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On coalgebra based on classes

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Abstract

The category Class of classes and functions is proved to have a number of properties suitable for algebra and coalgebra: every endofunctor is set-based, it has an initial algebra and a terminal coalgebra, the categories of algebras and coalgebras are complete and cocomplete, and every endofunctor generates a free completely iterative monad. A description of a terminal coalgebra for the power-set functor is provided.

Keywords: Category of classes; Set-based endofunctor; Terminal coalgebra; Iterative monad

1. Introduction: do not be afraid of classes

This paper does not, despite its title, concern the foundations. It concerns coalgebra in a surprisingly coalgebra-friendly category

Class

of classes and functions—and the main message is that one almost does not need foundations for that, or just the reasonable minimum of foundations. What is the definition of reasonable minimum? In category theory one always works with "large" and "small"—and this is all one needs. Thus, "large" refers to, say, set theory which is a model of ZFC (Zermelo–Fraenkel axioms including the axiom of choice). And "small"

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means that a universe (of small sets) is once for all chosen within the given universe (of all sets). This is all one needs: the chosen universe of all small sets is itself a large set, and we denote by

$$\aleph_\infty$$

its cardinality. This means that the category Set of all small sets is, obviously, equivalent to the category $\mathsf{SET}_{<\aleph_\infty}$ of all large sets of cardinality smaller than $\aleph_\infty$. And if one forms, analogously, the category $\mathsf{SET}_{\leqslant\aleph_\infty}$ of all large sets of cardinality less or equal to $\aleph_\infty$, then Class is equivalent to it. Thus, one can think of the difference between Set and Class as of the difference between being smaller than, or smaller or equal to, $\aleph_\infty$. The cardinal $\aleph_\infty$ is strongly inaccessible (i.e., for every cardinal $\alpha<\aleph_\infty$ we have $2^\alpha<\aleph_\infty$), and conversely, for every choice of a strongly inaccessible uncountable cardinal $\aleph_\infty$ there is a universe of small sets with $\mathsf{Set}\simeq\mathsf{SET}_{<\aleph_\infty}$.

In what follows we work with the category of all sets of cardinality less than $\aleph_\infty$ as Set and with the category of all sets of cardinality at most $\aleph_\infty$ as the category Class.

The present paper generalizes results published as a preprint in [6].

2. All endofunctors are set-based

The concept of a set-based endofunctor of Class has been introduced by Peter Aczel and Nax Mendler [3] in order to prove their "Final Coalgebra Theorem", see the next section. An endofunctor $F:\mathsf{Class}\to\mathsf{Class}$ is called *set-based* provided that for every class X and every $x\in FX$ there exists a small subset $m:M\hookrightarrow X$ such that x lies in the image of Fm. It turns out that every endofunctor has this property. The following proof, based on ideas of Václav Koubek [15], uses a classical set-theoretical result of Waclaw Sierpiński and Alfred Tarski, see [22,23].

Theorem 2.1. *For every infinite cardinal λ there exists, on a set X of cardinality λ, an almost disjoint collection of subsets $X_i\subseteq X$, $i\in I$, i.e., a collection satisfying* $\mathrm{card}(X_i\cap X_j)<\lambda$ *for all $i\neq j$ in I, such that I has more than λ elements.*

Theorem 2.2. *Every endofunctor of* Class *is set-based.*

Remark. We prove a more general result: suppose that λ is an infinite cardinal such that

$$\lambda^\kappa=\lambda \quad \text{for all cardinals } 0<\kappa<\lambda. \tag{1}$$

(This is true when λ is strongly inaccessible. Under the general continuum hypothesis (1) holds for all infinite regular cardinals.) We will prove that for the category $\mathsf{SET}_{\leqslant\lambda}$ of all sets of cardinality at most λ we have

every endofunctor of $\mathsf{SET}_{\leqslant\lambda}$ is λ-accessible.

Recall from [8] that for an endofunctor F of $\mathsf{SET}_{\leqslant\lambda}$ the following conditions are equivalent:

(1) F is λ-accessible, i.e., preserves λ-filtered colimits.

(2) For every set X in $\mathsf{SET}_{\leqslant\lambda}$ and every element x of FX there exists a subset $m: M \hookrightarrow X$ of cardinality less than λ such that x lies in the image of Fm.
(3) F is a quotient of a λ-ary polynomial functor.
(Given a signature Σ, i.e., a set of operation symbols σ with prescribed arities $\mathrm{ar}(\sigma)$, finite or infinite, then the corresponding functor

$$H_\Sigma : X \mapsto \coprod_{\sigma\in\Sigma} X^{\mathrm{ar}(\sigma)}$$

is called *polynomial*. It is *λ-ary* if $\mathrm{ar}(\sigma) < \lambda$ for all $\sigma \in \Sigma$.)
In particular, set-based and $\aleph_\infty$-accessible are equivalent.

Proof of theorem and remark. Let λ be an infinite regular cardinal. Given $F: \mathsf{SET}_{\leqslant\lambda} \to \mathsf{SET}_{\leqslant\lambda}$ and a set X in $\mathsf{SET}_{\leqslant\lambda}$, then for every element $x \in FX$ we are to find a subset $m: M \hookrightarrow X$ with $\operatorname{card} M < \lambda$ and $x \in Fm[FM]$. If $\operatorname{card} X < \lambda$ there is nothing to prove, so assume $\operatorname{card} X = \lambda$. We can further assume, without loss of generality, that F preserves finite intersections. In fact, by a theorem of Věra Trnková, see, e.g., Theorem III.4.5 of [9], there exists a functor F' preserving finite intersections and such that the restrictions of F and F' to the full subcategory $\mathsf{SET}_{\leqslant\lambda}$ of all non-empty sets are naturally isomorphic. Since F is λ-accessible iff F' is, we can assume $F' = F$. By Theorem 2.1 there exists an almost disjoint collection of subsets $v_i: X_i \hookrightarrow X$, $i \in I$, with $\operatorname{card} I > \lambda$. Since the collection of all subsets of X of cardinality less than λ has cardinality λ (due to (1) above), we can suppose without loss of generality that each X_i has cardinality λ—in fact, by discarding all X_i's of cardinalities less than λ we still obtain an almost disjoint collection of more than λ members. Thus, for each $i \in I$ there exists an isomorphism

$$w_i : X \to X_i$$

and we put

$$y_i = F(v_i w_i)(x) \in FX.$$

Since F is an endofunctor of $\mathsf{SET}_{\leqslant\lambda}$, the set FX has cardinality smaller than that of I, consequently, the elements y_i are not pairwise distinct. Choose $i \neq j$ in I with

$$y_i = y_j$$

and form a pullback (i.e., an intersection of v_i and v_j):

$$\begin{tikzcd} & M \arrow[dl, "u_i"'] \arrow[dr, "u_j"] & \\ X_i \arrow[dr, "v_i"'] & & X_j \arrow[dl, "v_j"] \\ & X & \end{tikzcd}$$

Since F preserves this pullback and $Fv_i(Fw_i(x)) = Fv_j(Fw_j(x))$, there exists

$$y \in FM \quad \text{with } Fu_i(y) = Fw_i(x).$$

For the subobject $m = w_i^{-1}u_i : M \hookrightarrow X$ this implies

$$Fm(y) = x$$

and this concludes the proof. $\square$

Remark 2.3. Denote by $J : \mathsf{Set} \to \mathsf{Class}$ the inclusion functor. That a functor $F : \mathsf{Class} \to \mathsf{Class}$ is set-based can be equivalently restated as being naturally isomorphic to a left Kan extension $\mathrm{Lan}_J K$ for some functor $K : \mathsf{Set} \to \mathsf{Class}$.

