finite-power-set functor $\mathcal{P}_f$ a nice description of a final coalgebra has been presented by Michael Barr [B]: let $B_f$ denote the coalgebra of all finitely branching extensional trees. This is a small subcoalgebra of our (large) coalgebra $B$. We call two trees $b$, $b'$ in $B_f$ *Barr-equivalent*, notation $b \sim_0 b'$ provided that for every natural number $n$ the tree $b|_n$ obtained by cutting $b$ at level $n$ has the same extensional reflection as the tree $b'|_n$. (An extensional reflection is obtained by identifying pairs of siblings which define identical subtrees until one gets an extensional tree.) For example

$$\Omega \sim_0 \Omega'$$

Barr proved that the quotient coalgebra

$$B_f/\sim_0$$

is a final $\mathcal{P}_f$-coalgebra—that is, $\sim_0$ is the bisimilarity equivalence on $B_f$.

### 5.3 The Bisimilarity Equivalence $\sim$

We define, for every small ordinal number $i$, the following equivalence relation $\sim_i$ on $B$:

- $\sim_0$ is the Barr-equivalence

and in case $i > 0$

- $t \sim_i s$ iff for all $j < i$ the following hold:
  1. for each child $t'$ of $t$ there exists a child $s'$ of $s$ such that $t' \sim_j s'$
  2. vice versa.

**Remark 5.3** We shall show below that the bisimilarity equivalence $\sim$ is the intersection of all $\sim_i$. Notice that this intersection is just the usual construction of a greatest fixed point. Indeed, consider the collection $\text{Rel}$ of all binary relations on $B$. This collection, ordered by set-inclusion, is a class-complete
lattice. Define $\Phi : \text{Rel} \to \text{Rel}$ as follows:

$$ t\Phi(R) s \iff \text{for every child } t' \text{ of } t \text{ there exists a child } s' \text{ of } s \text{ such that } t' R s', \text{ and vice versa.} $$

Observe that $\Phi$ is a monotone function. Moreover, a binary relation $R$ is a fixed point of $\Phi$ if and only if $R$ is a bisimulation on $B$. Notice that the definition of $\sim_i$ is just an iteration of $\Phi$ on the largest equivalence relation $\equiv_0$ (i.e., $B \times B$) shifted by $\omega$ steps: we have

$$ \sim_0 = \Phi^{(\omega)}(\equiv_0) $$

where for every relation $R$ the iterations $\Phi^{(i)}(R)$, $i \in \text{Ord}$, are defined inductively as follows: $\Phi^{(0)}(R) = R$, the isolated step is $\Phi^{(i+1)}(R) = \Phi(\Phi^{(i)}(R))$, and for limit ordinals $\Phi^{(\lambda)}(R) = \bigcap_{\alpha < \lambda} \Phi^{(\alpha)}(R)$. Consequently, $\sim_i = \Phi^{(\omega+i)}(\equiv_0)$.

That we are indeed constructing the largest fixed point for $\Phi$ follows from the following

**Lemma 5.4** $\Phi$ preserves intersections of descending $\text{Ord}$-chains.

**Proof.** Let $(R_i)_{i \in \text{Ord}}$ be a descending chain in $\text{Rel}$ and let

$$ R = \bigcap_{i \in \text{Ord}} R_i $$

be its intersection. We show that $\Phi(R) = \bigcap_{i \in \text{Ord}} \Phi(R_i)$. In fact, the inclusion from left to right is obvious. To show the inclusion from right to left, suppose that the pair $(t, s)$ is in the right-hand relation. Let $t'$ be any child of $t$. Then, for any ordinal number $i \in \text{Ord}$ there exists a child $s'_i$ of $s$ with $t R_i s'_i$. Since $s$ has only a small set of children the set $\{s'_i \mid i \in \text{Ord}\}$ is small, too. Therefore there is a cofinal subset $C$ of $\text{Ord}$ such that $\{s'_i \mid i \in C\}$ has only one element, $s'$ say. It follows that $t' R_i s'$ for all $i \in \text{Ord}$. Hence, $t \Phi(R) s$, as desired. \(\square\)

**Theorem 5.5** Two trees $t, s \in B$ are bisimilar iff $t \sim_i s$ holds for all small ordinals $i$.

**Proof.** It follows from Lemma 5.4 that the intersection of all $\sim_i = \Phi^{(i)}(\sim_0)$, $i \in \text{Ord}$ is a fixed point of $\Phi$.

Next form the quotient coalgebra $B/\sim$. Since $B$ is weakly final, so is $B/\sim$. In order to establish that $B/\sim$ is a final $\mathcal{P}^\infty$-coalgebra we must show that for any $\mathcal{P}^\infty$-coalgebra $(X, \xi)$ and any two coalgebra homomorphisms $h, k : (X, \xi) \to (B, \beta)$ we have $h(x) \sim k(x)$ for all $x \in X$. We show this by transfinite induction, i.e., we prove that $h(x) \approx_i k(x)$ holds for all $i \in \text{Ord}$.

