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Iterative Algebras for a Base

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Abstract

For algebras A whose type is given by an endofunctor, iterativity means that every flat equation morphism in A has a unique solution. In our previous work we proved that every object generates a free iterative algebra, and we provided a coalgebraic construction of those free algebras. Iterativity w.r.t. an endofunctor was generalized by Tarmo Uustalu to iterativity w.r.t. a "base", i.e., a functor of two variables yielding finitary monads in one variable. In the current paper we introduce iterative algebras in this general setting, and provide again a coalgebraic construction of free iterative algebras.

Keywords: free iterative theory, rational monad, coalgebra

1 Introduction

In our previous work we introduced, for every finitary[4] endofunctor H, the concept of an iterative H-algebra. The aim was to generalize and simplify the description of free iterative theories of Calvin Elgot and his coauthors [10], [11]. In that we followed the footsteps of Evelyn Nelson [16] who introduced iterative Σ-algebras (in Set) and simplified the description of the free iterative

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[4] "finitary" means: preserving filtered colimits

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theory R_Σ on Σ: that theory assigns to every set Y the algebra $R_\Sigma Y$ of all *rational* Σ*-trees* on Y, i.e., those Σ-trees that have finitely many subtrees up to isomorphism (a characterization provided by Susanna Ginali [12]). Whereas the original proof, using constructions on algebraic theories, occupied most of the technical papers [10], [8], [11], the proof due to Evelyn Nelson was concise and intuitive. In [5] we introduced iterativity for H-algebras for every finitary endofunctor H of a locally finitely presentable category $\mathcal{A}$ — this includes categories such as many-sorted sets $\mathcal{A} = \mathsf{Set}^S$ and the category $\mathsf{Fin}[\mathsf{Set}, \mathsf{Set}]$ of all finitary set functors. We proved that every object Y of $\mathcal{A}$ generates a free iterative H-algebra, RY, and we provided a coalgebraic construction of RY: for the case of $Y = 0$ (initial in $\mathcal{A}$), $R0$ is a colimit of all finite H-coalgebras, and in the general case we work with $H(-) + Y$ instead of H. And, again, we proved that the monad $R(-)$ of free iterative H-algebras is a free iterative monad on H.

In the present paper we work with a *base* instead of an endofunctor, and study iterativity of base algebras. By a base we understand a finitary endofunctor from $\mathcal{A}$ to $\mathsf{FM}(\mathcal{A})$, the category of finitary monads on $\mathcal{A}$. This was introduced by Tarmo Uustalu [18] under the name parametrized monad. The motivating idea is to study iterativity of Σ-algebras, for a signature Σ, where each operation symbol comes with the information in what places iteration is allowed to occur. Let us illustrate this on the simple example of a signature consiting of a single binary operation symbol $*$.

Case 1: full iterativity. This is the concept of iterative algebra of Evelyn Nelson [16]: An algebra $(A, *)$ is iterative if every system

$$\begin{aligned} x_1 &\approx t_1 \\ &\vdots \\ x_m &\approx t_m \end{aligned} \tag{1.1}$$

of equations in variables $X = \{ x_1, \ldots, x_m \}$ and with right-hand sides $t_i = x_j * x_k$ for x_j, x_k in X, or $t_i \in A$, has a unique solution in A. Equivalently every system (1.1) where each t_i is a term on $X + A$, $t_i \notin X$, has a unique solution. A free iterative algebra, RY, on a set Y is the algebra of all rational binary trees on Y.

Case 2: restricted iterativity. Here we require that the free variables are only allowed to occur on the left-hand position of $*$. Thus, an iterative algebra is one in which every system (1.1) with right-hand sides $t_i = x * a$ for $x \in X$ and $a \in A$, or $t_i \in A$ has a unique solution. A free iterative algebra, for iterativity w. r. t. these systems of equations, is the algebra of all right-wellfounded rational binary trees over Y, i. e., those which have the right-most path from every node finite. Observe that for an iterative algebra all right-hand sides

$$t_i = x_{i_1} * (x_{i_2} * (\cdots * (x_{i_n} * a)) \cdots) , \qquad x_{i_j} \in X, j = 1, \dots, n \tag{1.2}$$

can be allowed: every system (1.1) with such right-hand sides has a unique solution, too.

Case 3: no iterativity. Here no variable is allowed to occur on right-hand sides of systems (1.1), i. e., we are left with the "trivial" systems in which all right-hand sides lie in A. Every algebra is then iterative.

In order to formalize such parametrized iterativity, we move from finitary functors $H : \mathcal{A} \longrightarrow \mathcal{A}$, used for "classical" algebra, to finitary functors $H : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$, called parametrized endofunctors. In the case of one binary operation the "classical" polynomial endofunctor $H : \mathsf{Set} \longrightarrow \mathsf{Set}$, $HX = X \times X$, is now substituted by three parametrized endofunctors: $H(X, Y) = X \times X$ for Case 1, $H(X, Y) = X \times Y$ for Case 2, and $H(X, Y) = Y \times Y$ for Case 3. Let us denote by

$$X \,\square\, Y \qquad \text{(read "}X\text{ box }Y\text{")}$$

a free $H(X, -)$-algebra on Y (for all pairs of objects $X, Y \in \mathcal{A}$). More precisely, for every object X we denote by $X \,\square\, -$ the free monad on the endofunctor $H(X, -)$ (which, as proved by Micheal Barr [7] is just the monad of the free algebras of $H(X, -)$). This yields a base, i. e., a functor $\mathcal{A} \longrightarrow \mathsf{FM}(\mathcal{A})$ in the obvious way.

Example: for one binary operation with full iterativity, $H(X, -) = X \times X$ is the constant endofunctor whose free algebra on Y is

$$X \,\square\, Y = X \times X + Y \, .$$

The case of restricted iterativity, $H(X, -) = X \times -$ corresponds to unary operation symbols indexed by X — the free algebras are

$$X \,\square\, Y = X^* \times Y$$

where X^* is a free monoid on X.

Finally, the "trivial" case of no iterativity yields the base

$$X \,\square\, Y = \text{free algebra on } Y$$

independently of X.

In general, we use the uncurried form of a base, i. e., a functor

$$\Box : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$$

finitary in both variables and equipped with monad units $A \longrightarrow X \Box A$ and monad multiplications $X \Box (X \Box A) \longrightarrow X \Box A$ for all pairs of objects X, A satisfying some obvious compatibility conditions (see Section 2). A *base algebra* is a monad algebra of the monad $A \Box -$ on the object A. That is, a base algebra is given by a morphism

$$\alpha : A \Box A \longrightarrow A$$

satisfying the Eilenberg-Moore conditions in the second variable.

The bases $X \Box Y = X \times X + Y$ and $X \Box Y = X^* \times Y$ on Set yield the usual algebras on one binary operation as base algebras. However, iterativity is different, as we demonstrate below.

