Bases for Parametrized Iterativity

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Abstract

Parametrized iterativity of an algebra means the existence of unique solutions of all finitary recursive systems of equations where recursion is allowed to use only some variables (chosen as a parameter). We show how such algebras can be introduced in an arbitrary category $\mathcal{A}$ by employing a base, i.e., an operation interpreting objects of $\mathcal{A}$ as monads on $\mathcal{A}$. For every base we prove that free base algebras and free iterative base algebras exist. The main result is a coalgebraic construction of the latter: all equation morphisms form a diagram whose colimit is proved to be a free iterative base algebra.

Key words: Iterative algebra, monad, iterative theory, parametrized signature, locally presentable category.

One for the money
Two for the show
Three to get ready
And four to go.

Nursery rhyme

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1 Introduction

The idea of parametrized iterativity of Tarmo Uustalu [U] can be formulated on the level of algebras A by requiring that recursive equations in A, where recursion is allowed on a subset of varibles only (determined by a choice of a parameter), have unique solutions. This has led us to introducing the concept of a base on the category Set, i.e. a finitary [2] functor from Set to the category of all finitary monads on Set, see [AMV$_3$]. There we proved that for every base free iterative agebras exist. The aim of the present paper is to prove that free iterative algebras can be constructed coalgebraically, precisely as in the case of (non-parametrized) iterativity treated in [AMV$_1$]: a free iterative algebra RY for a base $\Box$ is a colimit of the diagram of all coalgebras for the endofunctor $- \Box Y$ with a finitely presentable carrier. This results in a monad R called the rational monad of the base $\Box$. In a subsequent paper we will apply the coalgebraic description to prove a "parametric generalization" of the main result of [AMV$_1$]: that the rational monad of every finitary endofunctor H is a free iterative monad on H in the sense of Calvin Elgot [E].

In the rest of the introduction we shortly recall the idea of (non-parametrized) iterativity, due to Evelyn Nelson [N] and Jerzy Tiuryn [T], for Σ-algebras and its generalization in [AMV$_1$] to H-algebras. We also recall the parametrization idea inspired by the work of Tarmo Uustalu [U], as presented in [AMV$_3$].

1.A Iterative Σ-Algebras

Let $\Sigma = \big(\Sigma(n)\big)_{n\in\mathbb{N}}$ be a signature and A be a Σ-algebra. We study recursive systems of equations [3] in A of the form

$$\begin{aligned} x_1 &\approx t_1(x_1, \dots, x_m, a_1, \dots, a_k)\\ x_2 &\approx t_2(x_1, \dots, x_m, a_1, \dots, a_k)\\ &\vdots\\ x_m &\approx t_m(x_1, \dots, x_m, a_1, \dots, a_k)\end{aligned} \tag{1.1}$$

where each right-hand side t_i is a term over $X + A$, in which $X = \{x_1, \dots, x_m\}$ is a set of variables and $a_1, \dots, a_k$ are elements of A. The system is called *guarded* if none of the right-hand sides t_j is a single variable $x_1, \dots, x_m$. Evelyn Nelson calls the algebra A *iterative* if every such system of equations

[2] Finitary functors are those preserving filtered colimits. Finitary monads are monads with a finitary underlying functor.
[3] We use $\approx$ for formal equations, whereas $=$ denotes the identity of both sides.

has a unique solution in A, i.e., if there exists a unique m-tuple $b_1, \ldots, b_m$ of elements of A with

$$b_j = t_j(b_1, \ldots, b_m, a_1, \ldots, a_k) \quad \text{for } j = 1, \ldots, m.$$

We can restrict ourselves to *flat* equations, which means that each t_i is either a "flat term" $\sigma(y_1, \ldots, y_n)$ for some n-ary operation symbol $\sigma \in \Sigma_n$ and some elements $y_1, \ldots, y_n$ of X, or a single element of A. In fact, there is an obvious process of flattening of equations which we illustrate by the next

Example 1.1. Suppose Σ consists of a single binary operation symbol $*$. If A is an iterative binary algebra, then for every element a there exists a unique element b with

$$b = b * (a * b).$$

This follows from the recursive equation

$$x \approx x * (a * x).$$

We can flatten it by introducing new variables y and z:

$$\begin{aligned} x &\approx x * y \\ y &\approx z * x \\ z &\approx a \end{aligned}$$

Notation 1.2. Let H_Σ be the *polynomial functor* corresponding to a finitary signature Σ:

$$H_\Sigma X = \Sigma(0) + \Sigma(1) \times X + \Sigma(2) \times X^2 + \ldots$$

Let A be a Σ-algebra with the structure morphism $\alpha : H_\Sigma A \longrightarrow A$. A flat system of equations with the set $X = \{x_1, \ldots, x_m\}$ of variables assigns to every variable x_i either a flat term, i.e., an element of $H_\Sigma X$, or an element of A. This is represented by a morphism

$$e : X \longrightarrow H_\Sigma X + A.$$

Such morphisms, where X is an arbitrary finite set, are called (finitary, flat) *equation morphisms*. A *solution* of e assigns to every variable from X an element of A, thus, it is represented by a morphism

$$e^\dagger : X \longrightarrow A.$$

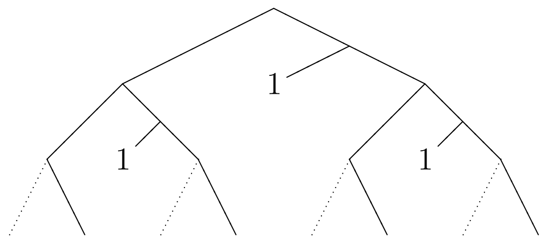
The relationship between equation morphisms and solutions is expressed by the commutativity of the following square

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & A \\ H_\Sigma X + A \arrow[r, "H_\Sigma e^\dagger + A"'] & H_\Sigma A + A \arrow[u, "{[\alpha, A]}"'] \end{tikzcd}$$

Example 1.3. If Σ consists of a single unary operation, then an algebra $\alpha : A \longrightarrow A$ is iterative if and only if α has a single fixed point and no other cycle, see [AMV$_1$].

For more general signatures there is no hope of a simple characterization of iterative algebras: there will always be a lot of mutually independent properties, as that illustrated in Example 1.1. However, a free iterative Σ-algebra on a set Y of generators is easy to describe. By a Σ-*tree on* Y we understand a rooted, ordered, finitely branching tree whose leaves are labelled in $\Sigma(0) + Y$ and nodes with $n > 0$ successors are labelled in $\Sigma(n)$; we consider trees always up to isomorphism. A Σ-tree is called *rational*, a concept provided by Susanna Ginali [G], if it has (up to isomorphism) finitely many subtrees only. The algebra $R_\Sigma Y$ of all rational Σ-trees on Y is a free iterative algebra on Y, see [N].

Example 1.4. The unique solution of $x \approx x * (1 * x)$ in $R_\Sigma\{1\}$ is the following tree $e^\dagger(x)$:



(1.2)

This tree t has, besides itself, just two subtrees: 1 and $1 * t$.

1.B Iterative H-algebras

We generalize 1.A naturally from H_Σ-algebras in Set to H-algebras in $\mathcal{A}$, where H is a finitary endofunctor of a locally finitely presentable category $\mathcal{A}$. Recall that if the functor $\mathcal{A}(A, -) : \mathcal{A} \longrightarrow \mathsf{Set}$ is finitary, then the object A is called *finitely presentable*. A category $\mathcal{A}$ is called *locally finitely presentable* provided that it is cocomplete and has a small set

$$\mathcal{A}_{fp}$$

of finitely presentable objects such that every object of $\mathcal{A}$ is a filtered colimit of objects in $\mathcal{A}_{fp}$. We usually take as $\mathcal{A}_{fp}$ a set representing all finitely presentable objects up to isomorphism.

By a (finitary, flat) *equation morphism* in an object A we understand a morphism

$$e : X \longrightarrow HX + A, \quad X \text{ finitely presentable.}$$

Let A be an H-algebra with the structure morphism $\alpha : HA \longrightarrow A$. Then a *solution* of e is a morphism $e^\dagger : X \longrightarrow A$ for which the square

$$\begin{tikzcd} X \arrow[rr, "e^\dagger"] \arrow[d, "e"'] & & A \\ HX + A \arrow[rr, "He^\dagger + A"'] & & HA + A \arrow[u, "{[\alpha, A]}"'] \end{tikzcd}$$

commutes. An H-algebra is called *iterative* if every equation morphism in it has a unique solution.

Every object Y of $\mathcal{A}$ generates a free iterative H-algebra RY. In case $\mathcal{A} = \mathsf{Set}$ and $H = H_\Sigma$, we can describe RY as a quotient algebra of the rational-tree algebra $R_\Sigma Y$, see [AM].

A general coalgebraic construction of the algebra RY is presented in [AMV$_1$]: consider the category of coalgebras for $H(-)+Y$, then the equation morphisms in Y form a full subcategory EQ_Y (of all coalgebras with a finitely presentable domain). We have an obvious forgetful functor

$$\mathrm{Eq}_Y : \mathsf{EQ}_Y \longrightarrow \mathcal{A}, \quad (e : X \longrightarrow HX + Y) \longmapsto X$$

which forms a filtered diagram in $\mathcal{A}$. Its colimit

$$RY = \operatorname{colim} \mathrm{Eq}_Y$$

is then a free iterative H-algebra on Y.

Applying this description of RY, we proved in [AMV$_1$] that the *rational monad*, i.e., the monad $Y \longmapsto RY$ of free iterative H-algebras, is iterative in the sense of Calvin Elgot [E]. In fact, R is characterized as a free iterative monad on H.

Example 1.5. The rational monad of the finite-powerset functor $\mathcal{P}_{\mathit{fin}} : \mathsf{Set} \longrightarrow \mathsf{Set}$, given by $\mathcal{P}_{\mathit{fin}} X = \{M \mid M \subseteq X, M \text{ finite}\}$, is the monad of all rational, strongly extensional, finitely branching trees on Y, see [AM]. Strongly extensional means that distinct children of a node never define bisimilar subtrees, and rational refers once more to the fact that there are (up to isomorphism) only finitely many subtrees.

In [AMV$_3$] we study Σ-algebras that are iterative in a weaker sense than in 1.A above: for every operation in Σ we choose (as a free parameter) the number of variables which can be used for recursion. Before formulating this precisely, let us consider the case of a single binary operation.

Example 1.6. For algebras on a single binary operation $*$ let us choose a parameter

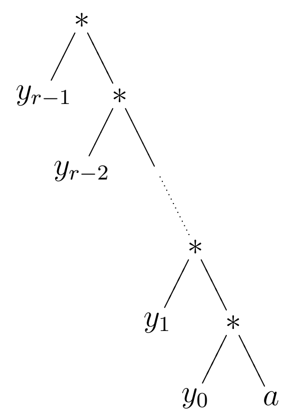
$$i = 2, 1, 0$$

which tells us the number and position of the variables of $y * z$ that can be used for recursion. In Example 1.1 above the choice is 2. If our choice were 0, there is no recursion: every algebra is "parametrized iterative" by default. Let us choose 1 — that is, either y or z is used for iteration, but due to symmetry, we consider the first case only. (Also later, for operations of larger arities: given the parameter i, we always assume that the *first* i variables from the left are those used for iteration. But any other choice of the i variables among the n possible ones would also work, of course.) What does this choice of the left-hand iteration mean in terms of the equations we are solving? Consider the system (1.1) above: the right-hand sides t_j are finite binary trees with leaves labelled in $X + A$ — and we now additionally require that

$$\text{every } X\text{-labelled leaf is the left-hand child of its parent.} \tag{1.3}$$

Thus, iterativity with parameter $i = 1$ means that every system (1.1) of recursive equations whose right-hand sides satisfy (1.3) has a unique solution. Algebras with this property are called *parametrized iterative.*

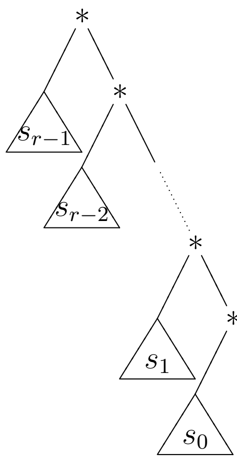
The flat variation here has the following form: the right-hand sides of (1.1) are trees of the form



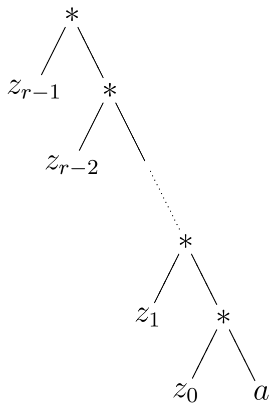
(1.4)

for $r = 0, 1, 2, \ldots$, where $y_0, \ldots, y_{r-1} \in X$ and $a \in A$. In fact, whenever a binary algebra has unique solutions of all systems (1.1) with right-hand sides of the form (1.4), then it is parametrized iterative. The flattening of a right-hand side t_j can be performed recursively (by adding new variables) as follows:

it is clear that (1.3) implies that t_j has the form

$t_j =$  (1.5)

for a in A and trees s_0, …, s_{r-1} satisfying (1.3). Introduce new variables z_0, …, z_{r-1}, replace t_j on the right-hand side of equations by



and add the equations

$$z_j \approx s_j, \quad j = 0, \ldots, r-1$$

continuing with flattening of these latter equations. Consequently, an algebra is parametrized iterative if and only if every system (1.1) of recursive equations with right-hand sides of the form (1.4) has a unique solution. This is strictly weaker than full iterativity, e.g., the empty algebra is parametrized iterative (but not iterative in the full sense where $x \approx x{*}x$ must have a unique solution). A free parametrized algebra on a set Y of generators is the algebra of all binary rational trees that are *right-well-founded*, i.e., the right-most path from every node is finite, see [AMV$_3$].