Thus, Theorem 2.2 says that restriction along J, i.e., the functor

$$\_ \cdot J : [\mathsf{Class}, \mathsf{Class}] \to [\mathsf{Set}, \mathsf{Class}], \quad F \mapsto F \cdot J$$

is an equivalence of categories.

In particular, every endofunctor of Set has an essentially unique extension to an endofunctor of Class.

Notation 2.4. For an endofunctor $F : \mathsf{Set} \to \mathsf{Set}$ we denote by

$$F^\infty : \mathsf{Class} \to \mathsf{Class}$$

the extension of F, more precisely, the essentially unique functor such that the following square

$$\begin{tikzcd} \mathsf{Set} \arrow[r, "F"] \arrow[d, "J"'] & \mathsf{Set} \arrow[d, "J"] \\ \mathsf{Class} \arrow[r, "F^\infty"'] & \mathsf{Class} \end{tikzcd}$$

commutes.

3. All endofunctors are varietors and covarietors

In the present section we show that endofunctors H of Class have a surprisingly simple structure, and they admit free H-algebras (i.e., are varietors) and cofree H-coalgebras (i.e., are covarietors)—moreover, these algebras and coalgebras can be explicitly described.

3.1. Polynomial endofunctors

Classical universal algebra deals with Σ-algebras in the category Set, where Σ is a (small) signature, i.e., a small set of operation symbols σ with prescribed arities $\mathrm{ar}(\sigma)$ which are (in general, infinite) small cardinal numbers. Thus, if

$$\mathsf{Card}$$

denotes the class of all small cardinal numbers, then a small signature is a small set Σ equipped with a function $\mathrm{ar} : \Sigma \to \mathsf{Card}$. And Σ-algebras are just algebras over the polynomial endofunctor H_Σ of Set given on objects, X, by

$$H_\Sigma X = \coprod_{\sigma\in\Sigma} X^{\mathrm{ar}(\sigma)}.$$

Quite analogously, in Class we work with (large) *signatures* as classes Σ equipped with a function $\mathrm{ar}: \Sigma \to \mathsf{Card}$ (thus, largeness refers to the possibility of having a proper class of operations, arities are small). Here, again, we obtain a *polynomial endofunctor* H_Σ defined on classes X by

$$H_\Sigma X = \coprod_{\sigma\in\Sigma} X^{\mathrm{ar}(\sigma)}$$

and analogously on morphisms.

Proposition 3.2. *Every endofunctor H of* Class *is a quotient of a polynomial functor. That is, there exists a natural epitransformation $\varepsilon: H_\Sigma \to H$ for some signature Σ.*

Proof. Let Σ be the signature which, for every small cardinal n has as n-ary symbols precisely the elements of Hn. Then the function

$$\varepsilon_X : \coprod_{n\in\mathsf{Card}} \coprod_{\sigma\in Hn} X^n \to HX$$

which to every $f: n \to X$ in the σth summand X^n assigns $Hf(\sigma)$ in HX is obviously a component of a natural transformation. And ε is pointwise surjective: for a small set M put $n = \mathrm{card}\, M$ and choose an isomorphism $f: n \to M$. Then every element of HM has the form $Hf(\sigma)$ for a unique $\sigma \in \Sigma$. Thus, ε_M is surjective. For a general class X use the fact that H is set-based (Theorem 2.2), thus for an element $x \in HX$ there exists a small subset $m: M \hookrightarrow X$ and $y \in HM$ such that $x = Hm(y)$. Since ε_M is surjective, there exists $z \in H_\Sigma M$ such that $\varepsilon_M(z) = y$. Define $t = H_\Sigma m(z) \in H_\Sigma X$. Due to naturality of ε it follows that $\varepsilon_X(t) = x$. $\square$

Example 3.3. The power-set functor $\mathscr{P}: \mathsf{Set} \to \mathsf{Set}$ extends to $\mathscr{P}^\infty: \mathsf{Class} \to \mathsf{Class}$, see Notation 2.4, which assigns to every class X the class of all small subsets of X. We can represent $\mathscr{P}^\infty$ as a quotient of H_{Σ^0} where Σ^0 is the signature which possesses, for every cardinal $n \in \mathsf{Card}$, a unique operation σ_n: here

$$\varepsilon_X : H_{\Sigma^0} = \coprod_{n\in\mathsf{Card}} X^n \to \mathscr{P}^\infty X$$

assigns to every $f: n \to X$ the image $f[n] \subseteq X$.

3.4. Algebras

Recall that for an endofunctor H of Class an *H-algebra* is a class A together with a function $\alpha: HA \to A$. Given another algebra $\beta: HB \to B$, a *homomorphism* from A

to B is a function $f : A \to B$ such that the following square

$$\begin{tikzcd} HA \arrow[r, "\alpha"] \arrow[d, "Hf"'] & A \arrow[d, "f"] \\ HB \arrow[r, "\beta"'] & B \end{tikzcd}$$

commutes. The category of all H-algebras and homomorphisms is denoted by

$$\mathsf{Alg}\, H.$$

Example 3.5. (1) $\mathsf{Alg}\, H_\Sigma$ is the category of Σ-algebras (i.e., classes A endowed, for every n-ary symbol σ, with an n-ary operation on A) which, except for the "size" of underlying sets, is just the classical category of universal algebra.

(2) $\mathsf{Alg}\, \mathscr{P}^\infty$ has as objects classes A together with a function $\alpha : \mathscr{P}^\infty A \to A$. This can be equivalently considered as a variety of Σ^0-algebras as follows: let E be the class of all equations given by the equation schemes

$$\sigma_n(x_i)_{i<n} \approx \sigma_m(y_j)_{j<m},$$

where n and m are small cardinals and the variables x_i and y_j are such that the sets $\{x_i \mid i<n\}$ and $\{y_j \mid j<m\}$ are equal. Then $\mathsf{Alg}\, \mathscr{P}^\infty$ is isomorphic to the variety of all Σ^0-algebras satisfying the above equations.