For a given base algebra A let us call morphisms

$$e : X \longrightarrow X \Box A\,, \qquad X \text{ finitely presentable,}$$

equation morphisms. For a fully iterative binary operation this is $e : X \longrightarrow X \times X + A$ expressing precisely a system (1.1), for the restricted iterativity we get $e : X \longrightarrow X^* \times A$ as in (1.2) above.

Definition 1.1 A base algebra $\alpha : A \Box A \longrightarrow A$ is *iterative* provided that for every equation morphism $e : X \longrightarrow X \Box A$ there exists a unique *solution*, i. e., a unique morphism $e^\dagger : X \longrightarrow A$ for which the square

$$\begin{array}{ccc} X & \xrightarrow{\;e^\dagger\;} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \alpha} \\ X \Box A & \xrightarrow[\;e^\dagger \Box A\;]{} & A \Box A \end{array}$$

commutes.

The main result of our paper is that

(i) free iterative base algebras always exist, and

(ii) the monad they present in $\mathcal{A}$ is a free iterative monad on the given base.

This means that the full strength of the results of [5], concerning (fully) iterative H-algebras of a given endofunctor H, generalize to the case of base algebras. In that special case one works with the base

$$X \Box Y = HX + Y$$

where iterativity of base algebras is the full iterativity above, and the monad of free iterative algebras was proved to be a free iterative monad on H.

The notion of a base (under the name parametrized monad) has been introduced by Tarmo Uustalu [18] who generalized some results of our paper [1]. In the present paper we continue in the same vein by generalizing results of [5] from H-algebras to base algebras. Although the structure of the present paper follows that of [5] closely, it turns out that all the proofs are substantially more difficult. Thus our original hope that we will just indicate how to modify the previous proof ideas to the present generality failed, and we feel obliged to present detailed proofs again. The concrete examples of bases below (see 2.5) and their free iterative algebras (see 3.3–3.6) were already considered in [18].

2 Bases and Base Algebras

Assumption 2.1 Throughout this section we assume that a locally finitely presentable category $\mathcal{A}$ is given. We denote by $\mathsf{FM}(\mathcal{A})$ the category of all finitary monads on $\mathcal{A}$ (i.e., monads whose underlying functor preserves filtered colimits) and monad morphisms.

Definition 2.2 By a *base* on $\mathcal{A}$ is understood a finitary functor from $\mathcal{A}$ to $\mathsf{FM}(\mathcal{A})$.

Notation 2.3

(i) Given a base, we have a finitary underlying functor from $\mathcal{A}$ to the category $\mathsf{Fin}[\mathcal{A},\mathcal{A}]$ of finitary endofunctors of $\mathcal{A}$. This is a curried form of a functor of two variables, finitary in each variable, which we denote by

$$\Box : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}.$$

(ii) The unit of the monad $X \Box -$ is denoted by $u^X : Id \longrightarrow X \Box -$; its components are

$$u^X_A : A \longrightarrow X \Box A.$$

(iii) The multiplication of the monad $X \Box -$ is denoted by m^X; its components are

$$m^X_A : X \Box (X \Box A) \longrightarrow X \Box A.$$

Remark 2.4 Explicitly, to specify a base means to specify a finitary functor of two variables

$$\Box : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$$

together with morphisms

$$u^X_A : A \longrightarrow X \Box A \quad \text{and} \quad m^X_A : X \Box (X \Box A) \longrightarrow X \Box A$$

for arbitrary objects X, A of $\mathcal{A}$ such that the following diagrams commute:

$$\begin{array}{ccccc} X \Box A & \xrightarrow{X \Box u^X_A} & X \Box (X \Box A) & \xleftarrow{u^X_{X \Box A}} & X \Box A \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle m^X_A} & \diagup\!\!\diagup & \\ & & X \Box A & & \end{array} \tag{2.1}$$

and

$$\begin{array}{ccc} X \Box (X \Box (X \Box A)) & \xrightarrow{X \Box m^X_A} & X \Box (X \Box A) \\ {\scriptstyle m^X_{X \Box A}}\downarrow & & \downarrow{\scriptstyle m^X_A} \\ X \Box (X \Box A) & \xrightarrow[m^X_A]{} & X \Box A \end{array} \tag{2.2}$$

expressing the monad axioms for each $X \Box -$, together with

$$\begin{array}{ccc} A & \xrightarrow{u^X_A} & X \Box A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h \Box f} \\ B & \xrightarrow[u^Y_B]{} & Y \Box B \end{array} \tag{2.3}$$

and

$$\begin{array}{ccc} X \Box (X \Box A) & \xrightarrow{m^X_A} & X \Box A \\ {\scriptstyle h \Box (h \Box f)}\downarrow & & \downarrow{\scriptstyle h \Box f} \\ Y \Box (Y \Box B) & \xrightarrow[m^Y_B]{} & Y \Box B \end{array} \tag{2.4}$$

which express the naturality of u^X and m^X and the fact that for every morphism $h : X \longrightarrow Y$ we have a monad morphism $h \Box (-) : X \Box (-) \longrightarrow Y \Box (-)$.

Examples 2.5

(i) We have "trivial" bases given by all constant functors from $\mathcal{A}$ to $\mathsf{FM}(\mathcal{A})$. That is, for every finitary monad $\mathbb{S}$ on $\mathcal{A}$ we form the trivial base

$$X \Box_{\mathbb{S}} A = SA$$

whose unit and multiplication is that of $\mathbb{S}$.

(ii) Coproduct is a base

$$X \Box A = X + A$$

with the obvious unit and multiplication

$$u^X_A = \mathsf{inr} : A \longrightarrow X + A \quad \text{and} \quad m^X_A = [\mathsf{inl}, \mathsf{inl}, \mathsf{inr}] : X + X + A \longrightarrow X + A.$$

(iii) Let X^* be a free monoid on the object X of $\mathcal{A}$ with unit $\eta_X : 1 \longrightarrow X^*$ and multiplication $\mu_X : X^* \times X^* \longrightarrow X^*$. Then we have the base

$$X \Box A = X^* \times A$$

with base unit $u_A^X = \eta_X \times A$ and base multiplication $m_A^X = \mu_X \times A$.

(iv) Let $\mathcal{B}$ be any locally finitely presentable category. Then so is the category $\mathcal{A} = \mathsf{Fin}[\mathcal{B}, \mathcal{B}]$ of all finitary endofunctors of $\mathcal{B}$. There we have a base

$$X \Box A = \mathcal{F}(X) \cdot A$$

where a free monad on X is denoted by $(\mathcal{F}(X), \eta^X, \mu^X)$ — it exists since X is finitary, see [7]. The base unit is $u_A^X = \eta^X A$ and the base multiplication $m_A^X = \mu^X A$.

Example 2.6 Let $\Box : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ be a base. We obtain other bases $\dot{\Box}$ by precomposing $\Box$ with finitary endofunctors $H : \mathcal{A} \longrightarrow \mathcal{A}$:

$$X \mathbin{\dot{\Box}} A = HX \Box A$$

with unit

$$\dot{u}_A^X = u_A^{HX} : A \longrightarrow HX \Box A$$

and multiplication

$$\dot{m}_A^X = m_A^{HX} : HX \Box (HX \Box A) \longrightarrow HX \Box A.$$

Of particular importance is the base obtained from the base $+$, i. e.,

$$X \mathbin{\dot{\Box}} A = HX + A.$$

We will see below that our previous results of [5] on H-algebras are special cases of the results concerning these bases.