Observe that the above trees (1.4) are elements of

$$X^* \times A.$$

That is, the equation systems (1.1) we consider here can be represented by morphisms of the form

$$e : X \longrightarrow X^* \times A, \quad X \text{ finite.}$$

A *solution* of e is represented by a morphism

$$e^\dagger : X \longrightarrow A$$

such that the square

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & A \\ X^* \times A \arrow[r, "(e^\dagger)^* \times A"'] & A^* \times A \arrow[u, "\widehat{\alpha}"'] \end{tikzcd} \tag{1.6}$$

commutes, where $\widehat{\alpha}$ extends the binary operation $*$ of A:

$$\widehat{\alpha}(y_{r-1}, y_{r-2}, \dots, y_0, a) = y_{r-1} * (y_{r-2} * (\dots * (y_0 * a)) \dots).$$

For general signatures Σ we choose for every n-ary symbol σ a parameter $i = 0, 1, \dots, n$ (called the *iterativity* of σ) which means that in the expression

$$\sigma(y_1, \dots, y_i, z_1, \dots, z_p), \quad \text{where } p = n - i,$$

the variables $y_1, \dots, y_i$ are used for recursion but $z_1, \dots, z_p$ are not. We call Σ with this choice of parameters a *parametrized signature*. And for parametrized iterativity we consider only those recursive systems (1.1) that employ only the allowed variables for recursion. To formulate this precisely, we form, for every finite set X of variables, the *derived signature* $\widehat{\Sigma}$ whose operation symbols have the form

$$\sigma(y_1, \dots, y_i, -, \dots, -) \quad \text{of arity } p = n - i.$$

Here σ is an n-ary operation symbol of iterativity i, and $(y_1, \dots, y_i)$ is an arbitrary (but fixed) i-tuple of elements of X. We denote, for every pair (X, A) of sets, by

$$X \,\square\, A$$

a free algebra of the derived signature on Set and by

$$u_A^X : A \longrightarrow X \,\square\, A$$

the universal arrow. Example: if Σ consists of a single binary operation symbol $*$ of iterativity 1, the derived signature consists of unary operations $y * -$ for $y \in X$, and the free algebra is

$$X \,\square\, A = X^* \times A$$

with $u_A^X : A \longrightarrow X^* \times A$ given by $a \longmapsto (\varepsilon, a)$. Observe that the tree (1.4) is a typical element of $X \,\square\, A$. In general, the recursive systems (1.1) we consider here are precisely those whose right-hand sides t_j lie in $X \,\square\, A$. Thus, given a Σ-algebra A, we call it *iterative* w.r.t. the parametrized signature Σ provided that every morphism

$$e : X \longrightarrow X \,\square\, A, \quad X \text{ is finite,}$$

has a unique solution. To define solution, denote by

$$\widehat{\alpha} : A \,\square\, A \longrightarrow A$$

the unique homomorphism expressing the algebraic structure of A: for $X = A$ we consider A as an (obvious) $\widehat{\Sigma}$-algebra and $\widehat{\alpha}$ is the unique homomorphism with $\widehat{\alpha} \cdot u_A^A = id$. Then a solution of $e : X \longrightarrow X \,\square\, A$ is a morphism

$$e^\dagger : X \longrightarrow A$$

such that the square

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & A \\ X \,\square\, A \arrow[r, "e^\dagger \square A"'] & A \,\square\, A \arrow[u, "\widehat{\alpha}"'] \end{tikzcd} \tag{1.7}$$

commutes. The arrow $e^\dagger \,\square\, A$ is well-defined in the sense that $\square$ is actually a functor $\mathsf{Set} \times \mathsf{Set} \longrightarrow \mathsf{Set}$. More precisely, the definition of $\square$ is such that for every set X we have a finitary monad $X \,\square\, -$ on Set: the free-algebra monad of the derived signature $\widehat{\Sigma}$. It is easy to see that this is an object-assignment of a functor

$$\square : \mathsf{Set} \longrightarrow \mathsf{FM}(\mathsf{Set})$$

from Set to the category of finitary monads on Set which, when uncurried, yields a functor $\mathsf{Set} \times \mathsf{Set} \longrightarrow \mathsf{Set}$.

In [AMV$_3$] we described free iterative algebras for every parametrized signature in Set, and (more generally), free bases on every parametrized endofunctor of a locally finitely presentable category $\mathcal{A}$. In the present paper we work with arbitrary bases on $\mathcal{A}$, and at the end we come back to parametrized signatures in $\mathcal{A}$ and the corresponding free bases.

1.D Bases on Locally Finitely Presentable Categories

Let $\mathcal{A}$ be a locally finitely presentable category, and denote by $\mathsf{FM}(\mathcal{A})$ the category of all finitary monads on $\mathcal{A}$. We are going to study finitary functors

$$\square : \mathcal{A} \longrightarrow \mathsf{FM}(\mathcal{A})$$

and call them *bases*. This approach is inspired by the work of Tarmo Uustalu [U] who considered similar functors to study complete iterativity.

We use the uncurried notation $X \,\Box\, A$ for X, A in $\mathcal{A}$, see Remark 2.4 for details. An important case is a free base on a *parametrized endofunctor*, i.e., a finitary functor $H : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$. Here

$$X \,\Box\, A = \text{ free } H(X, -)\text{-algebra on } A,$$

and we again denote by $u^X_A : A \longrightarrow X \,\Box\, A$ the universal arrow. This generalizes the above notation, where every parametrized signature Σ is expressed via the *parametrized polynomial endofunctor* H_Σ defined by

$$H_\Sigma(X, A) = \coprod_{\sigma \in \Sigma} X^i \times A^p \quad \text{for } \sigma \text{ of iterativity } i \text{ and arity } i + p.$$

Observe that a Σ-algebra is simply an algebra of $H(A, -)$ on A. In general, we study algebras on parametrized endofunctors, given by an object A and a morphism

$$\alpha : H(A, A) \longrightarrow A.$$

The algebraic structure can also be expressed (as used in 1.C above) by the unique homomorphism

$$\widehat{\alpha} : A \,\Box\, A \longrightarrow A.$$

extending id_A. This allows us to speak about parametrized iterative algebras analogously to the case of Σ-algebras above: we introduce algebras on an object A as the Eilenberg-Moore algebras of the monad $A \,\Box\, -$ on $\mathcal{A}$. As above, an equation system is expressed by a morphism $e : X \longrightarrow X \,\Box\, A$, where X is now an arbitrary finitely presentable object. And A is called *iterative* if every such morphism e has a unique solution given by commutativity of (1.7). In this generality we present a coalgebraic construction of free iterative algebras — for an arbitrary base on an arbitrary locally finitely presentable category.

1.E Synopsis of the paper

Section 2 presents examples of bases and a description of the free base algebras as the initial algebras of the derived endofunctor.

Iterative base algebras are studied in Section 3 where also the existence of free iterative base algebras is proved. The main result of Section 4 is the description of the free iterative base algebras as filtered colimits of all equation morphisms.

Finally, Section 5 is devoted to bases given by parametric signatures.

Related Work. This paper, which is an expanded version of [AMV$_2$], was inspired by the work of Tarmo Uustalu [U].

2 Bases And Base Algebras

Assumption 2.1. Throughout this section we assume that a locally finitely presentable category $\mathcal{A}$ is given. We denote by $\mathsf{FM}(\mathcal{A})$ the category of all finitary monads on $\mathcal{A}$ and monad morphisms. We also choose a small full subcategory representing all finitely presentable objects of $\mathcal{A}$ and denote it by $\mathcal{A}_{fp}$.

Definition 2.2. By a finitary *base* on $\mathcal{A}$ is understood a finitary functor from $\mathcal{A}$ to $\mathsf{FM}(\mathcal{A})$.

Notation 2.3.

(1) Given a base, we have a finitary underlying functor from $\mathcal{A}$ to the category $\mathsf{Fin}[\mathcal{A},\mathcal{A}]$ of finitary endofunctors of $\mathcal{A}$. This is a curried form of a functor of two variables, finitary in both of them, which we denote by

$$\Box : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}.$$

(2) The unit of the monad $X \Box -$ is denoted by $u^X : Id \longrightarrow X \Box -$; its components are

$$u^X_A : A \longrightarrow X \Box A.$$

(3) The multiplication of the monad $X \Box -$ is denoted by

$$m^X : X \Box (X \Box -) \longrightarrow X \Box -;$$

its components are

$$m^X_A : X \Box (X \Box A) \longrightarrow X \Box A.$$

Remark 2.4. Explicitly, to specify a base means to specify a finitary functor of two variables

$$\Box : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$$

and morphisms

$$u^X_A : A \longrightarrow X \Box A \quad \text{and} \quad m^X_A : X \Box (X \Box A) \longrightarrow X \Box A$$

for arbitrary objects X and A of $\mathcal{A}$ such that the following four diagrams commute: the first two

$$\begin{tikzcd} X \Box A \arrow[r, "X \Box u^X_A"] \arrow[dr, equal] & X \Box (X \Box A) \arrow[d, "m^X_A"] & X \Box A \arrow[l, "u^X_{X \Box A}"'] \arrow[dl, equal] \\ & X \Box A & \end{tikzcd} \tag{2.1}$$

and

$$
\begin{tikzcd}
X \,\Box\, (X \,\Box\, (X \,\Box\, A)) \arrow[r, "X \Box m^X_A"] \arrow[d, "m^X_{X \Box A}"'] & X \,\Box\, (X \,\Box\, A) \arrow[d, "m^X_A"] \\
X \,\Box\, (X \,\Box\, A) \arrow[r, "m^X_A"'] & X \,\Box\, A
\end{tikzcd}
\tag{2.2}
$$

express the monad axioms for $X \,\Box\, -$, and the latter two

$$
\begin{tikzcd}
A \arrow[r, "u^X_A"] \arrow[d, "f"'] & X \,\Box\, A \arrow[d, "h \Box f"] \\
B \arrow[r, "u^Y_B"'] & Y \,\Box\, B
\end{tikzcd}
\tag{2.3}
$$

and

$$
\begin{tikzcd}
X \,\Box\, (X \,\Box\, A) \arrow[r, "m^X_A"] \arrow[d, "h \Box (h \Box f)"'] & X \,\Box\, A \arrow[d, "h \Box f"] \\
Y \,\Box\, (Y \,\Box\, B) \arrow[r, "m^Y_B"'] & Y \,\Box\, B
\end{tikzcd}
\tag{2.4}
$$

express the naturality of u^X and m^X and the fact that for every morphism $h : X \longrightarrow Y$ we have a monad morphism $h \,\Box\, (-) : X \,\Box\, (-) \longrightarrow Y \,\Box\, (-)$.

Examples 2.5.

(1) Coproduct is a base

$$X \,\Box\, A = X + A$$

with the obvious unit [4]

$$u^X_A = \mathsf{inr} \, : A \longrightarrow X + A$$

and multiplication

$$m^X_A = [\mathsf{inl}\,, \mathsf{inl}\,, \mathsf{inr}\,] : X + X + A \longrightarrow X + A.$$

(2) Let X^* denote a free monoid on object X of $\mathcal{A}$ with unit $\eta_X : 1 \longrightarrow X^*$ and multiplication $\mu_X : X^* \times X^* \longrightarrow X^*$. Then we have the base

$$X \,\Box\, A = X^* \times A$$

with base unit

$$u^X_A = \eta_X \times A : A \longrightarrow X^* \times A$$

and base multiplication

$$m^X_A = \mu_X \times A : X^* \times X^* \times A \longrightarrow X^* \times A.$$

[4] We denote by inl and inr the coproduct injections of binary coproducts.

(3) Let $\mathcal{B}$ be a locally finitely presentable category. Then so is the category $\mathcal{A} = \mathsf{Fin}[\mathcal{B}, \mathcal{B}]$ of all finitary endofunctors of $\mathcal{B}$. Here we have the base

$$X \,\square\, A = \mathcal{F}(X) \cdot A$$

where, for every X in $\mathsf{Fin}[\mathcal{B}, \mathcal{B}]$, a free monad on X is denoted by

$$(\mathcal{F}(X), \eta^X, \mu^X).$$

(It exists since X is finitary, see [B].) The base unit is

$$u_A^X = \eta^X A : A \longrightarrow \mathcal{F}(X) \cdot A$$

and the base multiplication is

$$m_A^X = \mu^X A : \mathcal{F}(X) \cdot \mathcal{F}(X) \cdot A \longrightarrow \mathcal{F}(X) \cdot A.$$

(4) Examples (1)–(3) above are instances of the following general construction. Assume that we are given a tensor product

$$\otimes : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$$

making $\mathcal{A}$ a monoidal category, and $\otimes$ is finitary in both variables. For every finitary functor

$$H : \mathcal{A} \longrightarrow \mathsf{Mon}(\mathcal{A}, \otimes)$$

into the category of all monoids in $(\mathcal{A}, \otimes)$ we obtain the following base

$$X \,\square\, A = HX \otimes A.$$

In fact, each $HX \otimes -$ is a finitary monad on $\mathcal{A}$ and the functoriality is obvious.

The above examples (1)–(3) are obtained as follows:

(1) $\otimes$ is the coproduct on $\mathcal{A}$ and $HX = X$ with the obvious structure of a monoid,
(2) $\otimes$ is the product on $\mathcal{A}$ and $HX = X^*$,
(3) $\otimes$ is the composition on $\mathsf{Fin}[\mathcal{B}, \mathcal{B}]$ and, like in (2), HX is the free monoid on X.

(5) In [AMV$_3$], Proposition 4.6, we studied free bases on parametrized signatures in Set. For example, the parametrized signature of one ternary operation of iterativity 1 leads to the base

$X \,\square\, A =$ all labelled finite binary trees with inner nodes labelled in X and leaves labelled in A.

Example 2.6. There are various ways how one can construct new bases from given ones. For example:

(1) Let $\Box : \mathcal{A}\times\mathcal{A} \longrightarrow \mathcal{A}$ be a base. We obtain other bases $\dot{\Box}$ by precomposing $\Box$ with finitary endofunctors $H : \mathcal{A} \longrightarrow \mathcal{A}$:

$$X \mathbin{\dot{\Box}} A = HX \Box A$$

More precisely:

$$\dot{\Box} \equiv \mathcal{A} \xrightarrow{H} \mathcal{A} \xrightarrow{\Box} \mathsf{FM}(\mathcal{A})$$

with base unit

$$\dot{u}_A^X = u_A^{HX} : A \longrightarrow HX \Box A$$

and base multiplication

$$\dot{m}_A^X = m_A^{HX} : HX \Box (HX \Box A) \longrightarrow HX \Box A.$$

Of particular importance are the bases obtained from the base $+$ of Example 2.5(1), i.e.,

$$X \mathbin{\dot{\Box}} A = HX + A.$$

We will show below that our previous results of [AMV$_1$] on H-algebras are special cases of the results concerning these bases.