The first step $i = 0$ is obvious and for the induction step suppose that $i > 0$ is any small ordinal number and that for all $x \in X$, $k(x) \approx_j h(x)$ for all $j < i$, where $\approx_j$ denotes $\Phi^{(j)}(\equiv_0)$. Consider any child $s'$ of $k(x)$, i.e., $s' = k(x')$ for some $x' \in \xi(x)$ since $k$ is a coalgebra homomorphism. Because $h$ is a coalgebra homomorphism $t' = h(x')$ is a child of $h(x)$ such that $s' \approx_j t'$ for all $j < i$, whence $k(x) \approx_i h(x)$. \(\square\)
Remark 5.6 Barr showed that $\sim_0$ is the bisimilarity equivalence on the set of finitely branching trees. However, it is not even a bisimulation on $B$. In order to see this notice that is suffices to find trees that are in $\sim_0$ but not in $\sim_1$. Consider the following trees

$$s_0 = \quad \cdots \quad \text{and} \quad t_0 = \quad \cdots$$

We clearly have $t \sim_0 s$. But $t_0 \not\sim_1 s_0$, since $s_0$ has a child which is an infinite path while $t_0$ does not.

Moreover, none of the relations $\sim_i$, $i < \omega$ is a bisimulation. This is easily seen by induction. The base case is the above example, and if $t_i$ and $s_i$ are trees with $t_i \sim_i s_i$ and $t_i \not\sim_{i+1} s_i$, then

$$t_{i+1} = \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad...
Lemma 5.7 The coalgebra \( BY \) of all small extensional trees is a weakly final coalgebra of \( \mathcal{P}^\omega(\_ ) + Y \).

Proof. Given a coalgebra

\[
\xi : A \longrightarrow \mathcal{P}^\omega A + Y
\]

(with \( A \cap Y = \emptyset \), for simplicity), we define for every \( a \in A \) a labelled tree \( t_a \) all of whose nodes are labelled in \( A + Y \) as follows:

the root of \( t_a \) is labelled by \( a \);

given a node of \( t_a \) labelled by \( x \in A \), then the children of that node correspond to the elements of \( \xi(x) \), in case \( \xi(x) \subseteq A \), and in case \( \xi(x) \in Y \), the node is a leaf.

Let \( h : A \longrightarrow BY \) assign to \( a \in A \) the tree \( h(a) \in BY \) obtained from \( t_a \) by deleting all the labels in \( A \). Then \( h \) is easily seen to be a homomorphism. \( \square \)

Definition 5.8 Two trees \( t, s \) in \( BY \) are called Barr-similar, notation

\[
t \sim_0 s
\]

provided that for every \( n \in \omega \) we have \( C_n(t) = C_n(s) \) (where \( C_n \) denotes the extensional reflection of the cutting at level \( n \), leaving all new leaves unlabelled).

For every small ordinal number \( i > 0 \) we denote by \( \sim_i \) the equivalence on \( BY \) with

\[
t \sim_i s \text{ iff for every } j < i \text{ and every child } t' \text{ of } t \text{ there is a child } s' \text{ of } s \text{ with } t' \sim_j s', \text{ and vice versa.}
\]

Theorem 5.9 A final coalgebra for \( \mathcal{P}^\omega(\_ ) + Y \) is a quotient

\[
T^i Y = BY/\sim
\]

of the coalgebra \( BY \) modulo the bisimilarity equivalence given by

\[
t \sim s \text{ iff } t \sim_i s \text{ for all small ordinals } i.
\]

Proof. Analogous to that of Theorem 5.5. \( \square \)

Corollary 5.10 Every guarded system of equations \( e : X \longrightarrow T^i(X + Y) \) has a unique solution \( e^! : X \longrightarrow T^i Y \). In particular, every system of equations

\[
(3) \quad x = A_x \quad x \in X
\]

where the right-hand sides are subsets \( A_x \subseteq X \), has a solution, i.e., a system \( x^! \) (\( x \in X \)) of extensional trees such that

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{
x^! \ar@{~}[r] & \cdots & y \ar@{~}[l] \\
\end{array}
\end{array}
\]

holds for all \( x \in X \), and these trees are unique up to bisimilarity.

Example 5.11 The equation

\[
x = \{x\}
\]

26
has as a solution the tree $\Omega$ of Example 5.1. And also the tree $\Omega'$.

**Remark 5.12** The possibility of uniquely solving all systems of equations (3) is the basis of non-well-founded set theory. In fact, every system (3) describes a graph on the set $X$ (with edges those pairs $(x,y)$ where $y \in A_x$) and a solution, provided that it is formed by sets rather than extensional trees, is precisely Aczel's concept of decoration of the graph. And Aczel's Antifoundation Axiom states that every graph has a unique decoration.

Now extensional trees are closely related to (well-founded) sets: In well-founded set theory

(a) every set has a graph of the elementhood relation which is extensional and has no infinite paths (i.e., is “well-founded” as a graph)

and

(b) two well-founded, extensional graphs are bisimilar if and only if they are equal.

Thus, non-well-founded set theory extends the concept of set so as to retain (a) and (b) for not necessarily well-founded graphs. Our concept of bisimilarity class of extensional graphs thus exactly corresponds to the concept of non-well-founded set.

**References**