Definition 2.7 Given a base $\Box$, by a *base algebra* is understood an object A of $\mathcal{A}$ together with a monadic algebra on A of the monad $A \Box -$.

That is, a base algebra is given by an object A and a morphism $\alpha : A \Box A \longrightarrow A$ such that the following two diagrams

$$\begin{array}{ccc} A & \xrightarrow{u_A^A} & A \Box A \\ & \searrow^{=} & \downarrow{\scriptstyle \alpha} \\ & & A \end{array} \qquad \begin{array}{ccc} A \Box (A \Box A) & \xrightarrow{A \Box \alpha} & A \Box A \\ {\scriptstyle m_A^A}\downarrow & & \downarrow{\scriptstyle \alpha} \\ A \Box A & \xrightarrow[\alpha]{} & A \end{array} \tag{2.5}$$

commute.

Notation 2.8 We denote by

$$\mathsf{Alg}(\mathcal{A}, \Box)$$

the category of all base algebras and all *homomorphisms* from (A, α) to (B, β), i.e., morphisms $h : A \longrightarrow A'$ of $\mathcal{A}$ such that the square

$$\begin{array}{ccc} A \,\Box\, A & \xrightarrow{\ \alpha\ } & A \\ {\scriptstyle h \Box h}\downarrow & & \downarrow{\scriptstyle h} \\ B \,\Box\, B & \xrightarrow[\ \beta\ ]{} & B \end{array} \tag{2.6}$$

commutes.

Examples 2.9

(i) Algebras of the base $X \,\Box\, A = X + A$ are given by an object A and an endomorphism of A (i.e., these are just unary algebras in $\mathcal{A}$). Homomorphisms are also the usual homomorphisms of unary algebras.

(ii) Algebras of the base $X \,\Box\, A = HX + A$ are the usual H-algebras, i.e., pairs consisting of an object A and a morphism $\alpha : HA \longrightarrow A$. Also, homomorphisms are the usual H-algebra homomorphisms. Thus,

$$\mathsf{Alg}(\mathcal{A}, \Box) = \mathsf{Alg}\, H$$

is the category of H-algebras and homomorphisms.

(iii) Algebras of the base on Set given by

$$X \,\Box\, A = X^* \times A \qquad \text{(see 2.5(iii))}$$

are precisely the usual algebras on one binary operation. In fact, given the latter, say,

$$\oplus : A \times A \longrightarrow A$$

define $\alpha : A^* \times A \longrightarrow A$ by

$$\alpha(a_1 a_2 \dots a_n, a) = a_1 \oplus (a_2 \oplus \dots (a_n \oplus a) \dots).$$

This satisfies (2.5). Conversely, given $\alpha : A^* \times A \longrightarrow A$ satisfying (2.5), it is given by the above formula where $\oplus$ denotes the restriction of α to all pairs in $A \times A$. Consequently, the bases

$$X \,\Box\, A = (X \times X) + A \quad \text{and} \quad X \,\Box\, A = X^* \times A$$

on Set define the same categories of algebras.

(iv) Let $\mathcal{B}$ be any locally finitely presentable category and put $\mathcal{A} = \mathsf{Fin}[\mathcal{B}, \mathcal{B}]$ with the base

$$X \,\Box\, A = \mathcal{F}(X) \cdot A \qquad \text{(see 2.5(iv))}$$

An algebra is a pair (A, α) consisting of a finitary endofunctor $A : \mathcal{B} \longrightarrow \mathcal{B}$ and a natural transformation $\alpha : A \cdot A \longrightarrow A$.

More precisely: each such pair defines a unique natural transformation $\overline{\alpha}$ from A to $\langle A, A\rangle$ (the right Kan extension of A along A). Since $\langle A, A\rangle$ is always a monad and $\mathcal{F}(A)$ is a free monad on A, $\overline{\alpha}$ yields a unique monad morphism

$$\widetilde{\alpha} : \mathcal{F}(A) \longrightarrow \langle A, A\rangle.$$

The unique natural transformation

$$\widehat{\alpha} : \mathcal{F}(A) \cdot A \longrightarrow A$$

corresponding to $\widetilde{\alpha}$ defines an algebra of our base — in fact, the condition (2.5) above is equivalent to $\widetilde{\alpha}$ being a monad morphism.

Proposition 2.10 *The category* $\mathsf{Alg}(\mathcal{A}, \Box)$ *is locally finitely presentable, and its forgetful functor into* $\mathcal{A}$ *has a left adjoint, i.e., free base algebras exist on every object of* $\mathcal{A}$*.*

Proof. The endofunctor $SA = A \,\Box\, A$ of $\mathcal{A}$ is finitary, and thus the category $\mathsf{Alg}\, S$ is locally finitely presentable and its forgetful functor has a left adjoint, see [6].

The category $\mathsf{Alg}(\mathcal{A}, \Box)$ is a full subcategory of $\mathsf{Alg}\, S$, and it is easy to verify that it is closed under limits and filtered colimits in $\mathsf{Alg}\, S$. It follows that it is a reflective subcategory, see Theorem 2.48 in [6]. Since the forgetful functor of $\mathsf{Alg}(\mathcal{A}, \Box)$ is a restriction of that of $\mathsf{Alg}\, S$, the proposition follows.

$\Box$

3 Iterative Base Algebras

Assumption 3.1 Throughout this section $\Box$ denotes a base on a locally finitely presentable category $\mathcal{A}$.

Definition 3.2

(i) By a (finitary, flat) *equation morphism* in an object A is meant a morphism

$$e : X \longrightarrow X \,\Box\, A, \quad X \text{ finitely presentable.}$$

(ii) Suppose that A is the underlying object of a base algebra $\alpha : A \Box A \longrightarrow A$. Then by a *solution* of e is meant a morphism $e^\dagger : X \longrightarrow A$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{e^\dagger} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \alpha} \\ X \,\square\, A & \xrightarrow[e^\dagger \square A]{} & A \,\square\, A \end{array} \tag{3.1}$$

commutes.

(iii) A base algebra is called *iterative* provided that every equation morphism has a unique solution.

Example 3.3 Consider the base $X \,\square\, A = X + A$ in Set. Then a base algebra, i. e., a unary algebra $\alpha : A \longrightarrow A$, is iterative if and only if α^k has a unique fixed point for all k, see [5].

Example 3.4 Algebras of the base

$$X \,\square\, A = (X \times X) + A$$

on $\mathcal{A} = \mathsf{Set}$ are the usual algebras on one binary operation: see Example 2.9(ii) and put $HX = X \times X$. There is no easy criterion for an algebra to be iterative. But there are nice examples of iterative algebras, see [5], e.g.,

$$\begin{array}{ll} A = \{1, 2, 3, \dots\} \cup \{\infty\} & \text{with addition} \\ A = (0, \infty] & \text{with addition} \\ A = (1, \infty] & \text{with multiplication} \end{array}$$

A free iterative algebra on a set Y (of generators) can be described as the algebra RY of all rational binary trees on Y, see Section 1.