(2) Let $F \dashv U : \mathcal{A} \longrightarrow \mathcal{B}$ be a finitary adjunction. Then the functor $[U, F] : \mathsf{Fin}[\mathcal{A}, \mathcal{A}] \longrightarrow \mathsf{Fin}[\mathcal{B}, \mathcal{B}]$ is lax monoidal and therefore it lifts to the respective categories of finitary monads. For every base $\Box$ on $\mathcal{A}$ we have a new base

$$\dot{\Box} \equiv \mathcal{B} \xrightarrow{F} \mathcal{A} \xrightarrow{\Box} \mathsf{FM}(\mathcal{A}) \xrightarrow{[U,F]} \mathsf{FM}(\mathcal{B})$$

on $\mathcal{B}$. That is, the new base is given on objects by

$$X \mathbin{\dot{\Box}} A = U(FX \Box FA)$$

This is a special case of the following:

(3) Every finitary lax monoidal functor $P : \mathcal{V} \longrightarrow \mathsf{Fin}[\mathcal{A}, \mathcal{A}]$ ($\mathcal{V}$ a monoidal category) lifts canonically to the respective categories of monoids (monoids in $\mathsf{Fin}[\mathcal{A}, \mathcal{A}]$ being, of course, finitary monads). Thus, one may construct a new base by giving a finitary functor $H : \mathcal{A} \longrightarrow \mathsf{Mon}(\mathcal{V})$ to the category of monoids in $\mathcal{V}$. The new base $\dot{\Box}$ is a composite

$$\dot{\Box} \equiv \mathcal{A} \xrightarrow{H} \mathsf{Mon}(\mathcal{V}) \xrightarrow{P} \mathsf{FM}(\mathcal{A})$$

In fact, every base $\Box : \mathcal{A} \longrightarrow \mathsf{FM}(\mathcal{A})$ can be (trivially) decomposed in the above manner: it is easy to prove that giving a base is equivalent to giving a lax monoidal functor $P : \mathcal{A} \longrightarrow \mathsf{Fin}[\mathcal{A}, \mathcal{A}]$, where $\mathcal{A}$ is endowed with a monoidal structure of binary coproduct. Since $\mathsf{Mon}(\mathcal{A}) = \mathcal{A}$, it then suffices to put $H = Id$.

Definition 2.7. Given a base $\Box$, by a *base algebra* is understood an object A of $\mathcal{A}$ together with an Eilenberg-Moore algebra on A of the monad $A \Box -$.

That is, a base algebra is given by a morphism $\alpha : A \,\Box\, A \longrightarrow A$ such that the following two diagrams

$$\begin{tikzcd} A \arrow[r, "u_A^A"] \arrow[dr, equal] & A \,\Box\, A \arrow[d, "\alpha"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} A \,\Box\, (A \,\Box\, A) \arrow[r, "A\Box\alpha"] \arrow[d, "m_A^A"'] & A \,\Box\, A \arrow[d, "\alpha"] \\ A \,\Box\, A \arrow[r, "\alpha"'] & A \end{tikzcd} \tag{2.5}$$

commute.

Notation 2.8. We denote by

$$\mathsf{Alg}\,\Box$$

the category of all base algebras and all *homomorphisms* from (A, α) to (B, β), i.e., morphisms $h : A \longrightarrow B$ such that the square

$$\begin{tikzcd} A \,\Box\, A \arrow[r, "\alpha"] \arrow[d, "h\Box h"'] & A \arrow[d, "h"] \\ B \,\Box\, B \arrow[r, "\beta"'] & B \end{tikzcd} \tag{2.6}$$

commutes.

Examples 2.9.

(1) Algebras for the base $X \,\Box\, A = X + A$ are given by an object A and an endomorphism of A (i.e., these are just unary algebras in $\mathcal{A}$). Homomorphisms are also the usual homomorphisms of unary algebras.

(2) Algebras for the base $X \,\Box\, A = HX + A$ are the usual H-algebras, i.e., pairs consisting of an object A and a morphism $\alpha : HA \longrightarrow A$. Also, homomorphisms are the usual H-algebra homomorphisms. Thus,

$$\mathsf{Alg}\,\Box = \mathsf{Alg}\,H$$

is the category of H-algebras and homomorphisms.

(3) Algebras for the base on Set given by

$$X \,\Box\, A = X^* \times A$$

(see Example 2.5(2)) are precisely the usual algebras on one binary operation. In fact, given such an algebra, say,

$$\alpha : A \times A \longrightarrow A$$

define $\widehat{\alpha} : A^* \times A \longrightarrow A$ by

$$\widehat{\alpha}(a_1 a_2 \dots a_n, a) = \alpha(a_1, \alpha(a_2, \dots (\alpha(a_n, a)) \dots)). \tag{2.7}$$

This satisfies (2.5). Conversely, given $\widehat{\alpha} : A^* \times A \longrightarrow A$ satisfying (2.5), it is given by the above formula (2.7) where α denotes the restriction of $\widehat{\alpha}$ to all pairs in $A \times A$. Consequently, the bases

$$X \mathbin{\Box} A = (X \times X) + A \quad \text{and} \quad X \mathbin{\Box} A = X^* \times A$$

on Set define the same categories of algebras.

(4) Let $\mathcal{B}$ be any locally finitely presentable category and put $\mathcal{A} = \mathsf{Fin}[\mathcal{B}, \mathcal{B}]$ with the base

$$X \mathbin{\Box} A = \mathcal{F}(X) \cdot A$$

(see Example 2.5(3)). A base algebra is a pair (A, α) consisting of a finitary endofunctor $A : \mathcal{B} \longrightarrow \mathcal{B}$ and a natural transformation $\alpha : A \cdot A \longrightarrow A$.

More precisely: each such pair defines a unique natural transformation $\overline{\alpha}$ from A to $\langle A, A\rangle$ (the right Kan extension of A along A). Since $\langle A, A\rangle$ is always a monad, see [K], and $\mathcal{F}(A)$ is a free monad on A, $\overline{\alpha}$ yields a unique monad morphism

$$\widetilde{\alpha} : \mathcal{F}(A) \longrightarrow \langle A, A\rangle.$$

The unique natural transformation

$$\widehat{\alpha} : \mathcal{F}(A) \cdot A \longrightarrow A$$

corresponding to $\widetilde{\alpha}$ defines an algebra for our base — in fact, the condition (2.5) above is equivalent to $\widetilde{\alpha}$ being a monad morphism.

Proposition 2.10. *The category* $\mathsf{Alg}\,\Box$ *is locally finitely presentable, and its forgetful functor into* $\mathcal{A}$ *has a left adjoint, i.e., free base algebras exist on every object of* $\mathcal{A}$.

Proof. The endofunctor $SA = A \mathbin{\Box} A$ of $\mathcal{A}$ is finitary, and thus the category $\mathsf{Alg}\, S$ is locally finitely presentable and its forgetful functor has a left adjoint, see Corollary 2.75 of [AR].

The category $\mathsf{Alg}\,\Box$ is a full subcategory of $\mathsf{Alg}\, S$, and it is easy to verify that it is closed under limits and filtered colimits in $\mathsf{Alg}\, S$. It then follows that it is a reflective subcategory, see Theorem 2.48 in [AR]. Since the forgetful functor of $\mathsf{Alg}\,\Box$ is a restriction of that of $\mathsf{Alg}\, S$, the proposition follows. ☐

Examples 2.11.

(1) Free algebras for $X + A$ are the free unary algebras $A \longmapsto \mathbb{N} \bullet A$ (a countable copower of A).

(2) Free algebras for $HX + A$ are the usual free H-algebras. For example, the algebra of all finite binary trees on A if $HX = X \times X$.

(3) Free algebras of $X^* \times A$ in Set are the free binary algebras on A (of all finite binary trees on A).
(4) The base of Example 2.5(5) yields the free ternary algebras (of all finite ternary trees on A).

Notation 2.12. We denote by $\mathbb{F} = (F, \eta, \mu)$ the finitary monad on $\mathcal{A}$ given by free algebras for the given base. The structure of a free base algebra on X is denoted by

$$\varphi_X : FX \,\square\, FX \longrightarrow FX. \tag{2.8}$$

Then the multiplication μ_X can be described as the extension of $id : FX \longrightarrow FX$ along η_{FX} to a homomorphism $\mu_X : (FFX, \varphi_{FX}) \longrightarrow (FX, \varphi_X)$.

In the non-parametrized case of algebras for a given finitary endofunctor H, the category $\mathsf{Alg}\, H$ of algebras is monadic, thus, isomorphic to the category of monadic algebras of the free monad on H, see [B]. This holds in the parametric case too:

Proposition 2.13. *For every base $\square$ the forgetful functor*

$$U_\square : \mathsf{Alg}\,\square \longrightarrow \mathcal{A}$$

is monadic.

Proof. The functor $U_\square$ has a left adjoint by Proposition 2.10. We prove that $U_\square$ creates coequalizers of pairs which have an absolute coequalizer in $\mathcal{A}$. This proves monadicity by Beck's Theorem, see [ML].

Consider a parallel pair

$$(A, \alpha) \underset{g}{\overset{f}{\rightrightarrows}} (B, \beta)$$

of morphisms in $\mathsf{Alg}\,\square$ such that its image under $U_\square$ has an absolute coequalizer $c : B \longrightarrow C$ in $\mathcal{A}$.

Since, for the finitary functor $SA = A \,\square\, A$, the forgetful functor $\mathsf{Alg}\, S \longrightarrow \mathcal{A}$ is monadic (see [B]), there exists a unique structure $\gamma : C \,\square\, C \longrightarrow C$ of an S-algebra on C such that c is a homomorphism, and c is a coequalizer in $\mathsf{Alg}\, S$. Thus, it suffices to show that (C, γ) is an algebra for $\square$:

(1) To prove $\gamma \cdot u_C^C = id_C$, consider the commutative diagram

$$\begin{tikzcd}
B \arrow[rr, equal] \arrow[dr, "u_B^B"] \arrow[dd, "c"'] & & B \arrow[dd, "c"] \\
& B \,\square\, B \arrow[ur, "\beta"'] \arrow[d, "c\square c"'] & \\
C \arrow[r, "u_C^C"'] & C \,\square\, C \arrow[r, "\gamma"'] & C
\end{tikzcd}$$

The desired equality follows since c is an epimorphism.

(2) To prove $\gamma \cdot (C \,\square\, \gamma) = \gamma \cdot m_C^C$, consider the diagram

$$\begin{tikzcd}
B \,\square\, B \arrow[ddd, "\beta"'] \arrow[ddr, "c\square c"] & & & B \,\square\, (B \,\square\, B) \arrow[lll, "m_B^B"'] \arrow[rrr, "B\square\beta"] \arrow[d, "c\square(c\square c)"] \arrow[ddll, phantom, "(2.4)"] \arrow[ddrr, phantom, "(2.4)"] & & & B \,\square\, B \arrow[ddd, "\beta"] \arrow[ddl, "c\square c"'] \\
& & & C \,\square\, (C \,\square\, C) \arrow[dll, "m_C^C"'] \arrow[drr, "C\square\gamma"] & & & \\
& C \,\square\, C \arrow[drr, "\gamma"] & & & & C \,\square\, C \arrow[dll, "\gamma"'] & \\
B \arrow[rrr, "c"'] \arrow[uur, phantom, "(2.6)"] & & & C & & & B \arrow[lll, "c"] \arrow[uul, phantom, "(2.6)"']
\end{tikzcd}$$

whose perimeter commutes, since (B, β) is a base algebra. The left-hand side and the right-hand side commute as well as the two upper parts commute as indicated. Now use the fact that $c \,\square\, (c \,\square\, c)$ is an epimorphism to see that the lower diamond commutes as desired.

□

Proposition 2.14. *The functor $X \longmapsto FX \,\square\, X$ is naturally isomorphic to F via the composite*

$$j_X \equiv FX \,\square\, X \xrightarrow{FX \square \eta_X} FX \,\square\, FX \xrightarrow{\varphi_X} FX \tag{2.9}$$

Proof. (1) It is obvious that j_X is natural in X, we prove that it has an inverse.