Remark 3.6. Recall from [9] that a *basic equation* is an equation between two flat terms, i.e., terms of the form $\sigma(x_i)_{i<n}$ where σ is an n-ary operation symbol and x_i are (not necessarily distinct) variables. The example $\mathsf{Alg}\, \mathscr{P}^\infty$ above is quite typical: every category $\mathsf{Alg}\, H$ is a variety presented by basic equations (and vice versa)—this has been shown for Set in [9], let us recall it and extend to the present ambient:

Given a functor H represented as in Proposition 3.2, consider all the basic equations

$$\sigma(x_i)_{i<n} \approx \rho(y_j)_{j<m},$$

where $\sigma, \rho \in \Sigma$ and for the set $V = \{x_i \mid i<n\} \cup \{y_j \mid j<m\}$ of variables we have $\varepsilon_V(\sigma(x_i)) = \varepsilon_V(\rho(y_j))$, more precisely:

ε_V merges the n-tuple $(x_i)_{i<n}$ in the σ-summand of $H_\Sigma V$ with the m-tuple $(y_j)_{j<m}$ in the ρ-summand of $H_\Sigma V$.

Then $\mathsf{Alg}\, H$ is equivalent to the variety of all Σ-algebras presented by the above equations.

Conversely, given a class E of basic equations in signature Σ, there is a quotient H of H_Σ such that the variety of Σ-algebras presented by E is isomorphic to $\mathsf{Alg}\, H$.

Corollary 3.7. *For every endofunctor H of* Class *an initial H-algebra exists.*

In fact, we can describe an initial H-algebra, I, in two substantially different ways:

(1) I is a quotient of the initial Σ-algebra modulo the congruence generated by the given basic equations.

Recall here the description of initial Σ-algebras well-known in universal algebra: it is the algebra of all well-founded Σ-trees, i.e., Σ-trees in which every branch is finite. This remains unchanged in case of large signatures, the only difference is that all Σ-trees do not form a small set (but each Σ-tree is small, by definition). That is, by a *Σ-tree* we mean an ordered, labelled tree on a small set of nodes, where labels are operation symbols, and every node labelled by an n-ary symbol has precisely n children. The algebra

$$I_\Sigma$$

of all *well-founded* Σ-trees has operations given by tree-tupling. This is an initial algebra in $\mathsf{Alg}\, H_\Sigma$.

Given a quotient $\varepsilon : H_\Sigma \to H$, form the smallest congruence $\sim$ on I_Σ which is generated by all the basic equations corresponding to ε. Then $I_\Sigma/\!\sim$ is an initial algebra of $\mathsf{Alg}\, H$.

(2) I is a colimit of the transfinite chain $W : \mathsf{Ord} \to \mathsf{Class}$ (where Ord is the chain of all small ordinals) given by iterating H on the initial object $\emptyset$ of Class:

$$W_i = H^i(\emptyset)$$

and

$$I = \operatorname*{colim}_{i \in \mathsf{Ord}} W_i.$$

More precisely, there is a unique chain W for which we have

First step: $W_0 = \emptyset$, $W_1 = H(\emptyset)$ and $W_{0,1} : \emptyset \to H(\emptyset)$ unique.

Isolated step: $W_{i+1} = H(W_i)$ and $W_{i+1,j+1} = H(W_{i,j})$.

Limit step: $W_j = \mathrm{colim}_{i<j}\, W_i$ with colimit cocone $(W_{i,j})_{i<j}$.

A colimit of this chain exists (see Observation 3.9 below) and is preserved by H, see 2.3, therefore, if $I = \mathrm{colim}\, W_i$ then

$$HI \cong \operatorname*{colim}_{i \in \mathsf{Ord}} H(W_i) = \operatorname*{colim}_{i \in \mathsf{Ord}} W_{i+1} \cong I$$

and the canonical isomorphism $HI \to I$ defines an initial H-algebra, see [4].

Example 3.8. An initial $\mathscr{P}^\infty$-algebra. Whereas the power-set functor $\mathscr{P}$ has no initial algebra, $\mathscr{P}^\infty$ does (since every endofunctor of Class does). The above chain W_i coincides with the chain of sets defined by the cumulative hierarchy:

$$W_0 = \emptyset,$$

$$W_{i+1} = \exp W_i$$

and

$$W_j = \bigcup_{i<j} W_i \quad \text{for limit ordinals } j.$$

Consequently, we can describe an initial $\mathscr{P}^\infty$-algebra as

$$I = \mathsf{Set} = \text{the class of all small sets}$$

with the structure map $\mathscr{P}^{\infty} I \to I$ given by the union. This has been first observed by Jan Rutten and Danielle Turi [21].

The other option of describing I is also interesting: let us first form the initial Σ^0-algebra. Since operations of any arity are unique, we can first forget the labelling, thus

$$I_{\Sigma^0} = \text{the algebra of all well-founded trees.}$$

To every well-founded tree t let us assign the corresponding non-ordered tree (obtained by forgetting the linear ordering of children of any node) and recall that a non-ordered tree is called *extensional* provided that every pair of distinct siblings defines a pair of non-isomorphic subtrees. For every $t \in I_{\Sigma^0}$ denote by $[t]$ the *extensional quotient* of the (non-ordered version of) t; that is, the extensional tree obtained from t by iteratively merging any pair of siblings defining isomorphic subtrees. Then an initial $\mathscr{P}^{\infty}$-algebra can be described as

$$I_{\Sigma^0}/\sim \quad \text{where } t \sim t' \text{ iff } [t] = [t'].$$

This follows from the above result of [21] due to the axiom of extensionality for Set.

Observation 3.9. The category Class has all small limits and all class-indexed colimits. That is, given a functor $D : \mathscr{D} \to \mathsf{Class}$ then
(a) if $\mathscr{D}$ is small then $\lim D$ exists
and
(b) if $\mathscr{D}$ has only a class of morphisms then $\operatorname{colim} D$ exists.
In fact, the strongly inaccessible cardinal $\aleph_\infty$ which is the cardinality of all classes, satisfies

$$(\aleph_\infty)^n = \aleph_\infty \quad \text{for all } n \in \mathsf{Card}.$$

Consequently, a cartesian product of a small collection of classes is a class—thus, Class has small products. And since small limits are always subobjects of small products, it follows that Class has all small limits. Analogously, since

$$\aleph_\infty \cdot \aleph_\infty = \aleph_\infty$$

it follows that Class has all class-indexed coproducts: a disjoint union of a class of classes is a class. Since class-indexed colimits are always quotients of class-indexed coproducts, it follows that Class has class-indexed colimits.

Example 3.10. Class-indexed limits do not exist, in general. For example, if I is a proper class then 2^I (a cartesian product of I copies of the two-element set $2 = \{0, 1\}$) is not a class, having cardinality $2^{\aleph_\infty} > \aleph_\infty$. It follows that a product of I copies of 2 does not exist in the category Class: it is trivial that if $(\pi_i : L \to 2)_{i \in I}$ were such a product, then for every subclass $J \subseteq I$ we have the unique $u_J : 1 \to L$ with $\pi_i \cdot u_J$ given by 0 for $i \in J$ and 1 for $j \in I \backslash J$. Then the u_J's are pairwise distinct, thus, $\operatorname{card} L > \aleph_\infty$, a contradiction.