Example 3.5 Consider the base $X \,\square\, A = X^* \times A$ on $\mathcal{A} = \mathsf{Set}$, see Example 2.5(iii). Although its algebras are, again, the usual binary algebras, the concept of iterative algebras differs from the above example. Recall that an algebra $(A, \oplus)$ leads to $\alpha : A^* \times A \longrightarrow A$ with

$$\alpha(a_1 a_2 \dots a_n, a) = a_1 \oplus (a_2 \oplus \dots (a_n \oplus a) \dots).$$

It is iterative if and only if for every equation morphism $e : X \longrightarrow X^* \times A$ (X finite) there exists a unique $e^\dagger : X \longrightarrow A$ such that for every variable x we have that

(i) $e(x) = (\varepsilon, a)$ implies $e^\dagger(x) = a$, and

(ii) $e(x) = (x_1 \dots x_n, a)$ implies $e^\dagger(x) = e^\dagger(x_1) \oplus (e^\dagger(x_2) \oplus \dots (e^\dagger(x_n) \oplus a) \dots)$.

Thus, for example, the empty algebra is iterative in the present sense (but it is not iterative for $(X \times X) + A$).

A free iterative algebra, $\widehat{R}Y$, on a set Y can be described as a subalgebra of the above algebra RY of all rational binary trees on Y. Let us call a binary tree *right-wellfounded* if from every node the right-most path is always finite (i.e., it leads to a leaf). It is obvious that the set $\widehat{R}Y$ of all rational right-wellfounded trees is a subalgebra of RY. This subalgebra is iterative (w.r.t. the present base $X^* \times A$), in fact, $\widehat{R}Y$ is a free iterative algebra on Y.

Example 3.6 We know that the classical Σ-algebras for a signature $\Sigma = (\Sigma_n)_{n\in\mathbb{N}}$ are just algebras of the corresponding "polynomial" endofunctor H_Σ of Set. The above Example 3.4 immediately generalizes to the base $X \mathbin{\dot\Box} Y = H_\Sigma X + Y$ of Example 2.6: a free iterative algebra on a set Y is the algebra

$$R_\Sigma Y$$

of all rational Σ-trees on Y, see Section 1.

Open Problem 3.7 Describe, for $\mathcal{A} = \mathsf{Fin}[\mathsf{Set}, \mathsf{Set}]$, free iterative algebras of the bases

$$X \Box A = (X \times X) + A$$

and

$$X \Box A = \mathcal{F}(X) \cdot A \,.$$

Example 3.8 For the trivial bases $\Box_{\mathbb{S}}$, see Example 2.5(i), all algebras are iterative. In fact, given an Eilenberg-Moore algebra $\alpha : SA \longrightarrow A$ and an equation morphism $e : X \longrightarrow SA$, the unique solution is $e^\dagger = \alpha \cdot e : X \longrightarrow A$.

Notation 3.9 Let $e : X \longrightarrow X \Box A$ be an equation morphism with parameters in A. Every morphism $h : A \longrightarrow B$ in $\mathcal{A}$ yields an equation morphism

$$h \bullet e \equiv X \xrightarrow{\ e\ } X \Box A \xrightarrow{X \Box h} X \Box A \,. \tag{3.2}$$

Lemma 3.10 *Let (A, α) and (B, β) be iterative algebras. Then a morphism $h : A \longrightarrow B$ in $\mathcal{A}$ is a homomorphism if and only if it preserves solutions, i. e., for every equation morphism $e : X \longrightarrow X \Box A$ the solution of $h \bullet e$ in B is $h \cdot e^\dagger$.*

Lemma 3.10 explains that the choice of "plain" homomorphisms between iterative algebras is adequate.

Proposition 3.11 *Every object of $\mathcal{A}$ generates a free iterative algebra.*

Proof. It is easy to prove that iterative algebras are closed under limits and filtered colimits in $\mathsf{Alg}(\mathcal{A}, \Box)$. By Theorem 2.48 in [6] they form a reflective subcategory of $\mathsf{Alg}(\mathcal{A}, \Box)$. Now apply Proposition 2.10. □

4 A Coalgebraic Construction

We know from the preceding section that, for every base, free iterative algebras exist. The aim of the present section is to show that a free iterative algebra RY on an object Y can be constructed as a filtered colimit of all equation morphisms $e : X \longrightarrow X \,\Box\, Y$ (where X ranges through a set $\mathcal{A}_{fp}$ representing all finitely presentable objects of $\mathcal{A}$ up to isomorphism).

More precisely, consider the coalgebras of the endofunctor $- \,\Box\, Y$ together with the usual coalgebra homomorphisms. We denote by

$$\mathsf{EQ}_Y$$

the category of all equation morphisms as the full subcategory of $\mathsf{Coalg}\,(-\,\Box\, Y)$ on all objects from $\mathcal{A}_{fp}$. Since $\mathsf{Coalg}\,(- \,\Box\, Y)$ is cocomplete, with colimits formed on the level of $\mathcal{A}$, it is obvious that EQ_Y is closed in it under finite colimits. In particular, EQ_Y is a filtered category. We also denote by

$$\mathrm{Eq}_Y : \mathsf{EQ} \longrightarrow \mathcal{A}, \quad (e : X \longrightarrow X \,\Box\, Y) \mapsto X$$

the forgetful functor. This defines a filtered diagram in $\mathcal{A}$.

Notation 4.1 RY denotes a (filtered) colimit of the diagram Eq_Y with colimit cocone

$$e^\sharp : X \longrightarrow RY \quad (\text{for all } e : X \longrightarrow X \,\Box\, Y \text{ in } \mathsf{EQ}_Y).$$

Remark 4.2 The aim of the present section is to prove that RY carries a structure of an iterative algebra making it a free iterative algebra on Y. We proceed in two steps: we first assume that Y is a finitely presentable object. This enables us, for example, to define the universal arrow immediately: observe that $u_Y^Y : Y \longrightarrow Y \,\Box\, Y$ is an object of EQ_Y and denote by

$$\eta_Y = (u_Y^Y)^\sharp : Y \longrightarrow RY$$

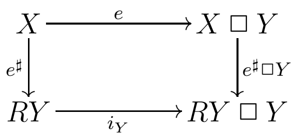
the corresponding colimit morphism.

An extension of all the results to arbitrary objects Y is then easy. For example, η_Y is defined as follows: express Y as a colimit of a filtered diagram of finitely presentable objects Y_t, $(t \in T)$, then it is easy to verify that RY is a filtered colimit of RY_t, $(t \in T)$, and we put $\eta_Y = \operatorname*{colim}_{t\in T} \eta_{Y_t}$.