(2) We will prove that $FX \,\square\, X$ has an algebra structure ω_X such that j_X becomes a homomorphism into (FX, φ_X). We define ω_X to be the following composite

$$(FX \,\square\, X) \,\square\, (FX \,\square\, X) \xrightarrow{j_X \square (FX \square X)} FX \,\square\, (FX \,\square\, X) \xrightarrow{m_X^{FX}} FX \,\square\, X\,. \tag{2.10}$$

The unit law for algebras on $\square$ follows from the commutative diagram

$$
\begin{tikzcd}
FX \square X \arrow[r, "u^{FX\square X}_{FX\square X}"] \arrow[d, equal] & (FX \square X) \square (FX \square X) \arrow[d, "j_X\square(FX\square X)"] \arrow[dd, bend left=60, "\omega_X"] \\
FX \square X \arrow[r, "u^{FX}_{FX\square X}"] \arrow[dr, equal] \arrow[ur, phantom, "(2.3)"] \arrow[r, phantom, shift right=4ex, "(2.1)"] & FX \square (FX \square X) \arrow[d, "m^{FX}_X"] \\
& FX \square X
\end{tikzcd}
$$

while the associativity law follows from the diagram below, in which we denote, for short, $D = FX \square X$:

$$
\begin{tikzcd}
D \square (D \square D) \arrow[r, "D\square(j_X\square D)"] \arrow[rr, bend left, "D\square\omega_X"] \arrow[dr, "j_X\square(j_X\square D)"] \arrow[dd, "m^D_D"'] & D \square (FX \square D) \arrow[r, "D\square(D\square m^{FX}_X)"] & D \square D \arrow[d, "j_X\square D"] \\
\arrow[r, phantom, "(2.4)"] & FX \square (FX \square D) \arrow[r, "FX\square m^{FX}_X"] \arrow[d, "m^{FX}_D"'] \arrow[dr, phantom, "(2.2)"] & FX \square D \arrow[d, "m^{FX}_X"] \\
D \square D \arrow[r, "j_X\square D"'] \arrow[rr, bend right, "\omega_X"'] & FX \square D \arrow[r, "m^{FX}_X"'] & D
\end{tikzcd}
$$

The two lower parts commute as indicated and the upper part commutes by the naturality of $\square$. We now verify that $j_X : FX \square X \longrightarrow FX$ is indeed a homomorphism. In fact, denote $C = FX$ for short and consider the diagram below:

$$
\begin{tikzcd}
(C \square X) \square (C \square X) \arrow[r, "(C\square\eta_X)\square(C\square\eta_X)"] \arrow[rr, bend left, "j_X\square j_X"] \arrow[d, "(C\square\eta_X)\square id"'] \arrow[ddd, bend right=60, "\omega_X"'] & (C \square C) \square (C \square C) \arrow[r, "\varphi_X\square\varphi_X"] \arrow[d, "\varphi_X\square id"] & C \square C \arrow[ddd, "\varphi_X"] \\
(C \square C) \square (C \square X) \arrow[ur, "(C\square C)\square(C\square\eta_X)"] \arrow[d, "\varphi_X\square id"'] & C \square (C \square C) \arrow[ur, "id\square\varphi_X"'] \arrow[dd, "m^C_C"] \arrow[ddr, phantom, "(2.5)"] & \\
C \square (C \square X) \arrow[ur, "C\square(C\square\eta_X)"] \arrow[d, "m^C_X"'] \arrow[r, phantom, "(2.4)"] & & \\
C \square X \arrow[r, "C\square\eta_X"'] \arrow[rr, bend right, "j_X"'] & C \square C \arrow[r, "\varphi_X"'] & C
\end{tikzcd}
$$

Finally, the equality $j_X \cdot u_X^{FX} = \eta_X$ follows from the diagram

$$\begin{tikzcd}
FX \square X \arrow[r, "FX\square\eta_X"] \arrow[rr, bend left, "j_X"] & FX \square FX \arrow[r, "\varphi_X"] & FX \\
X \arrow[u, "u_X^{FX}"] \arrow[r, "\eta_X"'] \arrow[ur, phantom, "(2.3)"] & FX \arrow[u, "u_{FX}^{FX}"'] \arrow[ur, equal] \arrow[ur, phantom, "(2.5)", near end] &
\end{tikzcd}$$

(3) By freeness of (FX, φ_X), we obtain a unique homomorphism

$$i_X : (FX, \varphi_X) \longrightarrow (FX \square X, \omega_X)$$

such that $i_X \cdot \eta_X = u_X^{FX}$. It follows that $j_X \cdot i_X = id$.

To prove that $i_X \cdot j_X = id$, consider the following diagram

$$\begin{tikzcd}
FX \square X \arrow[r, "i_X\square X"] \arrow[d, "FX\square\eta_X"'] \arrow[dd, bend right=60, "j_X"'] & (FX \square X) \square X \arrow[d, "(FX\square X)\square u_X^{FX}"] \arrow[dd, bend left=60, "j_X\square X"] \\
FX \square FX \arrow[r, "i_X\square i_X"] \arrow[d, "\varphi_X"'] & (FX \square X) \square (FX \square X) \ (*) \arrow[d, "\omega_X"] \\
FX \arrow[r, "i_X"'] & FX \square X
\end{tikzcd}$$

where both middle squares commute by definition of i_X, and the area $(*)$ commutes, since the diagram

$$\begin{tikzcd}
(FX \square X) \square X \arrow[r, "(FX\square X)\square u_X^{FX}"] \arrow[d, "(FX\square\eta_X)\square X"'] \arrow[dd, bend right=60, "j_X\square X"'] & (FX \square X) \square (FX \square X) \arrow[d, "(FX\square\eta_X)\square(FX\square X)"] \arrow[ddd, bend left=60, "\omega_X"] \\
(FX \square FX) \square X \arrow[r, "(FX\square FX)\square u_X^{FX}"] \arrow[d, "\varphi_X\square X"'] & (FX \square FX) \square (FX \square X) \arrow[d, "\varphi_X\square(FX\square X)"] \\
FX \square X \arrow[r, "FX\square u_X^{FX}"] \arrow[dr, equal] & FX \square (FX \square X) \arrow[d, "m_X^{FX}"] \\
& FX \square X
\end{tikzcd}$$

does. $\square$

Definition 2.15. The isomorphism $j_X : FX \square X \longrightarrow FX$ is called a *constructor* of the base $\square$.

Corollary 2.16. *The functor*

$$SX = FX \square X$$

carries a monad structure $(S, \overline{\eta}, \overline{\mu})$ for which the constructor is a monad isomorphism.

Explicitly:

$$\overline{\eta}_X \equiv X \xrightarrow{u_X^{FX}} FX \,\square\, X$$

and

$$\overline{\mu}_X \equiv \begin{tikzcd} SSX = F(FX \,\square\, X) \,\square\, (FX \,\square\, X) \arrow[d, "Fj_X \square id"] \\ FFX \,\square\, (FX \,\square\, X) \arrow[d, "\mu_X \square id"] \\ FX \,\square\, (FX \,\square\, X) \arrow[d, "m_X^{FX}"] \\ SX = FX \,\square\, X \end{tikzcd}$$

Proof. Firstly observe that both $\overline{\eta}_X$ and $\overline{\mu}_X$ are natural in X by definition. The equality $\overline{\mu}_X \cdot \overline{\eta}_{SX} = id_{SX}$ follows from the commutative diagram

$$\begin{tikzcd}
FX \,\square\, X \arrow[r, "u_{FX \square X}^{F(FX \square X)}"] \arrow[d, equal] \arrow[dr, phantom, "(2.3)"] & F(FX \,\square\, X) \,\square\, (FX \,\square\, X) \arrow[d, "Fj_X \square id"] \arrow[ddd, bend left=60, "\overline{\mu}_X"] \\
FX \,\square\, X \arrow[r, "u_{FX \square X}^{FFX}"] \arrow[d, equal] \arrow[dr, phantom, "(2.3)"] & FFX \,\square\, (FX \,\square\, X) \arrow[d, "\mu_X \square (FX \square X)"] \\
FX \,\square\, X \arrow[r, "u_{FX \square X}^{FX}"] \arrow[dr, equal] \arrow[dr, phantom, "(2.1)"] & FX \,\square\, (FX \,\square\, X) \arrow[d, "m_X^{FX}"] \\
& FX \,\square\, X
\end{tikzcd}$$

and the equality $\overline{\mu}_X \cdot S\overline{\eta}_X = id_{SX}$ follows from

$$\begin{tikzcd}
FX \,\square\, X \arrow[r, "Fu_X^{FX} \square u_X^{FX}"] \arrow[d, equal] \arrow[dr, phantom, "(2.9)"] & F(FX \,\square\, X) \,\square\, (FX \,\square\, X) \arrow[d, "Fj_X \square id"] \arrow[ddd, bend left=60, "\overline{\mu}_X"] \\
FX \,\square\, X \arrow[r, "F\eta_X \square u_X^{FX}"] \arrow[d, equal] & FFX \,\square\, (FX \,\square\, X) \arrow[d, "\mu_X \square (FX \square X)"] \\
FX \,\square\, X \arrow[r, "FX \square u_X^{FX}"] \arrow[dr, equal] \arrow[dr, phantom, "(2.1)"] & FX \,\square\, (FX \,\square\, X) \arrow[d, "m_X^{FX}"] \\
& FX \,\square\, X
\end{tikzcd}$$

where the lower square commutes by monad law for η and μ, and the upper one commutes since $\eta_X = j_X \cdot u_X^{FX}$ (which follows easily from (2.9), (2.3) and (2.5)).

The associativity of $\overline{\mu}$ is proved in analogously straightforward manner.

Since the forgetful functor $\mathsf{FM}(\mathcal{A}) \longrightarrow \mathsf{Fin}[\mathcal{A},\mathcal{A}]$, being monadic, reflects isomorphisms, it follows that the constructor j_X is indeed an isomorphism of monads. $\square$

Remark 2.17. It is a trivial observation about algebras for an endofunctor H that

$$\text{a free } H\text{-algebra on } X \equiv \text{an initial algebra for } H(-) + X.$$

This follows immediately from the fact that an algebra for $H(-) + X$ on an object A is precisely a pair consisting of an H-algebra on A and a morphism $X \longrightarrow A$. We are going to prove the same result for base algebras. It turns out that the proof is more involved.

Theorem 2.18. *A free base algebra on an object X is precisely an initial algebra for the endofunctor $- \,\square\, X$. More precisely:*

(a) The free base algebra FX is an initial algebra for $- \,\square\, X$ via the constructor j_X.
(b) Let $F'X$ denote an initial algebra for $- \,\square\, X$ with the structure morphism

$$j'_X : F'X \,\square\, X \longrightarrow F'X.$$

Then $F'X$ carries the following structure of a base algebra

$$\begin{tikzcd} F'X \,\square\, F'X \arrow[d, "F'X \square j'^{-1}_X"] \arrow[ddd, bend right=90, "\varphi_X"'] \\ F'X \,\square\, (F'X \,\square\, X) \arrow[d, "m^{F'X}_X"] \\ F'X \,\square\, X \arrow[d, "j'_X"] \\ F'X \end{tikzcd} \tag{2.11}$$

which is free with the universal arrow

$$\eta_X \equiv X \xrightarrow{u^{F'X}_X} F'X \,\square\, X \xrightarrow{j'_X} F'X \tag{2.12}$$

Remark. Observe that whereas in the previous text the role of X (in $X \,\square\, A$) was to fix the monad $X \,\square\, -$ (of variable object A), the above theorem switches the roles: here the right-hand variable is fixed and called X. An algebra for $- \,\square\, X$ is simply an object C together with a morphism $\gamma : C \,\square\, X \longrightarrow C$.

Proof. (a) Let FX be a free base algebra. For every algebra

$$\gamma : C \Box X \longrightarrow C$$

for $- \Box X$ we prove that there exists a unique homomorphism from (FX, j_X) to (C, γ).

Let us prove that $C \Box X$ is a base algebra with the structure morphism

$$\overline{\gamma} \equiv (C \Box X) \Box (C \Box X) \xrightarrow{\gamma \Box (C \Box X)} C \Box (C \Box X) \xrightarrow{m^C_X} C \Box X. \tag{2.13}$$

In fact, the triangle of (2.5) commutes due to the commutative diagram

$$\begin{tikzcd}
C \Box X \arrow[r, "u^{C\Box X}_{C\Box X}"] \arrow[dr, "u^{C}_{C\Box X}"'] \arrow[dr, phantom, "(2.3)", near end, shift left=3] \arrow[ddr, bend right, "id"'] & C \Box C \arrow[d, "\gamma\Box(C\Box X)"] \\
(2.1) & C \Box (C \Box X) \arrow[d, "m^C_X"] \\
 & C \Box X
\end{tikzcd}$$

and the square of (2.5) commutes due to the commutative diagram

$$\begin{tikzcd}
 & (C\Box X)\Box((C\Box X)\Box(C\Box X)) \arrow[rr, "m^{C\Box X}_{C\Box X}"] \arrow[d, "(C\Box X)\Box(\gamma\Box(C\Box X))"] \arrow[dr, "\gamma\Box(\gamma\Box(C\Box X))"] \arrow[dd, bend right=80, "(C\Box X)\Box\overline{\gamma}"'] & & (C\Box X)\Box(C\Box X) \arrow[d, "\gamma\Box(C\Box X)"] \arrow[dd, bend left=80, "\overline{\gamma}"] & \\
(2.13) & (C\Box X)\Box(C\Box(C\Box X)) \arrow[r, "\gamma\Box id"'] \arrow[d, "(C\Box X)\Box m^C_X"] & C\Box(C\Box(C\Box X)) \arrow[r, "m^C_{C\Box X}"'] \arrow[d, "C\Box m^C_X"] \arrow[ur, phantom, "(2.4)"] \arrow[dr, phantom, "(2.2)"] & C\Box(C\Box X) \arrow[d, "m^C_X"] & (2.13) \\
 & (C\Box X)\Box(C\Box X) \arrow[r, "\gamma\Box(C\Box X)"'] \arrow[rr, bend right, "\overline{\gamma}"'] & C\Box(C\Box X) \arrow[r, "m^C_X"'] & C\Box X &
\end{tikzcd}$$

Now invoke the universal property of FX to obtain a unique homomorphism k of base algebras from (FX, φ_X) to $(C \Box X, \overline{\gamma})$ such that $k \cdot \eta_X = u^C_X$:

$$\begin{tikzcd}
FX \Box FX \arrow[r, "\varphi_X"] \arrow[d, "k\Box k"'] & FX \arrow[d, "k"] & X \arrow[l, "\eta_X"'] \arrow[dl, "u^C_X"] \\
(C \Box X) \Box (C \Box X) \arrow[r, "\overline{\gamma}"'] & C \Box X &
\end{tikzcd} \tag{2.14}$$

We form the morphism

$$h \equiv FX \xrightarrow{k} C \Box X \xrightarrow{\gamma} C$$

and show that it is a homomorphism of $(- \Box X)$-algebras:

$$
\begin{tikzcd}
FX\Box X \arrow[r, "FX\Box\eta_X"] \arrow[d, "k\Box X"'] \arrow[rrr, bend left, "j_X"] \arrow[dd, bend right=60, "h\Box X"'] \arrow[dr, phantom, "(2.14)"] & FX\Box FX \arrow[rr, "\varphi_X"] \arrow[d, "k\Box k"] \arrow[drr, phantom, "(2.14)"] & & FX \arrow[d, "k"] \arrow[dd, bend left=60, "h"] \\
(C\Box X)\Box X \arrow[r, "id\Box u^C_X"] \arrow[d, "\gamma\Box X"'] & (C\Box X)\Box(C\Box X) \arrow[r, "\gamma\Box id"] & C\Box(C\Box X) \arrow[r, "m^C_X"] \arrow[d, phantom, "(2.1)"] & C\Box X \arrow[d, "\gamma"] \\
C\Box X \arrow[urr, "C\Box u^C_X"] \arrow[urrr, bend right, "id"'] \arrow[rrr, "\gamma"'] & & & C
\end{tikzcd}
\tag{2.15}
$$