Recall from [9] that an endofunctor H of Class is called a *varietor* provided that every object of Class generates a free H-algebra. Equivalently, if the forgetful functor $\mathsf{Alg}\,H \to \mathsf{Class}$ has a left adjoint.

And an initial algebra of the functor

$$H(\_) + A$$

is precisely a free H-algebra on A. The former exists by Corollary 3.7, thus we obtain

Corollary 3.11. *Every endofunctor of* Class *is a varietor*.

Remark 3.12. The category $\mathsf{Alg}\,H$ has all small limits and all class-indexed colimits for every endofunctor H of Class: the limits are (obviously) created by the forgetful functor. The existence of coequalizers follows from the fact that (Epi, Mono) is a factorization system in Class and every endofunctor H preserves epimorphisms (since they split), see [16, Corollary 8.6]. Since H is a varietor, $\mathsf{Alg}\,H$ is monadic, and thus has all colimits which Class has, see [17].

3.13. Coalgebras

Coalgebras of an endofunctor H of Class are classes A together with a function $\alpha : A \to HA$. Given another coalgebra $\beta : B \to HB$, a *homomorphism* from A to B is a function $f : A \to B$ such that the following square

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & HA \arrow[d, "Hf"] \\ B \arrow[r, "\beta"'] & HB \end{tikzcd}$$

commutes. The category of all H-coalgebras and homomorphisms is denoted by

$$\mathsf{Coalg}\,H.$$

Example 3.14. (1) Coalgebras over polynomial functors describe deterministic dynamic systems, see [20]. For example, if Σ consists of a binary symbol and a nullary one, then a coalgebra

$$A \to A \times A + 1$$

describes a system with the state-set A and two deterministic inputs (0 and 1, say) with exceptions: to every state a the pair (a_0, a_1) of states is assigned, representing the reaction of a to 0 and 1, respectively—unless a is an exception, mapped to the unique element of 1.

(2) $\mathscr{P}^\infty$-coalgebras can be identified with large small-branching graphs, i.e., classes A endowed with a binary relation (represented by the function $A \to \mathscr{P}^\infty A$ assigning to every node the small set of its descendants).

Theorem 3.15. *Every endofunctor H of* Class *has a terminal coalgebra.*

Several proofs of this theorem are known and we provide another one in 3.18 below. The first one is due to Peter Aczel and Nax Mendler [3]. Their Final Coalgebra Theorem states that every set-based endofunctor has a terminal coalgebra—but we know from Section 2 that all endofunctors are set-based. Another proof follows, as Michael Barr has noticed in [10], from the theory of accessible categories in the monograph [18]. A third proof can be derived from the result of James Worell [24] that every λ-accessible endofunctor H of Set has a terminal coalgebra obtained by 2λ steps of the dual chain of 3.7(2). That is, define a chain V of (in general, large) sets as follows:

First step: $V_0 = 1$, $V_1 = H(1)$ and $V_{0,1} : H(1) \to 1$ unique.

Isolated step: $V_{i+1} = H(V_i)$ and $V_{i+1,j+1} = H(V_{i,j})$.

Limit step: $V_j = \lim_{i<j} V_i$ with limit cone $(V_{i,j})_{i<j}$.

Then $V_{2\lambda}$ is a terminal coalgebra of H.

By applying this to $\lambda = \aleph_\infty$ we “almost” obtain a construction of terminal coalgebras of endofunctors of Class (first, one has to extend the endofunctor to the category of all large sets but this brings no difficulty). There is a catch here: although the resulting limit $V_{2\aleph_\infty}$ is indeed a class (which follows from Worell’s result), the intermediate step $V_{\aleph_\infty}$ can “slip” outside the scope of classes.

Example 3.16. A terminal coalgebra T of $\mathscr{P}^\infty$ has, in the non-well-founded set theory of Peter Aczel, see [1] or [11], a beautiful description: T is the class of all non-well-founded sets. This has been proved in [21]. However, we work here in the well-founded set theory ZFC. An explicit (but certainly not very beautiful) description of T is presented in Section 5 below.

Here we just observe that the chain V above “jumps out” of the realm of classes: if we put $V_0 = 1$, $V_{i+1} = \exp V_i$ and $V_j = \lim_{i<j} V_i$ for all ordinals in Ord, then we cannot form

$$V_{\aleph_\infty} = \lim_{i \in \mathsf{Ord}} V_i$$

within Class. The reason is that for all $i \leqslant \aleph_\infty$ we can easily prove by transfinite induction that

$$\operatorname{card} V_i \geqslant 2^i.$$

Thus, $V_{\aleph_\infty}$ is not a class.

Remark 3.17. (a) In spite of the three proofs mentioned above, we present a new proof, based on ideas of Peter Gumm and Tobias Schröder [12] since it is the shortest and clearest one, and it gives a sort of concrete description: a terminal H-coalgebra is obtained from a terminal H_Σ-coalgebra via a suitable congruence. Recall that for every H-coalgebra $\alpha: A \to HA$ a *congruence* is an equivalence relation $\sim$ on A (or, equivalently, a quotient in Class represented by the epimorphism $e: A \to A/{\sim}$ with $e(x) = [x]$) for which a (necessarily unique) structure map $\bar{\alpha}: A/{\sim} \to H(A/{\sim})$ exists turning e into a homomorphism:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "e"'] & HA \arrow[d, "He"] \\ A/{\sim} \arrow[r, "\bar{\alpha}"'] & H(A/{\sim}) \end{tikzcd}$$

(b) Recall further the concept of a (strong) *bisimulation* between coalgebras $\alpha: A \to HA$ and $\beta: B \to HB$: it is a relation $R \subseteq A \times B$ on which a structure of a coalgebra $R \to HR$ exists such that both projections $R \to A$ and $R \to B$ become coalgebra homomorphisms, see [20]. Every bisimulation $R \subseteq A \times A$ which is an equivalence relation is a congruence on A. The converse holds whenever H weakly preserves pullbacks (see 5.7 and 5.8 in [20]). In particular, the polynomial functors and $\mathscr{P}$ weakly preserve pullbacks.

(c) A nice description of terminal H_Σ-coalgebras is known, which works for large signatures as well as for small ones: let

$$T_\Sigma$$

be the class of all (small) Σ-trees. (In comparison to I_Σ, we just drop the well-foundedness.) This is, like I_Σ, a Σ-algebra w.r.t. tree tupling—and since in both cases tree-tupling is actually an isomorphism we can invert it to the structure map $\tau_\Sigma: T_\Sigma \to H_\Sigma T_\Sigma$ of a coalgebra. And that coalgebra is terminal.