Notation 4.3 For every object Y we denote by

$$i_Y : RY \longrightarrow RY \,\square\, Y$$

the unique morphism for which the squares



(4.1)

commute for every e in EQ_Y. This is well defined since the morphisms $(e^\sharp \square Y)\cdot e$ are easily seen to form a cocone of the diagram Eq_Y.

Lemma 4.4 *For every equation morphism e in EQ_Y there exists a unique morphism $e^\sharp : X \longrightarrow RY$ such that the square (4.1) commutes.*

Notation 4.5 We introduce notation here for "adding variables" to equations: given an equation morphism

$$e : X \longrightarrow X \,\square\, Y \quad \text{in } \mathsf{EQ}_Y$$

and an object Q, when are we able to form "canonically" an equation morphism

$$X + Q \longrightarrow (X + Q) \,\square\, Y \quad \text{in } \mathsf{EQ}_Y?$$

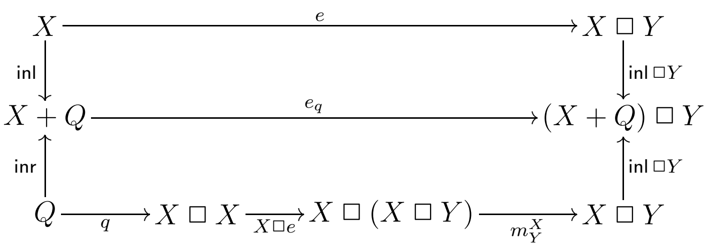
One possibility is to assume that a morphism

$$q : Q \longrightarrow X \,\square\, X$$

is given. Then we can define an equation morphism

$$e_q = (\mathsf{inl} \,\square\, Y) \cdot [X \,\square\, Y, m_Y^X \cdot (X \,\square\, e)] \cdot (e + q)$$

i.e., e_q is given by the following diagram



Theorem 4.6 *For every finitely presentable object Y there exists a unique structure of a base algebra $\rho_Y : RY \,\Box\, RY \longrightarrow RY$ such that the squares*

$$\begin{array}{ccc} Q & \xrightarrow{\quad \mathsf{inr} \quad} & X + Q \\ {\scriptstyle q}\downarrow & & \\ X \,\Box\, X & & \downarrow{\scriptstyle e_q{}^{\sharp}} \\ {\scriptstyle e^{\sharp}\Box e^{\sharp}}\downarrow & & \\ RY \,\Box\, RY & \xrightarrow[\quad \rho_Y \quad]{} & RY \end{array} \tag{4.2}$$

(where e and q are as in Notation 4.5) commute.

Remark 4.7 In the proof below we will denote by $\mathrm{Eq}_Y \,\Box\, \mathrm{Eq}_Y : \mathsf{EQ}_Y \longrightarrow \mathcal{A}$ the diagram given by objects $X \,\Box\, X$, where $e : X \longrightarrow X \,\Box\, Y$ ranges over EQ_Y, and by morphisms $h \,\Box\, h$, for h in EQ_Y. Since $\Box$ is finitary in both variables and the diagonal $\mathsf{EQ}_Y \longrightarrow \mathsf{EQ}_Y \times \mathsf{EQ}_Y$ is cofinal, the colimit of $\mathrm{Eq}_Y \,\Box\, \mathrm{Eq}_Y$ is $RY \Box RY$ with colimit injections $e^\sharp \Box e^\sharp$. Analogously, $X \Box RY = \mathrm{colim}\, X \Box \mathrm{Eq}_Y$, etc.

Proof. We establish that there is a unique morphism ρ_Y for which (4.2) commutes, and postpone the verification that this yields a base algebra to the full version of our paper.

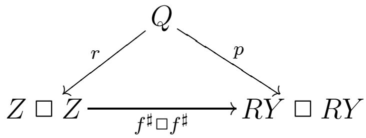
(1) Consider an arbitrary morphism $p : Q \longrightarrow RY \,\Box\, RY$ where $Q \in \mathcal{A}_{fp}$. In every locally finitely presentable category each object is a canonical filtered colimit of morphisms from finitely presentable objects, thus, $RY \,\Box\, RY$ is a colimit of the corresponding diagram D with the colimit cocone formed by all p's. We show that each p factors as

$$p = (e^\sharp \,\Box\, e^\sharp) \cdot q$$

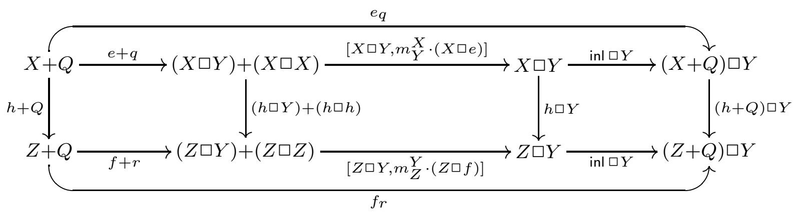
for some e and q, and that the morphisms $e_q{}^\sharp \cdot \mathsf{inr} : Q \longrightarrow RY$ (which, as we show, are independent of the choice of such a factorization) form a cocone of D.

The existence of the above factorization follows from the above filtered colimit $RY \,\Box\, RY = \mathrm{colim}(Eq_Y \,\Box\, \mathrm{Eq}_Y)$. The morphism $p : Q \longrightarrow RY \,\Box\, RY$, having a finitely presentable domain, factors through one of the colimit injections.

We claim that the upper passage $e_q{}^\sharp \cdot \mathsf{inr}$ of the square (4.2) is independent of the choice of the factorization. Thus, let $f : Z \longrightarrow Z \,\Box\, Y$ be an equation morphism and let



be another factorization of p through a colimit morphism of the diagram $\mathrm{Eq}_Y \,\square\, \mathrm{Eq}_Y$. Since that diagram is filtered, we can assume, without loss of generality, that a morphism h from e to f in EQ_Y exists, and that the equation $r = (h \,\square\, h) \cdot q$ holds. It follows that $h + Q$ is a morphism from e_q to f_r:



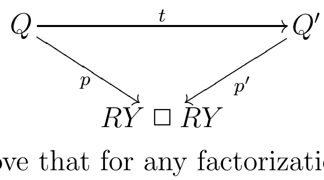
Consequently,

$$e_q{}^\sharp = f_r{}^\sharp \cdot (h + Q).$$

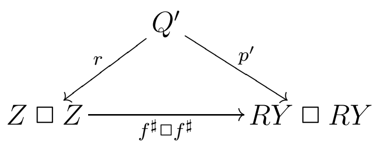
This proves the desired independence:

$$e_q{}^\sharp \cdot \mathsf{inr} \;=\; f_r{}^\sharp \cdot (h + Q) \cdot \mathsf{inr} \;=\; f_r{}^\sharp \cdot \mathsf{inr}\,.$$

(2) The above morphisms $e_q{}^\sharp \cdot \mathsf{inr} : Q \longrightarrow RY$ form a cocone of the diagram D. That is, given an arrow $p' : Q' \longrightarrow RY \,\square\, RY$, $Q' \in \mathcal{A}_{fp}$, and given a morphism