Next we prove the uniqueness of h. So suppose that h is any homomorphism of $(- \Box X)$-algebras from (FX, j_X) to (C, γ). By Proposition 2.14, $j_X : FX \Box X \longrightarrow FX$ is a homomorphism of base algebras with $j_X \cdot u_X^{FX} = \eta_X$ and it is an isomorphism in $\mathcal{A}$, whence in $\mathsf{Alg}\,\Box$ (since the forgetful functor $U_\Box : \mathsf{Alg}\,\Box \longrightarrow \mathcal{A}$ reflects isomorphisms). Denote as before by i_X the inverse of j_X. We prove that

$$FX \xrightarrow{i_X} FX \Box X \xrightarrow{h\Box X} C \Box X$$

is a homomorphism of base algebras such that $(h \Box X) \cdot i_X \cdot \eta_X = u_X^C$. In fact, the latter is clear by (2.3) since $i_X \cdot \eta_X = u_X^{FX}$. For the former we only need to establish that $h \Box X$ is a homomorphism:

$$
\begin{tikzcd}
(FX \Box X) \Box (FX \Box X) \arrow[rr, "(h\Box X)\Box(h\Box X)"] \arrow[d, "(FX\Box\eta_X)\Box id"'] \arrow[dd, bend left=60, "j_X\Box id"] \arrow[ddd, bend right=80, "\omega_X"'] \arrow[dr, phantom, "(2.9)"] & & (C \Box X) \Box (C \Box X) \arrow[dd, "\gamma\Box id"] \arrow[ddd, bend left=80, "\overline{\gamma}"] \arrow[ddl, phantom, "(2.15)"] \\
(FX \Box FX) \Box (FX \Box X) \arrow[d, "\varphi_X\Box id"'] & & \\
FX \Box (FX \Box X) \arrow[rr, "h\Box(h\Box X)"] \arrow[d, "m_X^{FX}"'] \arrow[l, phantom, "(2.10)"] \arrow[drr, phantom, "(2.4)"] & & C \Box (C \Box X) \arrow[d, "m_X^C"] \arrow[r, phantom, "(2.13)"] & {} \\
FX \Box X \arrow[rr, "h\Box X"'] & & C \Box X
\end{tikzcd}
$$

Now invoke the universal property of the free base algebra FX to conclude that $k = (h \Box X) \cdot i_X$, from which we obtain

$$h = \gamma \cdot (h \Box X) \cdot i_X = \gamma \cdot k$$

as desired.

(b) We now verify that (2.11) above defines a base algebra. The triangle of (2.5) follows from the commutative diagram

$$
\begin{tikzcd}
F'X \arrow[rr, "u^{F'X}_{F'X}"] \arrow[dr, "j'^{-1}_X"] \arrow[rrddd, bend right, "id"'] & & F'X \,\square\, F'X \arrow[d, "FX\square j'^{-1}_X"] \arrow[ddd, bend left, "\varphi_X"] \\
& F'X \,\square\, X \arrow[r, "u^{F'X}_{F'X\square X}"] \arrow[dr, equal] \arrow[ur, phantom, "(2.3)"] & F'X \,\square\, (F'X \,\square\, X) \arrow[d, "m^{F'X}_X"] \arrow[d, phantom, shift right=8ex, "(2.1)"] \arrow[d, phantom, shift left=10ex, "(2.11)"] \\
& & F'X \,\square\, X \arrow[d, "j'_X"] \\
& & F'X
\end{tikzcd}
$$

The square of (2.5) follows from the commutative diagram

$$
\begin{tikzcd}
F'X\square(F'X\square F'X) \arrow[r, "F'X\square(F'X\square j'^{-1}_X)"] \arrow[ddd, "m^{F'X}_{F'X}"'] \arrow[dddr, phantom, "(2.4)"] & F'X\square(F'X\square(F'X\square X)) \arrow[r, "F'X\square m^{F'X}_X"] \arrow[ddd, "m^{F'X}_{F'X\square X}"] \arrow[dddr, phantom, "(2.1)"] & F'X\square(F'X\square X) \arrow[r, "F'X\square j'_X"] \arrow[ddd, "m^{F'X}_X"] \arrow[dr, equal] & F'X\square F'X \arrow[d, "FX\square j'^{-1}_X"] \\
& & & F'X\square(F'X\square X) \arrow[d, "m^{F'X}_X"] \\
& & & F'X\square X \arrow[d, "j'_X"] \\
F'X\square F'X \arrow[r, "F'X\square j'^{-1}_X"'] & F'X\square(F'X\square X) \arrow[r, "m^{F'X}_X"'] & F'X\square X \arrow[r, "j'_X"'] \arrow[ur, equal] & F'X
\end{tikzcd}
$$

Now we check the universal property of $F'X$. Suppose we have a base algebra (C, γ) and a morphism $f : X \longrightarrow C$. We form the following algebra for the functor $- \,\square\, X$:

$$
\overline{\gamma} \equiv C \,\square\, X \xrightarrow{C\square f} C \,\square\, C \xrightarrow{\gamma} C
$$

Thus we obtain a unique $(- \,\square\, X)$-algebra homomorphism h from $(F'X, j'_X)$ to $(C, \overline{\gamma})$, i.e., we have the commutative diagram

$$
\begin{tikzcd}
F'X \,\square\, X \arrow[rr, "j'_X"] \arrow[d, "h\square X"'] \arrow[dr, "h\square f"] & & F'X \arrow[d, "h"] \\
C \,\square\, X \arrow[r, "C\square f"'] & C \,\square\, C \arrow[r, "\gamma"'] & C
\end{tikzcd}
\tag{2.16}
$$

To see that h extends f consider the diagram

$$\begin{tikzcd}
F'X \,\Box\, X \arrow[rr, "{j'}_X"] \arrow[ddd, "h \Box f"'] & \arrow[d, phantom, "(2.12)"] & F'X \arrow[ddd, "h"] \\
 & X \arrow[ul, "u_X^{F'X}"] \arrow[ur, "\eta_X"'] \arrow[d, "f"] \arrow[l, phantom, "(2.3)"] & \\
 & C \arrow[dl, "u_C^C"'] \arrow[dr, equal] \arrow[d, phantom, "(2.5)"] & \\
C \,\Box\, C \arrow[rr, "\gamma"'] & \, & C
\end{tikzcd}$$

The outward square commutes, thus we obtain the commutativity of the inner right-hand part, i.e., $h \cdot \eta_X = f$, since all other inner parts commute. To see that h is a base-algebra homomorphism, consider the commutative diagram

$$\begin{tikzcd}
F'X \,\Box\, F'X \arrow[rrr, "\varphi_X"] \arrow[dr, "F'X \Box j_X'^{-1}"] \arrow[ddd, "h \Box h"'] & & & F'X \arrow[ddd, "h"] \\
 & F'X \,\Box\, (F'X \,\Box\, X) \arrow[r, "m_X^{F'X}"] \arrow[d, "h \Box (h \Box f)"] \arrow[l, phantom, "(2.16)"] & F'X \,\Box\, X \arrow[ur, "j'_X"] \arrow[d, "h \Box f"] \arrow[r, phantom, "(2.16)"] \arrow[l, phantom, "(2.11)", shift left=3ex] \arrow[dl, phantom, "(2.4)"] & \, \\
 & C \,\Box\, (C \,\Box\, C) \arrow[r, "m_C^C"'] \arrow[dl, "C \Box \gamma"] & C \,\Box\, C \arrow[dr, "\gamma"] \arrow[dl, phantom, "(2.5)"] & \\
C \,\Box\, C \arrow[rrr, "\gamma"'] & & & C
\end{tikzcd} \tag{2.17}$$

To prove the uniqueness, let $k : (F'X, \varphi_X) \longrightarrow (C, \gamma)$ be a homomorphism of base algebras with $k \cdot \eta_X = f$, i.e., let the diagram

$$\begin{tikzcd}
F'X \,\Box\, X \arrow[r, "F'X \Box \eta_X"] \arrow[dr, "k \Box f"'] & F'X \,\Box\, F'X \arrow[r, "\varphi_X"] \arrow[d, "k \Box k"] & F'X \arrow[d, "k"] \\
 & C \,\Box\, C \arrow[r, "\gamma"'] & C
\end{tikzcd} \tag{2.18}$$

commute. It is our task to prove that (2.16) commutes (for k in place of h) and for this all we have to observe is that the upper horizontal passage in diagram (2.18) is j'_X:

$$\begin{tikzcd}
F'X \,\Box\, X \arrow[rr, "F'X \Box \eta_X"] \arrow[dr, "F'X \Box u_X^{F'X}"] \arrow[ddrr, bend right, "id"'] & \arrow[d, phantom, "(2.12)"] & F'X \,\Box\, F'X \arrow[r, "\varphi_X"] \arrow[d, "F'X \Box j_X'^{-1}"] & F'X \\
 & F'X \,\Box\, (F'X \,\Box\, X) \arrow[ur, "F'X \Box j'_X"] \arrow[r, equal] \arrow[d, phantom, "(2.1)"] & F'X \,\Box\, (F'X \,\Box\, X) \arrow[d, "m_X^{F'X}"] \arrow[r, phantom, "(2.11)"] & \, \\
 & \, & F'X \,\Box\, X \arrow[uur, bend right, "j'_X"'] &
\end{tikzcd}$$

$\Box$

Corollary 2.19. *A free base algebra on X can be constructed as a colimit of the following ω-chain in $\mathcal{A}$*

$$0 \xrightarrow{d} 0 \,\Box\, X \xrightarrow{d \Box X} (0 \,\Box\, X) \,\Box\, X \xrightarrow{(d\Box X)\Box X} ((0 \,\Box\, X) \,\Box\, X) \,\Box\, X \longrightarrow \cdots$$

where 0 is an initial object of $\mathcal{A}$, and $d : 0 \longrightarrow 0 \,\Box\, X$ is the unique morphism.

In fact, the above ω-chain is well-known to converge to the initial-algebra of $- \,\Box\, X$ (equivalently, a free base algebra on X) since $- \,\Box\, X$ is finitary, see [A].

3 Iterative Base Algebras

Assumption 3.1. Throughout the rest of the paper $\Box$ denotes a base on a locally finitely presentable category $\mathcal{A}$.

Definition 3.2.

(1) By a (finitary, flat) *equation morphism* in an object A is meant a morphism

$$e : X \longrightarrow X \,\Box\, A, \quad X \text{ finitely presentable.}$$

(2) Suppose that A is the underlying object of a base algebra $\alpha : A \Box A \longrightarrow A$. Then by a *solution* of an equation morphism e is meant a morphism $e^\dagger : X \longrightarrow A$ such that the square

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & A \\ X \,\Box\, A \arrow[r, "e^\dagger \Box A"'] & A \,\Box\, A \arrow[u, "\alpha"'] \end{tikzcd} \tag{3.1}$$

commutes.

(3) A base algebra is called *iterative* provided that every equation morphism has a unique solution.

Remark 3.3. The assumption that X be finitely presentable represents the fact that in the motivating example (see the system (1.1) in the introduction) we only have finitely many iterative variables.

Example 3.4. Let $\mathcal{A}$ be the category of complete metric spaces of diameter ≤ 1 and nonexpanding maps. Let $\Box$ be a *contracting base*, i.e., such that there exists $k < 1$ for which the distance $d_{\mathcal{A}(A,B)}(f, g) = \sup_{a \in A} d_B(f(a), g(a))$ between arbitrary parallel morphisms $f, g : A \longrightarrow B$ fulfills:

$$d_{\mathcal{A}(A \Box C, B \Box C)}(f \,\Box\, id_C, g \,\Box\, id_C) \leq k \cdot d_{\mathcal{A}(A,B)}(f, g) \quad \text{for all objects } C.$$

An example of a contractive base:

$$X \mathbin{\square} A = \text{disjoint union of } X \text{ and } A \text{ with all distances multiplied by } k.$$

Every nonempty base algebra (A, α) is iterative. In fact, given an equation morphism $e : X \longrightarrow X \mathbin{\square} A$, we define a sequence $e_n^\dagger : X \longrightarrow A$ by choosing $e_0^\dagger$ arbitrarily and defining $e_{n+1}^\dagger$ through the diagram

$$\begin{tikzcd} X \arrow[r, "e_{n+1}^\dagger"] \arrow[d, "e"'] & A \\ X \mathbin{\square} A \arrow[r, "e_n^\dagger \mathbin{\square} A"'] & A \mathbin{\square} A \arrow[u, "\alpha"'] \end{tikzcd}$$

Then $e_n^\dagger$ is a Cauchy sequence and $e^\dagger = \lim e_n^\dagger$ is the unique solution. The easy proof is left to the reader.

Example 3.5. Let $\mathcal{A} = \mathsf{Set}$.

(1) Algebras for the base $X \mathbin{\square} A = (X \times X) + A$ are the usual algebras on one binary operation: see Example 2.9(2) with $HX = X \times X$. There is no easy criterion for a base algebra to be iterative. But there are nice examples of iterative base algebras, see [AMV$_1$], e.g.,

$$\begin{aligned} A &= \{1, 2, 3, \dots\} \cup \{\infty\} && \text{with addition,} \\ A &= (0, \infty] && \text{with addition, or} \\ A &= (1, \infty] && \text{with multiplication.} \end{aligned}$$

A free iterative base algebra on a set Y (of generators) can be described as the algebra RY of all rational binary trees on Y, see 1.A.

(2) More generally, Σ-algebras, where Σ is a (non-parametrized) signature, are just algebras for the corresponding base $X \mathbin{\square}_\Sigma A = H_\Sigma X + A$, see Example 2.9(2). A free iterative base algebra on a set Y is the algebra

$$R_\Sigma Y$$

of all rational Σ-trees on Y, see 1.A.