Proposition 3.18. *Every endofunctor H of* Class, *represented as a quotient $\varepsilon: H_\Sigma \to H$ (as in Proposition 3.2) has a terminal coalgebra, viz, the quotient of the H-coalgebra*

$$T_\Sigma \xrightarrow{\tau_\Sigma} H_\Sigma T_\Sigma \xrightarrow{\varepsilon_{T_\Sigma}} HT_\Sigma$$

modulo the largest congruence.

Proof. (1) The largest congruence exists. In fact, the pushout of all congruences of T_Σ is easily seen (due to the universal property of pushouts) to be a congruence.

(2) Given the largest congruence $e: T_\Sigma \to T/{\sim}$, the corresponding coalgebra $\overline{\tau_\Sigma}: T_\Sigma/{\sim} \to H(T_\Sigma/{\sim})$ is terminal. In fact, given a coalgebra $\beta: B \to HB$ the uniqueness of a homomorphism from B to $T_\Sigma/{\sim}$ follows from the observation that given two homomorphisms $f_1, f_2: B \to T_\Sigma/{\sim}$, then a coequalizer $c: T_\Sigma/{\sim} \to T_\Sigma/{\approx}$ of f_1, f_2 in Class yields a congruence $ce: T_\Sigma \to T_\Sigma/{\approx}$, thus, $\approx$ and $\sim$ coincide, which means $f_1 = f_2$. The existence of a homomorphism is proved by choosing a splitting of the epimorphism ε_B:

$$u: HB \to H_\Sigma B \quad \text{with } \varepsilon_B u = id.$$

The unique homomorphism, f, of the H_Σ-coalgebra $B \xrightarrow{\beta} HB \xrightarrow{u} H_\Sigma B$ yields a homomorphism, ef, of H-coalgebras:

$$\begin{tikzcd}
B \arrow[r, "\beta"] \arrow[dd, "f"'] & HB \arrow[rr, equal] \arrow[dr, "u"'] & & HB \arrow[dd, "Hf"] \\
& & H_\Sigma B \arrow[ur, "\varepsilon_B"'] \arrow[d, "H_\Sigma f"] & \\
T_\Sigma \arrow[rr, "\tau_\Sigma"'] \arrow[d, "e"'] & & H_\Sigma T_\Sigma \arrow[r, "\varepsilon_{T_\Sigma}"'] & HT_\Sigma \arrow[d, "He"] \\
T_\Sigma/\sim \arrow[rrr, "\overline{\tau_\Sigma}"'] & & & H(T_\Sigma/\sim)
\end{tikzcd}$$

□

Corollary 3.19. *Every endofunctor H of* Class *is a covarietor, i.e., a cofree H-coalgebra on every class exists.*

In fact, a cofree H-coalgebra on A is just a terminal coalgebra of $H(\_) \times A$.

Remark 3.20. The category Coalg H has all small limits and all class-indexed colimits for every endofunctor H of Class: the colimits are (obviously) created by the forgetful functor. The existence of limits follows, if H preserves monomorphisms, from the dualization of Theorem 16.5 of [14]. For general H use the result of Věra Trnková cited in the proof of Theorem 2.2 above.

4. All endofunctors generate completely iterative monads

In this section, we assume that the reader is acquainted with the concept of an iterative theory (or iterative monad) of Calvin Elgot, and the coalgebraic treatment of completely iterative monads in [19] or [2]. In [2] we worked with endofunctors H such that a terminal coalgebra, TX, of the endofunctor $H(\_) + X$ exists for every X. Such functors were called *iteratable*. In the category of classes this concept need not be used.

Corollary 4.1. *Every endofunctor of* Class *is iteratable.*

This follows from Proposition 3.18 applied to $H(\_) + X$.

Recall from [19] or [2] that the coalgebra structure of TX, $TX \xrightarrow{\cong} HTX + X$, turns TX into a coproduct of HTX and X, where the coproduct inclusions are denoted by

$$\tau_X : HTX \to TX \quad (TX \text{ is an } H\text{-algebra})$$

and

$$\eta_X : X \to TX \quad (X \text{ is contained in } TX)$$

It turns out that this is part of a monad $\mathbb{T}=(T,\eta,\mu)$. This monad is *completely iterative*, i.e., for every "equation" morphism $e:X\to T(X+Y)$ which is guarded, i.e., it factorizes through the coproduct injection

$$HT(X+Y)+Y\hookrightarrow HT(X+Y)+X+Y=T(X+Y),$$

there exists a unique *solution*. That is, there exist a unique morphism $e^\dagger$ for which the following square

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & TY \\ T(X+Y) \arrow[r, "T[e^\dagger,\eta_Y]"'] & TTY \arrow[u, "\mu_Y"'] \end{tikzcd}$$

commutes. And in [2] it has been proved that $\mathbb{T}$ can be characterized as a free completely iterative monad on H.

Example 4.2. Let Σ be a (possibly large, infinitary) signature, i.e., a class of operation symbols together with a function ar(_) assigning a small cardinal to every symbol σ. Put $\Sigma_n=\{\sigma\mid \mathrm{ar}(\sigma)=n\}$. The polynomial functor $H_\Sigma:X\mapsto\coprod_{\sigma\in\Sigma}X^{\mathrm{ar}(\sigma)}$ generates the following completely iterative monad $\mathbb{T}_\Sigma=(T_\Sigma,\eta,\mu)$: $T_\Sigma Y$ is the Σ-algebra of all Σ-trees on Y, i.e., small trees with leaves labelled in Σ_0+Y and nodes with $n>0$ children labelled in Σ_n. The natural transformation η_Y is the singleton-tree embedding, and μ_Y is given by the usual tree substitution.

The fact that $\mathbb{T}_\Sigma$ is completely iterative just restates the well-known property of tree algebras: all iterative systems of equations that are guarded (i.e., do not contain equations $x\approx x'$ where x and x' are variables) have unique solutions.

Corollary 4.3. *All free completely iterative monads on* Class *are quotient monads of the tree-monads* $\mathbb{T}_\Sigma$ (*for all signatures* Σ).

In fact, every endofunctor H of Class is a quotient of H_Σ for a suitable signature Σ (denote by Σ_n the class Hn). It follows that a free completely iterative monad on H is a quotient of $\mathbb{T}_\Sigma$, see [5].

5. Terminal coalgebra of the power-set functor

We now concentrate on non-labelled transition systems, i.e., to coalgebras of the power-set functor $\mathscr{P}:\mathsf{Set}\to\mathsf{Set}$. It has been noticed by several authors, e.g., [3,10,13,21,24] that $\mathscr{P}^\infty$ has a very natural *weakly terminal* coalgebra B (i.e., such that every $\mathscr{P}$-coalgebra A has at least one homomorphism from A to B): the coalgebra of all (finite and infinite) small extensional trees (see 3.8). Throughout this section trees are always taken up to (graph) isomorphism. Thus, briefly, a tree is extensional if and only if distinct siblings always define distinct subtrees.