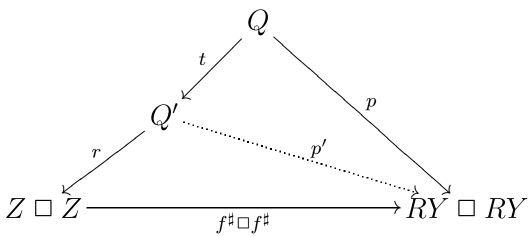
of the diagram D, we prove that for any factorization



we have

$$e_q{}^\sharp \cdot \mathsf{inr} \;=\; f_r{}^\sharp \cdot \mathsf{inr} \,\cdot t. \tag{4.3}$$

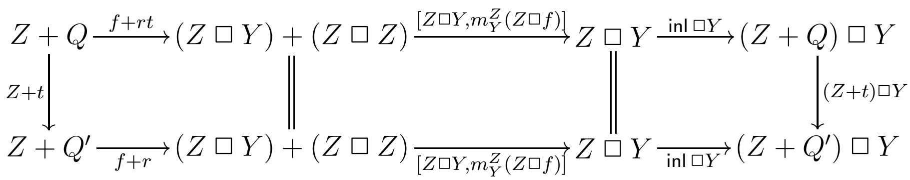
In fact, the following factorization of p:



yields, by the independence proved in part (1):

$$e_q{}^\sharp \cdot \mathsf{inr} = f_{rt}{}^\sharp \cdot \mathsf{inr}\,.$$

Moreover, $Z + t$ is an equation morphism from f_{rt} into f_r:



Consequently, $f_{rt}{}^\sharp = f_r{}^\sharp \cdot (Z + t)$, which implies (4.3):

$$e_q{}^\sharp \cdot \mathsf{inr} = f_r{}^\sharp \cdot (Z + t) \cdot \mathsf{inr} = f_r{}^\sharp \cdot \mathsf{inr} \cdot t.$$

The cocone $e_q{}^\sharp \cdot \mathsf{inr}$ has precisely one factorization through the colimit cocone — this proves that (4.2) defines a unique ρ_Y. □

Lemma 4.8 *For every finitely presentable object Y the morphism i_Y (see Diagram (4.1)) is an isomorphism with the inverse*

$$i_Y^{-1} = j_Y \equiv RY \,\square\, Y \xrightarrow{RY \square \eta_Y} RY \,\square\, RY \xrightarrow{\rho_Y} RY \tag{4.4}$$

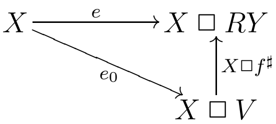
Theorem 4.9 *For every finitely presentable object Y, the algebra (RY, ρ_Y) is a free iterative algebra on Y w.r.t. the universal arrow $\eta_Y : Y \longrightarrow RY$.*

Proof. (1) (RY, ρ_Y) is iterative. In fact, every equation morphism

$$e : X \longrightarrow X \,\square\, RY\,, \quad X \text{ finitely presentable,}$$

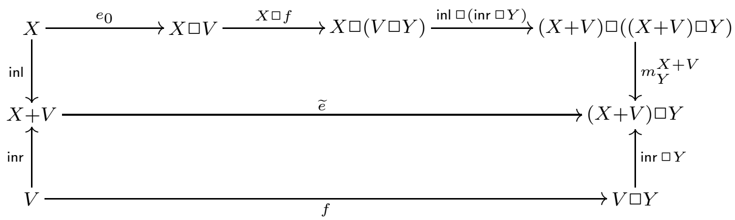
has a unique solution obtained as follows. Since $X \,\square\, RY = \operatorname{colim} X \,\square\, \mathrm{Eq}_Y$, see Remark 4.7, e factors through the colimit injection $X \,\square\, f^\sharp$ for some f :

$V \longrightarrow V \,\square\, Y$ in EQ_Y. Thus, we have a commutative triangle



(4.5)

We form an equation $\widetilde{e} : X + V \longrightarrow (X + V) \,\square\, Y$ as follows:



(4.6)

We will prove that the given equation morphism e has the solution

$$e^{\dagger} \equiv X \xrightarrow{\mathsf{inl}} X + V \xrightarrow{\widetilde{e}^{\sharp}} RY\,. \tag{4.7}$$

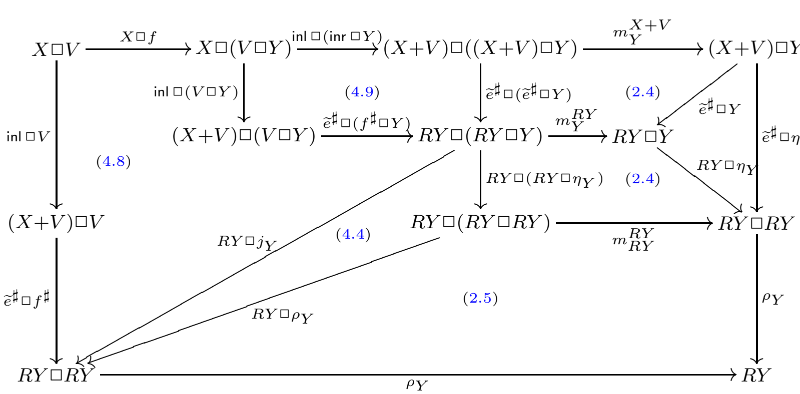
From (4.4) and (4.1) we have

$$j_Y \cdot (f^{\sharp} \,\square\, Y) \cdot f = i_Y^{-1} \cdot (f^{\sharp} \,\square\, Y) \cdot f = f^{\sharp}. \tag{4.8}$$

Furthermore, $\mathsf{inr} : V \longrightarrow X + V$ is a morphism of equations from f to $\widetilde{e}$ (see the lower square of (4.6)), thus,

$$\widetilde{e}^{\sharp} \cdot \mathsf{inr} = f^{\sharp}. \tag{4.9}$$

This proves that the diagram



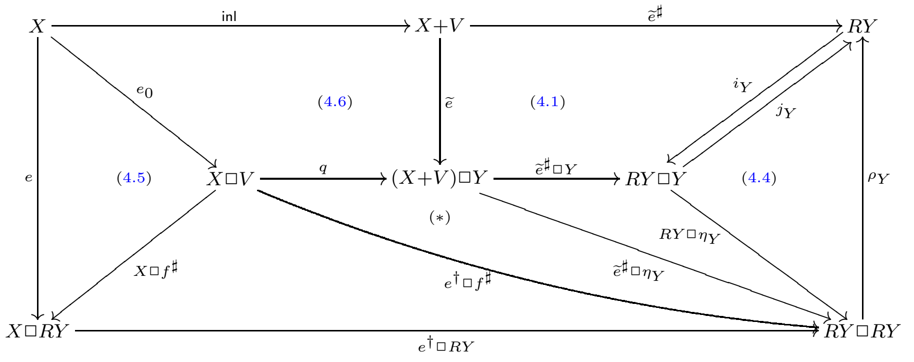
(4.10)

commutes. Denote by

$$q : X \,\square\, V \longrightarrow (X + V) \,\square\, Y$$

the upper horizontal morphism of (4.10).