Example 3.6. Consider the base $X \mathbin{\square} A = X^* \times A$ on $\mathcal{A} = \mathsf{Set}$, see Example 2.5(2). Although its base algebras are, again, the usual binary algebras, the concept of iterative base algebra differs from the above example. Recall that a base algebra (A, α) leads to $\widehat{\alpha} : A^* \times A \longrightarrow A$ with

$$\widehat{\alpha}(a_1 a_2 \dots a_n, a) = \alpha(a_1, \alpha(a_2, \dots \alpha(a_n, a) \dots)).$$

The base algebra (A, α) is iterative if and only if for every equation morphism

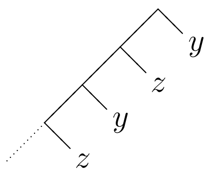
$e : X \longrightarrow X^* \times A$ there exists a unique $e^\dagger : X \longrightarrow A$ such that for every variable x we have that

(1) $e(x) = (\varepsilon, a)$ implies $e^\dagger(x) = a$, and
(2) $e(x) = (x_1 \dots x_n, a)$ implies $e^\dagger(x) = \widehat{\alpha}(e^\dagger(x_1)e^\dagger(x_2)\cdots e^\dagger(x_n), a)$.

Thus, for example, the empty algebra is iterative in the present sense (but it is not iterative for $(X \times X) + A$).

A free iterative algebra, $\widehat{R}Y$, on a set Y can be described as a subalgebra of the above algebra $R_\Sigma Y$ of all rational binary trees on Y. Let us call a binary tree *right-wellfounded* if from every node the right-most path is always finite (i.e., it leads to a leaf), see [AMV$_3$], or [U].

Example: the following rational tree



is right-wellfounded, the rational tree (1.2) is not.

Open Problem 3.7. Describe, for $\mathcal{A} = \mathsf{Fin}[\mathsf{Set}, \mathsf{Set}]$, free iterative algebras for the bases

$$X \mathbin{\Box} A = (X \times X) + A$$

and

$$X \mathbin{\Box} A = \mathcal{F}(X) \cdot A\,.$$

Notation 3.8. We denote by

$$\mathsf{Alg}_{\mathrm{it}}\,\Box$$

the full subcategory of $\mathsf{Alg}\,\Box$ formed by all iterative base algebras and by

$$U_{\mathrm{it}} : \mathsf{Alg}_{\mathrm{it}}\,\Box \longrightarrow \mathcal{A}$$

the canonical forgetful functor.

Notation 3.9. Let $e : X \longrightarrow X \mathbin{\Box} A$ be an equation morphism in A. Every morphism $h : A \longrightarrow B$ in $\mathcal{A}$ yields an equation morphism $h \bullet e$ in B defined by

$$h \bullet e \equiv X \xrightarrow{\ e\ } X \mathbin{\Box} A \xrightarrow{X \Box h} X \mathbin{\Box} B\,. \tag{3.2}$$

Definition 3.10. Let A and B be iterative base algebras. We say that a morphism $h : A \longrightarrow B$ in $\mathcal{A}$ *preserves solutions* provided that for every equation morphism e in A the solution of $h \bullet e$ is $h \cdot e^\dagger$, i.e., the triangle

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[dr, "(h\bullet e)^\dagger"'] & A \arrow[d, "h"] \\ & B \end{tikzcd}$$

commutes.

Remark 3.11. Let (A, α) and (B, β) be iterative algebras. Then a morphism $h : A \longrightarrow B$ in $\mathcal{A}$ is a homomorphism if and only if it preserves solutions. This is proved in [AMV$_3$], Lemma 3.11, for bases in Set but the proof in a general locally finitely presentable category is completely analogous. This explains that the choice of "plain" homomorphisms between iterative algebras is adequate.

Remark 3.12. In several proofs below we use the fact that since $\mathcal{A}$ is locally finitely presentable, every object A is a canonical colimit of the filtered diagram of all arrows $p : P \longrightarrow A$, P finitely presentable. More precisely, $A = \operatorname{colim} D_A$, where $D_A : \mathcal{A}_{fp}/A \longrightarrow \mathcal{A}$ is the forgetful functor of the comma category of A w.r.t. $\mathcal{A}_{fp}$.

Proposition 3.13. *Every object of $\mathcal{A}$ generates a free iterative base algebra, i.e., the forgetful functor $U_{\mathrm{it}} : \mathsf{Alg}_{\mathrm{it}}\,\square \longrightarrow \mathcal{A}$ has a left adjoint.*

Proof. Iterative algebras are closed in $\mathsf{Alg}\,\square$ under limits and filtered colimits. The proof is easy and completely analogous to the proof of the same result in the non-parametrized case (see Proposition 2.20 in [AMV$_1$]). Since the latter category is locally finitely presentable by Proposition 2.10, it follows from Theorem 2.48 in [AR] that $\mathsf{Alg}_{\mathrm{it}}\,\square$ forms a reflective subcategory of $\mathsf{Alg}\,\square$. Consequently, U_{it} has a left adjoint since U does (see Proposition 2.10 again). $\square$

Definition 3.14. By the *rational monad* of a base is meant the monad of free iterative base algebras.

More detailed: for every object Y we denote by (RY, ρ_Y) a free iterative base algebra with the universal arrow $\eta_Y : Y \longrightarrow RY$. We obtain the rational monad (R, η, μ) where $\mu_Y : RRY \longrightarrow RY$ is the unique homomorphism with $\mu_Y \cdot \eta_{RY} = id$:

$$\begin{tikzcd} RRY \,\square\, RRY \arrow[r, "\rho_{RY}"] \arrow[d, "\mu_Y \square \mu_Y"'] & RRY \arrow[d, "\mu_Y"'] & RY \arrow[l, "\eta_{RY}"'] \arrow[dl, equal] \\ RY \,\square\, RY \arrow[r, "\rho_Y"'] & RY & \end{tikzcd} \tag{3.3}$$

Examples 3.15.

(1) For the base $X \,\square\, A = X + A$ in Set we have, due to Example 1.3,

$$RY = \mathbb{N} \times Y + 1$$

where the right-hand summand is a fixed point of the unary operation, and on the left-hand summand the operation is given by $(n, y) \longmapsto (n + 1, y)$.

(2) Given a finitary signature Σ, the rational monad of the base $X \,\square\, A = H_\Sigma X + A$ is the monad of rational trees, see Example 3.5(2).

(3) The rational monad of the base $X \,\square\, A = X^* \times A$ is the monad of all right-wellfounded rational binary trees, see Example 3.6.

The base of Example 2.5(5) yields as RY the set of all right-wellfounded rational trees on Y, see Example 3.15(ii) in [AMV$_3$].

4 A Coalgebraic Construction

We know from the preceding section that, for every base, free iterative base algebras exist. The aim of the present section is to show that a free iterative base algebra RY on an object Y can be constructed as a filtered colimit of all equation morphisms $e : X \longrightarrow X \,\square\, Y$ (where X ranges through the subcategory $\mathcal{A}_{fp}$, see 2.1).

More precisely, consider the coalgebras of the endofunctor $- \,\square\, Y$ together with the usual coalgebra homomorphisms. We denote by

$$\mathsf{EQ}_Y$$

the category of all equation morphisms as the full subcategory of $\mathsf{Coalg}\,(- \,\square\, Y)$ on all objects from $\mathcal{A}_{fp}$. Since $\mathsf{Coalg}\,(- \,\square\, Y)$ is cocomplete, with colimits formed on the level of $\mathcal{A}$, it is obvious that EQ_Y is closed in it under finite colimits. In particular, EQ_Y is a filtered category. We also denote by

$$\mathrm{Eq}_Y : \mathsf{EQ}_Y \longrightarrow \mathcal{A}, \quad (e : X \longrightarrow X \,\square\, Y) \longmapsto X$$

the forgetful functor. This defines a small filtered diagram in $\mathcal{A}$.

Notation 4.1. RY denotes a (filtered) colimit of the diagram Eq_Y with colimit cocone

$$e^\sharp : X \longrightarrow RY \quad (\text{for all } e : X \longrightarrow X \,\square\, Y \text{ in } \mathsf{EQ}_Y).$$

This does not contradict our previous notation of Definition 3.14:

Remark 4.2. The aim of the present section is to prove that RY carries a structure of an iterative base algebra making it a free iterative base algebra on Y. We proceed in two steps: we first assume that Y is a finitely presentable object. This enables us, for example, to define the universal arrow immediately: observe that $u_Y^Y : Y \longrightarrow Y \,\Box\, Y$ is an object of EQ_Y, and denote by

$$\eta_Y = (u_Y^Y)^\sharp : Y \longrightarrow RY$$

the corresponding colimit morphism.

An extension of all the results to arbitrary objects Y is then easy; see Remark 4.12. For example, η_Y is defined as follows: express Y as a colimit of a filtered diagram of finitely presentable objects Y_t, $t \in T$, then it is easy to verify that RY is a filtered colimit of RY_t, $t \in T$, and we put $\eta_Y = \operatorname*{colim}_{t \in T} \eta_{Y_t}$.

The only exception of the above two-step program is the following notation, where Y is arbitrary:

Notation 4.3. For every object Y of $\mathcal{A}$ we denote by

$$i_Y : RY \longrightarrow RY \,\Box\, Y$$

the unique morphism for which the squares

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "e^\sharp"'] & X \,\Box\, Y \arrow[d, "e^\sharp \Box Y"] \\ RY \arrow[r, "i_Y"'] & RY \,\Box\, Y \end{tikzcd} \tag{4.1}$$

commute for all e in EQ_Y. This is well defined since the morphisms $(e^\sharp \,\Box\, Y) \cdot e$ are easily seen to form a cocone of the diagram Eq_Y.

Remark 4.4. We are going to prove that for every object X of $\mathcal{A}$ the morphism i_X above has an inverse, denoted by $j_X : RX \,\Box\, X \longrightarrow RX$ and called the rational constructor of the base. There is a deliberate notational clash with the notation of Lemma 2.14 for constructors of the base

$$j_X : FX \,\Box\, X \longrightarrow FX$$

The reason is that we want to stress the fact the analogy of the behaviour of free base algebras and free iterative base algebras.

Lemma 4.5. *For every equation morphism e in EQ_Y there exists a unique morphism $e^\sharp : X \longrightarrow RY$ such that the square* (4.1) *commutes.*

Proof. We are to prove that $f : X \longrightarrow RY$ is equal to $e^\sharp$ whenever the square

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "f"'] & X \,\square\, Y \arrow[d, "f \square Y"] \\ RY \arrow[r, "i_Y"'] & RY \,\square\, Y \end{tikzcd}$$

is commutative. Since X is finitely presentable, the morphism f factors through the colimit morphism $g^\sharp : V \longrightarrow RY$ for some $g : V \longrightarrow V \,\square\, Y$:

$$f = g^\sharp \cdot f_0.$$

In the following diagram

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "f_0"'] \arrow[dd, bend right=60, "f"'] & X \,\square\, Y \arrow[d, "f_0 \square Y"] \arrow[dd, bend left=60, "f \square Y"] \\ V \arrow[r, "g"] \arrow[d, "g^\sharp"'] & V \,\square\, Y \arrow[d, "g^\sharp \square Y"] \\ RY \arrow[r, "i_Y"'] & RY \,\square\, Y \end{tikzcd}$$

the outward square commutes, and so do all inner parts except for the upper square. This implies that $g^\sharp \,\square\, Y$ merges the two sides of that upper square. Now $g^\sharp \,\square\, Y$ is a colimit morphism of $RY \,\square\, Y$ (recall that $\mathsf{EQ}_Y \longrightarrow \mathcal{A}$ is a filtered diagram; thus $(-) \,\square\, Y$ preserves its colimit) and merges two parallel morphisms with the finitely presentable domain X. Therefore those morphisms are also merged by one of the connecting maps, say $p \,\square\, Y$, for some morphism p in EQ_Y:

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, "p"'] & V \,\square\, Y \arrow[d, "p \square Y"] \\ W \arrow[r, "h"'] & W \,\square\, Y \end{tikzcd}$$

That is, we have

$$(p \,\square\, Y) \cdot (f_0 \,\square\, Y) \cdot e = (p \,\square\, Y) \cdot g \cdot f_0.$$

Now p being a morphism of EQ_Y implies $g^\sharp = h^\sharp \cdot p$. Next, $p \cdot f_0$ is a morphism of EQ_Y from e to h. Consequently,

$$h^\sharp \cdot p \cdot f_0 = e^\sharp,$$

and therefore

$$f = g^\sharp \cdot f_0 = h^\sharp \cdot p \cdot f_0 = e^\sharp.$$

$\square$

Notation 4.6. Here we introduce notation for "adding variables" to equations: Given an equation morphism

$$e : X \longrightarrow X \mathbin{\square} Y \quad \text{in } \mathsf{EQ}_Y$$

and an object Q in $\mathcal{A}_{fp}$, when are we able to form "canonically" an equation morphism

$$X + Q \longrightarrow (X + Q) \mathbin{\square} Y \quad \text{in } \mathsf{EQ}_Y?$$

One possibility is to assume that a morphism

$$q : Q \longrightarrow X \mathbin{\square} X$$

is given. Then we can define an equation morphism

$$e_q = (\mathsf{inl} \mathbin{\square} Y) \cdot [X \mathbin{\square} Y, m^X_Y \cdot (X \mathbin{\square} e)] \cdot (e + q) : X + Q \longrightarrow (X + Q) \mathbin{\square} Y$$

That is, the diagram below defines e_q:

$$\begin{tikzcd}
X \arrow[rrr, "e"] \arrow[d, "\mathsf{inl}"'] & & & X \mathbin{\square} Y \arrow[d, "\mathsf{inl} \mathbin{\square} Y"] \\
X + Q \arrow[rrr, "e_q"] & & & (X + Q) \mathbin{\square} Y \\
Q \arrow[u, "\mathsf{inr}"] \arrow[r, "q"'] & X \mathbin{\square} X \arrow[r, "X \mathbin{\square} e"'] & X \mathbin{\square} (X \mathbin{\square} Y) \arrow[r, "m^X_Y"'] & X \mathbin{\square} Y \arrow[u, "\mathsf{inl} \mathbin{\square} Y"']
\end{tikzcd} \tag{4.2}$$

Theorem 4.7. *For every finitely presentable object Y there exists a unique structure of a base algebra $\rho_Y : RY \mathbin{\square} RY \longrightarrow RY$ such that the squares*

$$\begin{tikzcd}
Q \arrow[r, "\mathsf{inr}"] \arrow[d, "q"'] & X + Q \arrow[dd, "{e_q}^\sharp"] \\
X \mathbin{\square} X \arrow[d, "e^\sharp \mathbin{\square} e^\sharp"'] & \\
RY \mathbin{\square} RY \arrow[r, "\rho_Y"'] & RY
\end{tikzcd} \tag{4.3}$$

(where e and q are as in Notation 4.6) commute.