The weakly terminal coalgebra B has as elements all small extensional trees, and the coalgebra structure

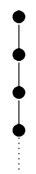
$$\beta : B \to \mathscr{P}^{\infty} B$$

is the inverse of tree tupling, i.e., β assigns to every tree t the set of all children of t.

We know from Theorem 3.15 that a terminal coalgebra for $\mathscr{P}^{\infty}$ exists. Since B is weakly terminal, it follows that a terminal coalgebra is a quotient of B modulo the largest congruence $\sim$, which is sometimes also called *bisimilarity equivalence* on B. Thus we shall refer to elements of B related by $\sim$ as *bisimilar*. We are going to describe the congruence $\sim$. We start by describing one interesting class.

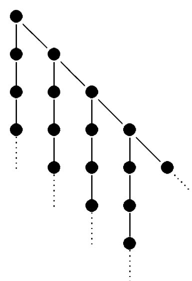
Example 5.1. An extensional tree t is bisimilar to the following tree:

Ω



if and only if all paths in t are infinite. Thus, for example, the following tree

Ω'



is bisimilar to Ω. This illustrates that the largest congruence is non-trivial. We prove $\Omega \sim \Omega'$ below.

Remark 5.2. For the finite-power-set functor $\mathscr{P}_{\mathrm{f}}$ a nice description of a terminal coalgebra has been presented by Michael Barr [10]: let B_{f} denote the coalgebra of all finitely branching extensional trees. This is a small subcoalgebra of our (large) coalgebra B. We call two trees b, b' in B *Barr-equivalent*, notation

$$b \sim_0 b'$$

provided that for every natural number n the tree $b|_n$ obtained by cutting b at level n has the same extensional quotient (see 3.8) as the tree $b'|_n$. For example

$$\Omega \sim_0 \Omega'$$

Barr proved that the quotient coalgebra

$$B_{\mathrm{f}}/{\sim_0}$$

is a terminal $\mathscr{P}_{\mathrm{f}}$-coalgebra—that is, $\sim_0$ is the largest congruence on B_{f}.

As before we denote by Ord the class of all small ordinal numbers. We define, for every $i \in \mathsf{Ord}$, the following equivalence relation $\sim_i$ on B:

$$\sim_0 \text{ is the Barr-equivalence}$$

and in case $i > 0$

$$t \sim_i s \text{ iff for all } j < i \text{ the following hold :}$$

(1) for each child t' of t there exists a child s' of s such that $t' \sim_j s'$ and
(2) vice versa.

Remark 5.3. We shall show below that the largest congruence $\sim$ is the intersection of all $\sim_i$. Notice that this intersection is just the usual construction of a greatest fixed point. Indeed, consider the collection Rel of all binary relations on B. This collection, ordered by set-inclusion, is a class-complete lattice. Define $\Phi : \mathsf{Rel} \to \mathsf{Rel}$ as follows: given $R \in \mathsf{Rel}$ then

$$t\,\Phi(R)\,s \text{ iff for every child } t' \text{ of } t \text{ there exists a child } s' \text{ of } s \text{ such that } t'\,R\,s', \text{ and vice versa.}$$

Observe that Φ is a monotone function. Moreover, a binary relation R is a fixed point of Φ if and only if R is a bisimulation on B. Notice that the definition of $\sim_i$ is just an iteration of Φ on the largest equivalence relation $\approx_0$ (i.e., $B \times B$) shifted by ω steps: we have

$$\sim_0 = \Phi^{(\omega)}(\approx_0),$$

where for every relation R the iterations $\Phi^{(i)}(R)$, $i \in \mathsf{Ord}$, are defined inductively as follows: $\Phi^{(0)}(R) = R$, the isolated step is $\Phi^{(i+1)}(R) = \Phi(\Phi^{(i)}(R))$, and for limit ordinals $\Phi^{(i)}(R) = \bigcap_{j<i} \Phi^{(j)}(R)$. Consequently, $\sim_i \, = \Phi^{(\omega+i)}(\approx_0)$.

It follows from the next result t we are indeed constructing the largest fixed point of Φ.

Lemma 5.4. *Φ preserves intersections of descending Ord-chains.*

Proof. Let $(R_i)_{i \in \mathsf{Ord}}$ be a descending chain in Rel and let

$$R = \bigcap_{i \in \mathsf{Ord}} R_i$$

be its intersection. We show that $\Phi(R) = \bigcap_{i\in\mathsf{Ord}} \Phi(R_i)$. In fact, the inclusion from left to right is obvious. To show the inclusion from right to left, suppose that $t\ \Phi(R_i)\ s$ holds for all $i\in\mathsf{Ord}$. Let t' be any child of t. Then, for any ordinal number $i\in\mathsf{Ord}$ there exists a child s_i' of s with $t\ R_i\ s_i'$. Since s has only a small set of children the set $\{s_i' \mid i\in\mathsf{Ord}\}$ is small, too. Therefore, there is a cofinal subset C of Ord such that $\{s_i' \mid i\in C\}$ has only one element, s' say. It follows that $t'\,R_i\,s'$ for all sufficiently large $i\in\mathsf{Ord}$. Hence, $t\Phi(R)s$, as desired. □

Theorem 5.5. *Two trees $t,s\in B$ are bisimilar iff $t\sim_i s$ holds for all small ordinals i.*

Proof. It follows from Lemma 5.4 that the intersection of all $\sim_i = \Phi^{(i)}(\sim_0)$, $i\in\mathsf{Ord}$ is a fixed point of Φ.

Next form the quotient coalgebra $B/\sim$. Since B is weakly terminal, so is $B/\sim$. In order to establish that $B/\sim$ is a terminal $\mathscr{P}^\infty$-coalgebra we must show that for any $\mathscr{P}^\infty$-coalgebra (X,ξ) and any two coalgebra homomorphisms $h,k:(X,\xi)\to(B,\beta)$ we have $h(x)\sim k(x)$ for all $x\in X$. We show this by transfinite induction. We write

$$\beta(k(x)) = \{s_j^x \mid j\in J_x\} \quad \text{and} \quad \beta(h(x)) = \{t_i^x \mid i\in I_x\}$$

for the sets of children of $k(x)$ and $h(x)$, respectively. Since h and k are coalgebra homomorphisms we have

$$\{s_j^x \mid j\in J_x\} = \{kx_\ell \mid \ell\in L_x\} \quad \text{and} \quad \{t_i^x \mid i\in I_x\} = \{hx_\ell \mid \ell\in L_x\},$$

where $\xi(x) = \{x_\ell \mid \ell\in L_x\}$.