The proof that (4.7) is a solution of e follows from the fact that the outward square of the diagram



commutes — in fact, all inner parts except (∗) commute and by (4.10) the triangle (∗) commutes when extended by ρ_Y, the right-hand vertical arrow.

It remains to prove that the solution $e^\dagger$ is unique. Given another solution

$$\begin{array}{ccc} X & \xrightarrow{\quad s \quad} & RY \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \rho_Y} \\ X \,\square\, RY & \xrightarrow[s\square RY]{} & RY \,\square\, RY \end{array} \tag{4.11}$$

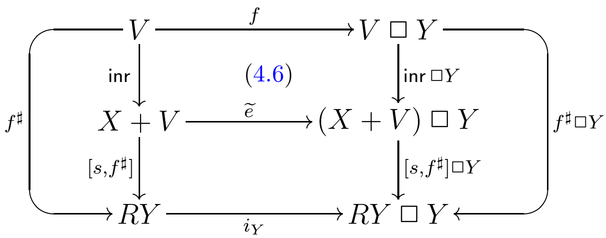
we prove that the square

$$\begin{array}{ccc} X + V & \xrightarrow{\quad \widetilde{e} \quad} & (X + V) \,\square\, Y \\ {\scriptstyle [s,f^\sharp]}\downarrow & & \downarrow{\scriptstyle [s,f^\sharp]\square Y} \\ RY & \xrightarrow[i_Y]{} & RY \,\square\, Y \end{array} \tag{4.12}$$

commutes. By Lemma 4.4, it follows that $\widetilde{e}^\sharp = [s, f^\sharp]$, thus

$$e^\dagger = \widetilde{e}^\sharp \cdot \mathsf{inl} = [s, f^\sharp] \cdot \mathsf{inl} = s\,.$$

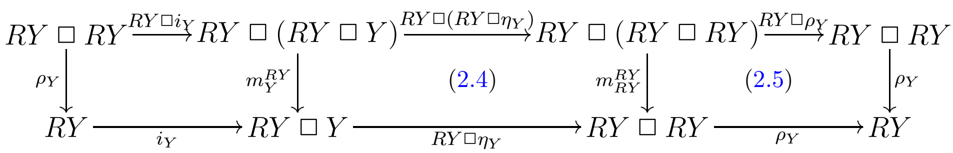
The right-hand component of (4.12) with domain V commutes:



For the left-hand component notice first that the equation

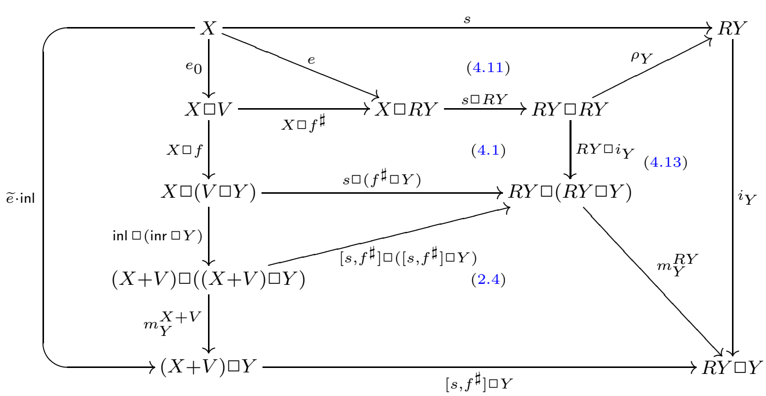
$$m_Y^{RY} \cdot (RY \,\Box\, i_Y) = i_Y \cdot \rho_Y \tag{4.13}$$

holds. In fact, in the diagram



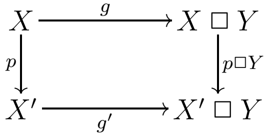
the horizontal morphisms are both identity morphisms (see (4.4)), which proves that the outward square commutes. Consequently, the left-hand square commutes, since $\rho_Y \cdot (RY \,\Box\, \eta_Y)$ is an isomorphism.

The left-hand component of (4.12) is, due to (4.6), the outward square of the commutative diagram



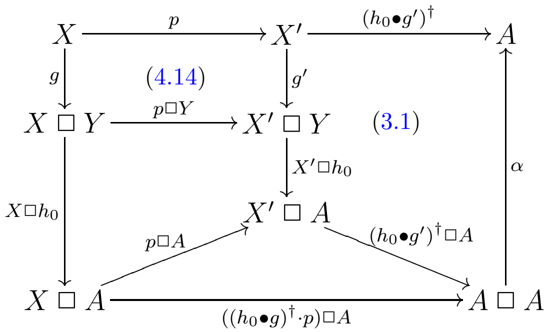
(2) RY is free, i.e., for every iterative algebra $\alpha : A \,\Box\, A \longrightarrow A$ and for every morphism $h_0 : Y \longrightarrow A$ there exist a unique homomorphism $h : RY \longrightarrow A$ with $h \cdot \eta_Y = h_0$.

(2a) Existence of h. For every equation morphism $g : X \longrightarrow X \,\Box\, Y$ in EQ_Y we form $h_0 \bullet g : X \longrightarrow X \,\Box\, A$, see Notation 3.9, and obtain the unique solution $(h_0 \bullet g)^\dagger : X \longrightarrow A$. This is a cocone of the diagram Eq_Y. In fact, given a morphism

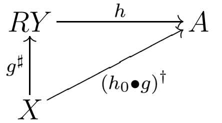


(4.14)

in EQ_Y, then $(h_0 \bullet g')^\dagger \cdot p$ is a solution of $h_0 \bullet g$:



Thus, $(h_0 \bullet g')^\dagger = (h_0 \bullet g)^\dagger \cdot p$ as desired. Consequently, we can define h by the commutativity of the triangles



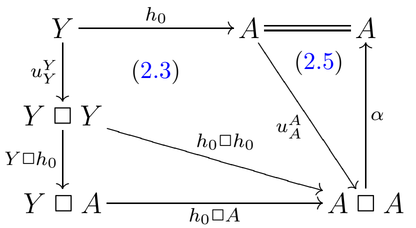
(4.15)

for all g in EQ_Y. We will show that this is the unique algebra homomorphism extending h_0.