Remark 4.8. In the proof below we will denote by $\mathrm{Eq}_Y \mathbin{\square} \mathrm{Eq}_Y : \mathsf{EQ}_Y \longrightarrow \mathcal{A}$ the diagram given by objects $X \mathbin{\square} X$, where $e : X \longrightarrow X \mathbin{\square} Y$ ranges over EQ_Y, and by morphisms $h \mathbin{\square} h$, for morphisms h in EQ_Y. Since $\square$ is finitary in both variables and the diagonal $\mathsf{EQ}_Y \longrightarrow \mathsf{EQ}_Y \times \mathsf{EQ}_Y$ is cofinal, the colimit of $\mathrm{Eq}_Y \mathbin{\square} \mathrm{Eq}_Y$ is $RY \mathbin{\square} RY$ with colimit injections $e^\sharp \mathbin{\square} e^\sharp$. Analogously, $RY \mathbin{\square} (RY \mathbin{\square} RY)$ is a colimit of a similar diagram $\mathrm{Eq}_Y \mathbin{\square} (\mathrm{Eq}_Y \mathbin{\square} \mathrm{Eq}_Y)$, and $X \mathbin{\square} RY = \operatorname{colim} X \mathbin{\square} \mathrm{Eq}_Y$.

Proof. (1) First we establish that there is a unique morphism ρ_Y for which (4.3) commutes.

(1a) Apply Remark 3.12 to $RY \,\square\, RY$: In order to prove (1), we show that each $p : Q \longrightarrow RY \,\square\, RY$ with Q in $\mathcal{A}_{fp}$ factors as

$$p = (e^\sharp \,\square\, e^\sharp) \cdot q$$

for some e and q, and that the morphisms ${e_q}^\sharp \cdot \mathsf{inr} : Q \longrightarrow RY$ (which, as we prove below immediately, are independent of the choice of such a factorization) form a cocone of the diagram $D_{RY \square RY}$. Then there is a unique ρ_Y with the requested property.

The existence of the above factorization of p follows from the above filtered colimit $RY \,\square\, RY = \mathrm{colim}(Eq_Y \,\square\, \mathrm{Eq}_Y)$. The morphism p having a finitely presentable domain, factors through one of the colimit injections $e^\sharp \,\square\, e^\sharp$.

We claim that the upper passage ${e_q}^\sharp \cdot \mathsf{inr}$ of the square (4.3) is independent of the choice of the factorization. Thus, let $f : Z \longrightarrow Z \,\square\, Y$ be an equation morphism and let

$$\begin{tikzcd}
 & Q \arrow[dl, "r"'] \arrow[dr, "p"] & \\
Z \,\square\, Z \arrow[rr, "f^\sharp \square f^\sharp"'] & & RY \,\square\, RY
\end{tikzcd}$$

be another factorization of p through a colimit morphism of the diagram $\mathrm{Eq}_Y \,\square\, \mathrm{Eq}_Y$. Since that diagram is filtered, we can assume, without loss of generality, that a morphism h from e to f in EQ_Y exists, and that the equation $r = (h \,\square\, h) \cdot q$ holds. It follows that in the notation (4.2) $h + Q$ is a morphism from e_q to f_r:

$$\begin{tikzcd}
X + Q \arrow[r, "e+q"] \arrow[d, "h+Q"'] \arrow[rrr, bend left, "e_q"] & (X \,\square\, Y) + (X \,\square\, X) \arrow[r, "{[X\square Y, m_Y^X \cdot (X \square e)]}"] \arrow[d, "(h\square Y)+(h\square h)"] & X \,\square\, Y \arrow[r, "\mathsf{inl} \,\square Y"] \arrow[d, "h\square Y"] & (X + Q) \,\square\, Y \arrow[d, "(h+Q)\square Y"] \\
Z + Q \arrow[r, "f+r"'] \arrow[rrr, bend right, "f_r"'] & (Z \,\square\, Y) + (Z \,\square\, Z) \arrow[r, "{[Z\square Y, m_Z^Y \cdot (Z \square f)]}"'] & Z \,\square\, Y \arrow[r, "\mathsf{inl} \,\square Y"'] & (Z + Q) \,\square\, Y
\end{tikzcd}$$

Consequently,

$${e_q}^\sharp = {f_r}^\sharp \cdot (h + Q).$$

This proves the desired independence:

$${e_q}^\sharp \cdot \mathsf{inr} = {f_r}^\sharp \cdot (h + Q) \cdot \mathsf{inr} = {f_r}^\sharp \cdot \mathsf{inr}\,.$$

(1b) The above morphisms ${e_q}^\sharp \cdot \mathsf{inr} : Q \longrightarrow RY$ form a cocone of the diagram $D_{RY \square RY}$, see Remark 3.12. That is, given an arrow $p' : Q' \longrightarrow RY \,\square\, RY$,

$Q' \in \mathcal{A}_{fp}$, and given a commutative triangle

$$\begin{tikzcd} Q \arrow[rr, "t"] \arrow[dr, "p"'] & & Q' \arrow[dl, "p'"] \\ & RY \,\square\, RY & \end{tikzcd}$$

we prove that for any factorization of p'

$$\begin{tikzcd} & Q' \arrow[dl, "r"'] \arrow[dr, "p'"] & \\ Z \,\square\, Z \arrow[rr, "f^\sharp \square f^\sharp"'] & & RY \,\square\, RY \end{tikzcd}$$

we have

$$e_q{}^\sharp \cdot \mathsf{inr} \;=\; f_r{}^\sharp \cdot \mathsf{inr} \;\cdot t. \tag{4.4}$$

In fact, the following factorization of p:

$$\begin{tikzcd} & & Q \arrow[dl, "t"'] \arrow[ddr, "p"] & \\ & Q' \arrow[dl, "r"'] \arrow[drr, dotted, "p'"] & & \\ Z \,\square\, Z \arrow[rrr, "f^\sharp \square f^\sharp"'] & & & RY \,\square\, RY \end{tikzcd}$$

yields, by the independence proved in part (1a), the equality

$$e_q{}^\sharp \cdot \mathsf{inr} \;=\; f_{rt}{}^\sharp \cdot \mathsf{inr}\,.$$

Moreover, $Z + t$ is an equation morphism from f_{rt} into f_r:

$$\begin{tikzcd} Z + Q \arrow[r, "f+rt"] \arrow[d, "Z+t"'] \arrow[rrr, bend left, "f_{rt}"] & (Z \,\square\, Y) + (Z \,\square\, Z) \arrow[r, "{[Z\square Y, m^Z_Y (Z\square f)]}"] \arrow[d, equal] & Z \,\square\, Y \arrow[r, "\mathsf{inl}\,\square Y"] \arrow[d, equal] & (Z + Q) \,\square\, Y \arrow[d, "(Z+t)\square Y"] \\ Z + Q' \arrow[r, "f+r"'] \arrow[rrr, bend right, "f_r"'] & (Z \,\square\, Y) + (Z \,\square\, Z) \arrow[r, "{[Z\square Y, m^Z_Y (Z\square f)]}"'] & Z \,\square\, Y \arrow[r, "\mathsf{inl}\,\square Y"'] & (Z + Q') \,\square\, Y \end{tikzcd}$$

Consequently, $f_{rt}{}^\sharp = f_r{}^\sharp \cdot (Z + t)$, which implies (4.4):

$$e_q{}^\sharp \cdot \mathsf{inr} \;=\; f_{rt}{}^\sharp \cdot \mathsf{inr} \;=\; f_r{}^\sharp \cdot (Z + t) \cdot \mathsf{inr} \;=\; f_r{}^\sharp \cdot \mathsf{inr} \;\cdot t.$$

The cocone $e_q{}^\sharp \cdot \mathsf{inr}$ has a unique factorization through the colimit cocone — this proves that (4.3) defines a unique ρ_Y.

(2) (RY, ρ_Y) is an algebra, i.e., we prove that the conditions (2.5) hold.

(2a) We verify first the left-hand triangle of (2.5) by proving that for every colimit morphism $e^\sharp : X \longrightarrow RY$ of Eq_Y we have

$$(\rho_Y \cdot u^{RY}_{RY}) \cdot e^\sharp = e^\sharp. \tag{4.5}$$

Using $q = u^X_X : X \longrightarrow X \,\square\, X$ in (4.3) we get

$${e_q}^\sharp \cdot \mathsf{inr} \;= \rho_Y \cdot (e^\sharp \,\square\, e^\sharp) \cdot u^X_X = \rho_Y \cdot u^{RY}_{RY} \cdot e^\sharp\,,$$

therefore, all we have to verify in order to prove (4.5) is

$${e_q}^\sharp \cdot \mathsf{inr} \;= e^\sharp.$$

In fact, we prove that

$${e_q}^\sharp = [e^\sharp, e^\sharp]$$

by using Lemma 4.5. Thus, it is sufficient to show the commutativity of the square

$$\begin{tikzcd}
X + X \arrow[r, "e_q"] \arrow[d, "{[e^\sharp,e^\sharp]}"'] & (X+X) \,\square\, Y \arrow[d, "{[e^\sharp,e^\sharp]\square Y}"] \\
RY \arrow[r, "i_Y"'] & RY \,\square\, Y
\end{tikzcd} \tag{4.6}$$

In fact, the left-hand component of (4.6) commutes due to the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[d, "e^\sharp"'] \arrow[rr, bend left, "e_q\cdot\mathsf{inl}"] \arrow[dr, phantom, "(4.1)"] & X \,\square\, X \arrow[r, "\mathsf{inl}\,\square Y"] \arrow[dr, "e^\sharp\square Y"'] \arrow[r, phantom, bend left, "(4.2)"] & (X+X) \,\square\, Y \arrow[d, "{[e^\sharp,e^\sharp]\square Y}"] \\
RY \arrow[rr, "i_Y"'] & & RY \,\square\, Y
\end{tikzcd}$$

For the right-hand component of (4.6) consider the diagram

$$\begin{tikzcd}
X \arrow[r, "u^X_X"] \arrow[drr, "e"'] \arrow[dd, "e^\sharp"'] \arrow[rrrr, bend left, "e_q\cdot\mathsf{inr}"] \arrow[ddr, phantom, "(4.1)"] & X \,\square\, X \arrow[r, "X\square e"] & X \,\square\, (X \,\square\, Y) \arrow[r, "m^X_Y"] \arrow[dr, phantom, "(2.1)"] & X \,\square\, Y \arrow[r, "\mathsf{inl}\,\square A"] \arrow[dl, equal] & (X+X) \,\square\, Y \arrow[dd, "{[e^\sharp,e^\sharp]\square Y}"] \\
& & X \,\square\, Y \arrow[u, "u^X_{X\square Y}"] \arrow[drr, "e^\sharp\square Y"'] & & \\
RY \arrow[rrrr, "i_Y"'] & & & & RY \,\square\, Y
\end{tikzcd}$$

(2b) We verify that the right-hand square of (2.5)

$$
\begin{tikzcd}
RY \,\square\, (RY \,\square\, RY) \arrow[r, "RY \square \rho_Y"] \arrow[d, "m^{RY}_{RY}"'] & RY \,\square\, RY \arrow[d, "\rho_Y"] \\
RY \,\square\, RY \arrow[r, "\rho_Y"'] & RY
\end{tikzcd}
\tag{4.7}
$$

commutes.

To do so, we prove that for every morphism

$$p : P \longrightarrow RY \,\square\, (RY \,\square\, RY), \quad P \text{ in } \mathcal{A}_{fp},$$

the composites of p with the two sides of (4.7) are equal. We first factor $p = e^\sharp \,\square\, (e^\sharp \,\square\, e^\sharp) \cdot p_0$ for some $e : X \longrightarrow X \,\square\, Y$ and some $p_0 : P \longrightarrow X \,\square\, (X \,\square\, X)$, using the fact that $RY \,\square\, (RY \,\square\, RY)$ is a filtered colimit of $\mathrm{Eq}_Y \,\square\, (\mathrm{Eq}_Y \,\square\, \mathrm{Eq}_Y)$. Next, express $X \,\square\, X$ as a filtered colimit of the diagram $D_{X \square X}$ of all morphisms $r : Q \longrightarrow X \,\square\, X$ with Q in $\mathcal{A}_{fp}$, see Remark 3.12. Since $\square$ is finitary in both variables, this yields $X \,\square\, (X \,\square\, X)$ as a filtered colimit of all $X \,\square\, r : X \,\square\, Q \longrightarrow X \,\square\, (X \,\square\, X)$ and we can factor p_0 through one of those, say, $X \,\square\, r_0$: we have $p_0 = (X \,\square\, r_0) \cdot p_1$ for some $p_1 : P \longrightarrow X \,\square\, Q$. The diagram

$$
\begin{tikzcd}
 & & RY \,\square\, (RY \,\square\, RY) \\
P \arrow[r, "p_1"] \arrow[urr, bend left, "p"] \arrow[drr, bend right, "p_0"'] & X \,\square\, Q \arrow[ur, "e^\sharp \square r"] \arrow[dr, "X \square r_0"] & \\
 & & X \,\square\, (X \,\square\, X) \arrow[uu, "e^\sharp \square (e^\sharp \square e^\sharp)"']
\end{tikzcd}
\tag{4.8}
$$

commutes, where

$$r = (e^\sharp \,\square\, e^\sharp) \cdot r_0. \tag{4.9}$$

Put

$$q \equiv P \xrightarrow{p_0} X \,\square\, (X \,\square\, X) \xrightarrow{m^X_X} X \,\square\, X.$$

Then the diagram

$$
\begin{tikzcd}
RY \square (RY \square RY) \arrow[rr, "m^{RY}_{RY}"] \arrow[dr, phantom, "(4.8)"] & & RY \square RY \arrow[r, "\rho_Y"] & RY \\
 & X \square (X \square X) \arrow[ul, "e^\sharp \square (e^\sharp \square e^\sharp)"'] \arrow[r, "m^X_X"'] \arrow[ur, phantom, "(2.4)"] & X \square X \arrow[u, "e^\sharp \square e^\sharp"'] \arrow[dr, phantom, "(4.3)"] & \\
P \arrow[uu, "p"] \arrow[ur, "p_0"] \arrow[rrr, "\mathsf{inr}"'] & & & X+P \arrow[uu, "e_q{}^\sharp"']
\end{tikzcd}
\tag{4.10}
$$

commutes.