First step, $i=0$: Recall from Example 3.8 the notation $[t]$ for the extensional quotient of a tree t. We will show that $k(x)\sim_0 h(x)$, i.e., $[k(x)|_n] = [h(x)|_n]$ for all $n<\omega$ by induction on n. The statement is obvious for $n=0$. For the induction step observe that

$$\begin{aligned}\{[s_j^x|_n] \mid j\in J_x\} &= \{[kx_\ell|_n] \mid \ell\in L_x\}\\ &= \{[hx_\ell|_n] \mid \ell\in L_x\}\\ &= \{[t_i^x|_n] \mid i\in I_x\}\end{aligned}$$

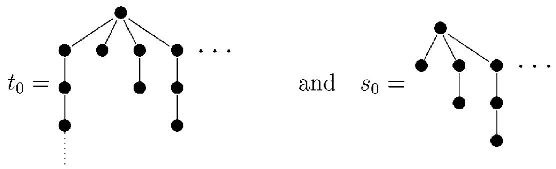
by the induction hypothesis. Hence, $[k(x)|_{n+1}]$ and $[h(x)|_{n+1}]$ have the same sets of children and therefore are equal.

Induction step: Suppose now that $i>0$ is any ordinal number and that for all $x\in X$, $k(x)\sim_j h(x)$ holds for all $j<i$. Consider any child s' of $k(x)$, i.e., $s'=kx_\ell$ for some $x_\ell\in\xi(x)$. Then $t'=hx_\ell$ is a child of $h(x)$ such that $s'\sim_j t'$ for all $j<i$.

Hence, we obtain $k(x)\sim_i h(x)$ for all $i\in\mathsf{Ord}$, which implies the desired result. □

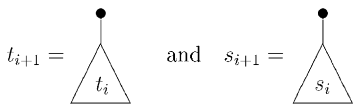
Remark 5.6. Barr showed that $\sim_0$ is the largest congruence on the set of finitely branching trees. However, it is not a congruence on B. In order to see this recall that the notions of congruence and bisimulation are equivalent in the current setting (see Remark 3.17), and notice that is suffices to find trees that are in $\sim_0$ but not in $\sim_1$.

Consider the following trees:

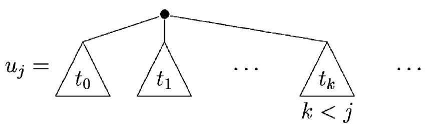


We clearly have $t \sim_0 s$. But $t_0 \not\sim_1 s_0$, since t_0 has a child which is an infinite path while s_0 does not.

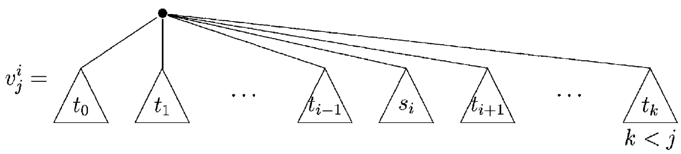
Definition 5.7. We define trees t_i and s_i for all small ordinals i for which we show below that they are equivalent under $\sim_i$ but not under $\sim_{i+1}$.
(1) We start with the trees t_0 and s_0 from the previous remark.
(2) Given t_i and s_i we define



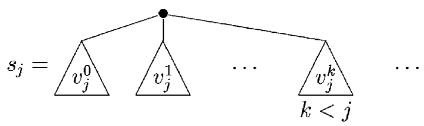
(3) For every limit ordinal j we use the following auxiliary trees (where $i<j$ is arbitrary)



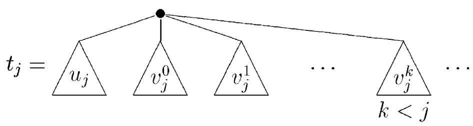
and



Define



and



Theorem 5.8. *None of the equivalences $\sim_i$ is a congruence.*

Proof. We prove that no $\sim_i$ is a bisimulation. To this end we show that $t_i \sim_i s_i$ but $t_i \not\sim_{i+1} s_i$ for all ordinals i. This proves the theorem.

(1) Proof of $t_i \sim_i s_i$. We proceed by transfinite induction on i.

Initial case: $t_0 \sim_0 s_0$—clear.

Isolated case: $t_i \sim_i s_i$ clearly implies $t_{i+1} \sim_{i+1} s_{i+1}$.

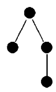
Limit case: Let j be a limit ordinal with $t_i \sim_i s_i$ for all $i<j$. Then, obviously, $u_j \sim_i v_j^i$ for all $i<j$, which implies $t_j \sim_j s_j$.

(2) We need some auxiliary facts about cuttings $w|_n$ of trees w at level n:

(a) For $n=1$ all the trees t_i, s_i, u_j, v_j^i cut to

because they all have more than one vertex—this is obvious.

(b) We have

$t_0|_2 = s_0|_2 =$ 

and

$t_i|_2 = s_i|_2 =$  for all $i \geq 1$

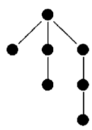
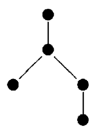
The first statement is obvious, and so is the second one for isolated ordinals i. For limit ordinals it follows from (a).

(c) We have

$u_j|_2 = v_j^i|_2 =$  for all limit ordinals j and all $i < j$

This follows from (a).

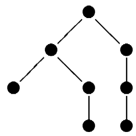
(d) We have

$t_0|_3 = s_0|_3 =$  $t_1|_3 = s_1|_3 =$ 

$t_i|_3 = s_i|_3 =$  for all $i \geq 2$

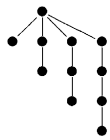
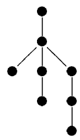
The last statement follows from (c).

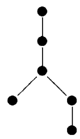
(e) We have

$u_j|_3 = v_j^i|_3 =$  for all $i < j$, j a limit ordinal

This follows from (b).

(f) We have

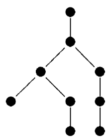
$t_0|_4 = s_0|_4 =$  $t_1|_4 = s_1|_4 =$ 

$t_2|_4 = s_2|_4 =$ 

and

$t_i|_4 = s_i|_4 =$  for all isolated $i \geq 3$

as well as

$t_j|_4 = s_j|_4 =$  for all limit ordinals j

The last statement follows from (e), the last but one from (d).

(3) We prove

$$t_i \not\sim_{i+2} t_k \quad \text{and} \quad s_i \not\sim_{i+2} t_k \quad \text{for all ordinals } i < k.$$

We proceed by transfinite induction on k:

(a) *Initial case*: there is nothing to prove if $k = 0$.

(b) *Isolated case*: $t_i \not\sim_{i+2} t_{k+1}$ is clear if $i = 0$; in fact, $t_0 \not\sim_0 t_{k+1}$ because $t_0|_2 \neq t_{k+1}|_2$, see (2b), and if i is a limit ordinal, $t_i \not\sim_0 t_{k+1}$ because $t_i|_4 \neq t_{k+1}|_4$, see (2f). If i is an isolated ordinal, then $t_{i-1} \not\sim_{i+1} t_k$ implies $t_i \not\sim_{i+2} t_{k+1}$. Analogously with $s_i \not\sim_{i+2} t_k$.

(c) *Limit case*: let k be a limit ordinal. We proceed by transfinite induction on i.