We first prove that

$$h_0 = h \cdot \eta_Y = h \cdot (u_Y^Y)^\sharp.$$

In fact, the commutative diagram



shows that h_0 is a solution of $h \bullet u_Y^Y$ in A, thus, by (4.15) we have

$$h_0 = (h \bullet u_Y^Y)^\dagger = h \cdot (u_Y^Y)^\sharp.$$

Next we will prove that h is a homomorphism. Due to Lemma 3.10, it is sufficient to prove that h preserves solutions, explicitly, for every equation morphism $e : X \longrightarrow X \,\square\, RY$ we form

$$\overline{e} = h \bullet e \equiv X \xrightarrow{\;e\;} X \,\square\, RY \xrightarrow{X \square h} X \,\square\, A \tag{4.16}$$

and we prove that

$$\overline{e}^\dagger = h \cdot e^\dagger. \tag{4.17}$$

Recall from (4.6) the equation morphisms $f : V \longrightarrow V \,\square\, Y$ and $\widetilde{e} : X + V \longrightarrow (X+V) \,\square\, Y$, and put

$$\overline{f} = h_0 \bullet f \equiv V \xrightarrow{\;f\;} V \,\square\, Y \xrightarrow{V \square h_0} V \,\square\, A\,. \tag{4.18}$$

Let us prove that

$$[\overline{e}^\dagger, \overline{f}^\dagger] = (h_0 \bullet \widetilde{e})^\dagger : X + V \longrightarrow A. \tag{4.19}$$

Then (4.17) follows: (4.19) implies due to (4.7) and (4.15) applied to $g = \widetilde{e}$ the equation

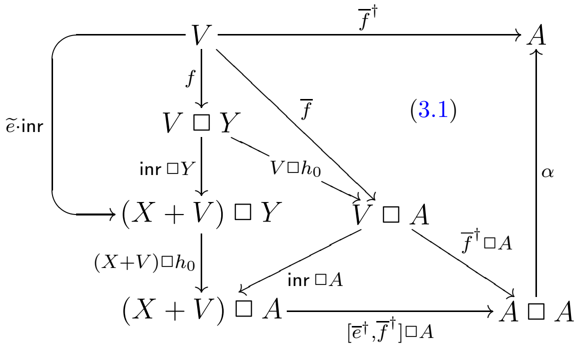
$$h \cdot e^\dagger = h \cdot \widetilde{e}^\sharp \cdot \mathsf{inl} = (h_0 \bullet \widetilde{e})^\dagger \cdot \mathsf{inl} = \overline{e}^\dagger.$$

Thus, the proof of (2a) will be complete by proving that the square

$$\begin{array}{ccc}
X + V & \xrightarrow{[\overline{e}^\dagger, \overline{f}^\dagger]} & A \\
\Big\downarrow{\scriptstyle \widetilde{e}} & & \\
(X+V) \,\square\, Y & & \Big\uparrow{\scriptstyle \alpha} \\
\Big\downarrow{\scriptstyle (X+V)\square h_0} & & \\
(X+V) \,\square\, A & \xrightarrow[{[\overline{e}^\dagger, \overline{f}^\dagger] \square A}]{} & A \,\square\, A
\end{array} \tag{4.20}$$

commutes.

For the right-hand component with domain V this follows from the commutative diagram



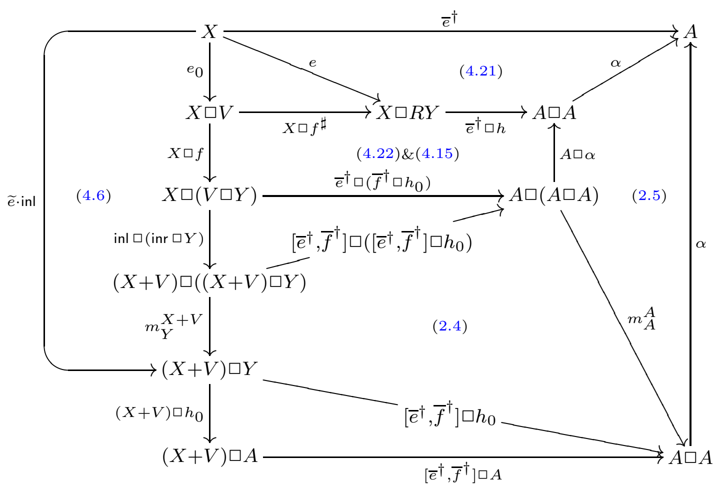
For the right-hand component with domain X, first observe that the definition of solution yields

$$\overline{e}^\dagger = \alpha\cdot(\overline{e}^\dagger \,\square\, A)\cdot(X \,\square\, h)\cdot e = \alpha\cdot(\overline{e}^\dagger \,\square\, h)\cdot e\,, \tag{4.21}$$

and

$$\overline{f}^\dagger = \alpha\cdot(\overline{f}^\dagger \,\square\, A)\cdot(X \,\square\, h_0)\cdot f = \alpha\cdot(\overline{f}^\dagger \,\square\, h_0)\cdot f. \tag{4.22}$$

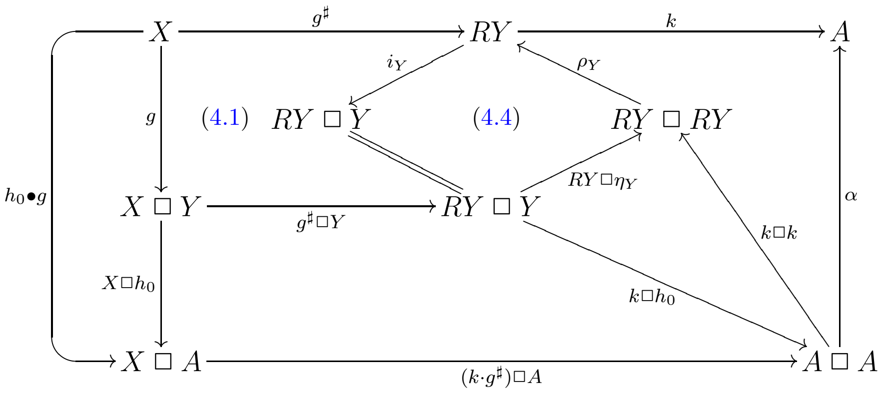
Therefore, we get a commutative diagram



which shows that the left-hand component of (4.20) commutes.

(2b) Uniqueness. Let $k : RY \longrightarrow A$ be a homomorphism (i.e., $k\cdot\rho_Y = \alpha\cdot(k\square k)$) with $h_0 = k\cdot\eta_Y$. Then for every equation morphism $g : X \longrightarrow X\square Y$

of EQ_Y the diagram



commutes, proving

$$k \cdot g^\sharp = (h_0 \bullet g)^\dagger = h \cdot g^\sharp\,,$$

see (4.15). Thus, $k = h$, since the morphisms $g^\sharp$'s form a colimit cocone. □

5 Conclusions and Future Work

In our paper we provided a coalgebraic construction of free iterative algebras of a general "environment" called base. This is analogous to (but, unfortunately, technically more involved than) the coalgebraic construction given in [5] for H-algebras, where H is an arbitrary finitary endofunctor. The next goal is to introduce iterative bases in the spirit of iterative theories of C. Elgot, and prove that the monad of free iterative algebras yields a free iterative base.

The technical result will serve us to describe algebraic trees of B. Courcelle [9] (i. e., trees resulting from a semantics of recursive program schemes) in a manner similar to the description of rational trees via R_Σ, the rational monad of Σ-algebras. For that, we need to introduce a suitable base on the category $\mathsf{FM}(\mathsf{Set})$ of all finitary monads on Set.

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