(c.1) *Initial case*: $t_0 \not\sim_2 t_k$ because $t_0|_2 \neq t_k|_2$, see (2b). Analogously $s_0 \not\sim_2 t_k$.

(c.2) *Isolated case*: $t_{i+1} \nsim_{i+3} t_k$ because $t_{i+1}|_4 \neq t_k|_4$, see (2f). Analogously $s_{i+1} \nsim_{i+3} t_k$.

(c.3) *Limit case*: let $j<k$ be a limit ordinal. Assuming $t_j \sim_{j-2} t_k$, we derive a contradiction. The child u_k of t_k must be $\sim_{j+1}$-equivalent to a child of t_j, i.e.,

$$\text{either } u_j \sim_{j+1} u_k, \text{ or } v_j^i \sim_{j+1} u_k \text{ for some } i<j.$$

The first possibility implies that the child t_j of u_k is $\sim_j$-equivalent to a child t_l of u_j, $l<j$. Thus, we have

$$t_l \sim_j t_j \text{ for } l<j<k.$$

This contradicts the fact that, by induction, $t_l \nsim_{l+2} t_j$ (and $l+2<j$). Analogously with the second possibility, $v_j^i \sim_{j+1} u_k$, where the only case that we have to consider extra is the child s_i of v_j^i—however,

$$s_i \sim_j t_j$$

is also a contradiction since, by induction, $s_i \nsim_{i+2} t_j$ (and $l+2<j$). Finally, assuming $s_j \sim_{j+2} t_k$, we derive a contradiction analogously.

(4) Proof of $t_i \nsim_{i+1} s_i$. We proceed by transfinite induction on i.

Initial case: $t_0 \nsim_1 s_0$ by our choice of trees t_0 and s_0.

Isolated case: From $t_i \nsim_{i+1} s_i$ it follows immediately that $t_{i+1} \nsim_{i+2} s_{i+1}$.

Limit case: Let j be a limit ordinal with $t_j \sim_{j+1} s_j$. We derive a contradiction. The child u_j of t_j is $\sim_j$-equivalent to a child of s_j. That is,

$$u_j \sim_j v_j^k \quad \text{for some } k<j.$$

This implies that the child t_k of u_j is $\sim_{k+2}$-equivalent to some child of v_j^k, i.e.,

$$\text{either } t_k \sim_{k+2} s_k \text{ or } t_k \sim_{k+2} t_l \text{ for some } l \neq k,\ l<j.$$

The first case does not happen: by induction hypothesis, $t_k \nsim_{k+1} s_k$. The second case contradicts to (3): if $k<l$, and for $l<k$ we know from (3) that $t_l \nsim_{l+2} t_k$, thus, again $t_l \nsim_{k+2} t_k$. □

Remark 5.9. We have described a terminal coalgebra of $\mathscr{P}^\infty$ as the coalgebra of all extensional trees modulo the congruence $\bigcap_{i\in \mathsf{Ord}} \sim_i$. Since none of the equivalences is a congruence, we see no hope in obtaining a nicer description of a terminal $\mathscr{P}^\infty$-coalgebra in well-founded set theory.

Acknowledgements

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References

[1] P. Aczel, Non-Well-Founded Sets, in: CSLI Lecture Notes, Vol. 14, Stanford University, Stanford, CA, 1988.
[2] P. Aczel, J. Adámek, S. Milius, J. Velebil, Infinite trees and completely iterative theories: a coalgebraic view, Theoret. Comput. Sci. 300 (2003) 1–45.
[3] P. Aczel, N. Mendler, A final coalgebra theorem, in: D.H. Pitt, D.E. Rydebeard, P. Dyjber, A.M. Pitts, A. Poigné (Eds.), Category Theory and Computer Science, Lecture Notes in Computer Science, Vol. 389, Springer, Berlin, 1989, pp. 357–365.
[4] J. Adámek, Free algebras and automata realization in the language of categories, Comment. Math. Univ. Carolin. 15 (1974) 589–602.
[5] J. Adámek, On a description of terminal coalgebras and iterative theories, Electron. Notes Theoret. Comput. Sci. 82 (1) (2003).
[6] J. Adámek, S. Milius, J. Velebil, Final coalgebras and a solution theorem for arbitrary endofunctors, Electron. Notes Theoret. Comput. Sci. 65.1 (2002).
[7] J. Adámek, H.-E. Porst, On varieties and covarieties in a category, Math. Structures Comput. Sci. 13 (2003) 201–232.
[8] J. Adámek, H.-E. Porst, On tree coalgebras and coalgebra presentations, Theoret. Comput. Sci. 311 (2004) 257–283.
[9] J. Adámek, V. Trnková, Automata and Algebras in a Category, Kluwer Publishing Company, Dordrecht, 1990.
[10] M. Barr, Terminal coalgebras in well-founded set theory, Theoret. Comput. Sci. 124 (1994) 182–192.
[11] J. Barwise, L. Moss, Vicious Circles, in: CSLI Lecture Notes, Vol. 60, Stanford University, Stanford, CA, 1996.
[12] H.P. Gumm, T. Schröder, Coalgebras of bounded type, Math. Structures Comput. Sci. 12 (2002) 565–578.
[13] P. Johnstone, J. Power, T. Tsujishita, H. Watanabe, J. Worell, On the structure of categories of coalgebras, Theoret. Comput. Sci. 260 (2001) 87–117.
[14] G.M. Kelly, A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on, Bull. Austral. Math. Soc. 22 (1980) 1–83.
[15] V. Koubek, Set functors II—contravariant case, Comment. Math. Univ. Carolin. 14 (1973) 47–59.
[16] V. Koubek, J. Reiterman, Categorical constructions of free algebras, colimits and completions of partial algebras, J. Pure Appl. Algebra 14 (1979) 195–231.
[17] F.E.J. Linton, Coequalizers in categories of algebras, in: Lecture Notes in Mathematics, Vol. 80, Springer, Berlin, 1969, pp. 75–90.
[18] M. Makkai, R. Paré, Accessible categories: the foundations of categorical model theory, Contemporary Mathematics, American Mathemal Society, Providence, 1989.
[19] L. Moss, Parametric corecursion, Theoret. Comput. Sci. 260 (1–2) (2001) 139–163.
[20] J.J.M.M. Rutten, Universal coalgebra: a theory of systems, Theoret. Comput. Sci. 249 (1) (2000) 3–80.
[21] J.J.M.M. Rutten, D. Turi, On the foundations of final coalgebra semantics: non-well-founded sets, partial orders, metric spaces, Math. Structures Comput. Sci. 8 (1998) 481–540.
[22] W. Sierpiński, Sur une decomposition d'ensembles, Monatsh. Math. Phys. 35 (1928) 239–242.
[23] A. Tarski, Sur la décomposition des ensembles en sous-ensembles prèsque disjoint, Fund. Math. 14 (1929) 205–215.
[24] J. Worrell, On coalgebras and final semantics, Ph.D. Thesis, Oxford University Computing Laboratory, 2000.