Next put

$$\overline{q} \equiv P \xrightarrow{p_1} X \,\square\, Q \xrightarrow{\mathsf{inl} \,\square\, \mathsf{inr}} (X+Q) \,\square\, (X+Q)\,,$$

and, using Notation 4.6,

$$f = e_{r_0} : X + Q \longrightarrow (X + Q) \,\square\, Y.$$

Observe that 4.6 yields

$$f \cdot \mathsf{inr} \, = (\mathsf{inl} \,\square\, Y) \cdot m_Y^X \cdot (X \,\square\, e) \cdot r_0. \tag{4.11}$$

Then by (4.3) we obtain

$$\rho_Y \cdot (f^\sharp \,\square\, f^\sharp) \cdot (\mathsf{inl} \,\square\, \mathsf{inr}\,) \cdot p_1 = f_{\bar{q}}{}^\sharp \cdot \mathsf{inr}\,. \tag{4.12}$$

Another application of (4.3) yields

$$\rho_Y \cdot (e^\sharp \,\square\, e^\sharp) \cdot r_0 = f^\sharp \cdot \mathsf{inr}\,. \tag{4.13}$$

It is easy to verify that $\mathsf{inl} \, : X \longrightarrow X + Q$ is a morphism of equations from e to $f = e_{r_0}$ — thus

$$e^\sharp = f^\sharp \cdot \mathsf{inl}\,. \tag{4.14}$$

This proves that the diagram

$$\begin{tikzcd}
RY\square(RY\square RY) \arrow[rrr, "RY\square\rho_Y"] & & \arrow[d, phantom, "(4.13)"] & RY\square RY \arrow[rr, "\rho_Y"] & & RY \\
& \arrow[dr, phantom, "(4.9)"] & X\square(X\square X) \arrow[ull, "e^\sharp\square(e^\sharp\square e^\sharp)"'] & \arrow[r, phantom, "(4.14)"] & (X+Q)\square(X+Q) \arrow[ul, "f^\sharp\square f^\sharp"'] & \\
\arrow[r, phantom, "(4.8)"] & & X\square Q \arrow[u, "X\square r_0"'] \arrow[uull, "e^\sharp\square r"'] \arrow[r, "X\square\mathsf{inr}"'] & X\square(X+Q) \arrow[uu, "e^\sharp\square f^\sharp"] \arrow[ur, "\mathsf{inl}\,\square(X+Q)"'] & \arrow[r, phantom, "(4.12)"] & \\
P \arrow[uuu, "p"] \arrow[urr, "p_1"'] \arrow[rrrrr, "\mathsf{inr}"'] & & & & & X+Q+P \arrow[uuu, "f_{\bar{q}}{}^\sharp"']
\end{tikzcd} \tag{4.15}$$

commutes.

Our task is to show that p merges the two sides of (4.7), in other words, that

$$\rho_Y \cdot (RY \,\square\, \rho_Y) \cdot p = \rho_Y \cdot m_{RY}^{RY} \cdot p$$

holds. The left-hand side is the upper passage of (4.15), the right-hand one is the upper passage of (4.10). Comparing the lower passages of (4.15) and (4.10), we see that what remains is to show that the triangle

$$\begin{tikzcd}
& RY & \\
X + P \arrow[ur, "e_q{}^\sharp"] \arrow[rr, "{[\mathsf{inl}\,,\mathsf{inr}\,]}"'] & & X + Q + P \arrow[ul, "f_{\bar{q}}{}^\sharp"']
\end{tikzcd} \tag{4.16}$$

commutes. This follows from the fact that

$$[\mathsf{inl}\,, \mathsf{inr}\,] : X + P \longrightarrow X + Q + P$$

is a morphism of equations from e_q to $f_{\overline{q}}$. To prove this, put $\widehat{X} = X + Q$. We are to show that the square

$$\begin{tikzcd}
X+P \arrow[r, "e+q"] \arrow[d, "{[\mathsf{inl}\,,\mathsf{inr}\,]}"'] \arrow[rrr, bend left, "e_q"] & (X\Box Y)+(X\Box X) \arrow[r, "{[X\Box Y, m^X_Y(X\Box e)]}"] & X\Box Y \arrow[r, "\mathsf{inl}\,\Box Y"] \arrow[d, dotted, "\mathsf{inl}\,\Box Y"'] & (X+P)\Box Y \arrow[d, "{[\mathsf{inl}\,,\mathsf{inr}\,]\Box Y}"] \\
\widehat{X}+P \arrow[r, "f+\overline{q}"'] \arrow[rrr, bend right, "f_{\overline{q}}"'] & (\widehat{X}\Box Y)+(\widehat{X}\Box \widehat{X}) \arrow[r, "{[\widehat{X}\Box Y, m^{\widehat{X}}_Y(\widehat{X}\Box f)]}"'] & \widehat{X}\Box Y \arrow[r, "\mathsf{inl}\,\Box Y"'] & (\widehat{X}+P)\Box Y
\end{tikzcd}$$

commutes. For the left-hand component, with domain X, this states that

$$f \cdot \mathsf{inl} = (\mathsf{inl} \ \Box\ Y) \cdot e : X \longrightarrow (X + Q) + Y$$

which follows immediately from $f = e_{r_0}$. To prove that the right-hand component commutes, recall that of $q = m^X_X \cdot p_0$ and $\overline{q} = (\mathsf{inl} \ \Box\ \mathsf{inr}\,) \cdot p_1$ and observe the commutativity of the following diagram:

$$\begin{tikzcd}
P \arrow[r, "p_0"] \arrow[dd, equal] \arrow[dd, phantom, shift left=6ex, "(4.8)"] \arrow[rrrrr, bend left, "e_q\cdot\mathsf{inr}"] & X\Box(X\Box X) \arrow[r, "m^X_X"] \arrow[dr, "X\Box(X\Box e)"'] & X\Box X \arrow[r, "X\Box e"] \arrow[d, phantom, "(2.4)"] & X\Box(X\Box Y) \arrow[r, "m^X_Y"] \arrow[d, phantom, "(2.2)"] & X\Box Y \arrow[r, "\mathsf{inl}\,\Box Y"] \arrow[dd, "\mathsf{inl}\,\Box Y"] \arrow[ddl, phantom, "(2.4)"] & (X+P)\Box Y \arrow[dd, "{[\mathsf{inl}\,,\mathsf{inr}\,]\Box Y}"] \\
 & & X\Box(X\Box(X\Box X)) \arrow[ur, "m^X_{X\Box Y}"] \arrow[r, "X\Box m^X_Y"] \arrow[d, phantom, "(4.11)"] & X\Box(X\Box Y) \arrow[ur, "m^X_Y"'] \arrow[d, "\mathsf{inl}\,\Box(\mathsf{inl}\,\Box Y)"] & & \\
P \arrow[r, "p_1"'] \arrow[rrrrr, bend right, "f_{\overline{q}}\cdot\mathsf{inr}"'] & X\Box Q \arrow[uu, "X\Box r_0"'] \arrow[r, "\mathsf{inl}\,\Box\mathsf{inr}"'] & \widehat{X}\Box\widehat{X} \arrow[r, "\widehat{X}\Box f"'] & \widehat{X}\Box(\widehat{X}\Box Y) \arrow[r, "m^{\widehat{X}}_Y"'] & \widehat{X}\Box Y \arrow[r, "\mathsf{inl}\,\Box Y"'] & (\widehat{X}+P)\Box Y
\end{tikzcd}$$

This completes the proof that (4.16) commutes. Thus, (4.7) commutes. $\square$

Lemma 4.9. *For every finitely presentable object Y the morphism i_Y (see Diagram (4.1)) is an isomorphism with the inverse*

$$i_Y^{-1} = j_Y \equiv RY \Box Y \xrightarrow{RY\Box\eta_Y} RY \Box RY \xrightarrow{\rho_Y} RY \tag{4.17}$$

Proof. (1) We prove $j_Y \cdot i_Y = id_{RY}$. Since $RY = \operatorname{colim} \mathrm{Eq}_Y$, it is sufficient to prove

$$j_Y \cdot i_Y \cdot e^\sharp = e^\sharp \tag{4.18}$$

for all $e : X \longrightarrow X \Box Y$ in EQ_Y.

We form a coproduct, f, of e and $u^Y_Y : Y \longrightarrow Y \Box Y$ in EQ_Y: we first denote by

$$X \xrightarrow{\mathsf{inl}} Z \xleftarrow{\mathsf{inr}} Y$$

a coproduct in the underlying category, and then denote by

$$f : Z \longrightarrow Z \Box Y$$

the unique morphism for which the squares

$$
\begin{tikzcd}
X \arrow[r, "e"] \arrow[d, "\mathsf{inl}"'] & X \,\Box\, Y \arrow[d, "\mathsf{inl}\,\Box Y"] \\
Z \arrow[r, "f"'] & Z \,\Box\, Y
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "u_Y^Y"] \arrow[d, "\mathsf{inr}"'] & Y \,\Box\, Y \arrow[d, "\mathsf{inr}\,\Box Y"] \\
Z \arrow[r, "f"'] & Z \,\Box\, Y
\end{tikzcd}
\tag{4.19}
$$

commute. Thus we have

$$
f^\sharp \cdot \mathsf{inl} = e^\sharp \quad \text{and} \quad f^\sharp \cdot \mathsf{inr} = (u_Y^Y)^\sharp = \eta_Y. \tag{4.20}
$$

We combine the right-hand square with

$$
(\mathsf{inr} \,\Box\, Y) \cdot u_Y^Y = u_Y^Z \tag{4.21}
$$

(applying (2.3) to $f = id$ and $h = \mathsf{inr}$) to conclude that the diagram

$$
\begin{tikzcd}
X \,\Box\, Y \arrow[rr, "X \Box u_Y^Y"] \arrow[dd, "\mathsf{inl}\,\Box \mathsf{inr}"'] \arrow[dr, "\mathsf{inl}\,\Box Y"] & & X \,\Box\, (Y \,\Box\, Y) \arrow[dd, "\mathsf{inl}\,\Box(\mathsf{inr}\,\Box Y)"] \\
& Z \,\Box\, Y \arrow[dr, "Z \Box u_Y^Z"] & \\
Z \,\Box\, Z \arrow[rr, "Z \Box f"'] & & Z \,\Box\, (Z \,\Box\, Y)
\end{tikzcd}
\tag{4.22}
$$

commutes. In fact, the outer square commutes by (4.19) and the right-hand triangle by (4.21).

Next define

$$
p \equiv X \xrightarrow{e} X \,\Box\, Y \xrightarrow{\mathsf{inl}\,\Box \mathsf{inr}} Z \,\Box\, Z
$$

and observe that $[Z, \mathsf{inl}] : Z + X \longrightarrow Z$ is a morphism of equations from f_p (see Notation 4.6) to f:

$$
\begin{tikzcd}
Z + X \arrow[r, "f + (\mathsf{inl}\,\Box \mathsf{inr})e"] \arrow[rrr, bend left, "f_p"] \arrow[d, "{[Z,\mathsf{inl}]}"'] & (Z \,\Box\, Y) + (Z \,\Box\, Z) \arrow[r, "{[Z \Box Y, m_Y^Z (Z \Box f)]}"] & Z \,\Box\, Y \arrow[r, "\mathsf{inl}\,\Box Y"] \arrow[dr, equal] & (Z + X) \,\Box\, Y \arrow[d, "{[Z,\mathsf{inl}]\Box Y}"] \\
Z \arrow[rrr, "f"'] & & & Z \,\Box\, Y
\end{tikzcd}
$$

In fact, the left-hand component with domain Z commutes trivially, for the right-hand one precompose (4.22) with e and postcompose with m_Y^Z:

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[dd, "\mathsf{inl}"'] & X \,\square\, Y \arrow[r, "\mathsf{inl}\,\square\,\mathsf{inr}"] \arrow[dr, "\mathsf{inl}\,\square\, Y"'] \arrow[ddl, phantom, "(4.19)"] & Z \,\square\, Z \arrow[r, "Z\square f"] \arrow[d, phantom, "(4.22)"] & Z \,\square\, (Z \,\square\, Y) \arrow[dd, "m_Y^Z"] \\
 & & Z \,\square\, Y \arrow[ur, "Z\square u_Y^Z"'] \arrow[dr, equal] \arrow[r, phantom, "(2.1)"] & {} \\
Z \arrow[rrr, "f"'] & & & Z \,\square\, Y
\end{tikzcd}$$

Therefore,

$$f^\sharp \cdot [Z, \mathsf{inl}\,] = {f_p}^\sharp .$$

From that and (4.20) we get

$$e^\sharp = f^\sharp \cdot \mathsf{inl} \; = f^\sharp \cdot [Z, \mathsf{inl}\,] \cdot \mathsf{inr} \; = {f_p}^\sharp \cdot \mathsf{inr} \, .$$

This implies, due to (4.3) and (4.20)

$$e^\sharp = {f_p}^\sharp \cdot \mathsf{inr} \; = \rho_Y \cdot (f^\sharp \,\square\, f^\sharp) \cdot (\mathsf{inl} \,\square\, \mathsf{inr}\,) \cdot e = \rho_Y \cdot (e^\sharp \,\square\, \eta_Y) \cdot e. \tag{4.23}$$

The proof of (4.18) now follows from this commutative diagram:

$$\begin{tikzcd}
RY \arrow[r, "i_Y"] & RY \,\square\, Y \arrow[rr, "j_Y"] \arrow[dr, "RY\square\eta_Y"] & \arrow[d, phantom, "(4.17)"] & RY \\
\arrow[r, phantom, "(4.1)"] & X \,\square\, Y \arrow[u, "e^\sharp\square Y"] \arrow[r, "e^\sharp\square\eta_Y"'] & RY \,\square\, RY \arrow[ur, "\rho_Y"'] \arrow[d, phantom, "(4.23)"] & \\
X \arrow[uu, "e^\sharp"] \arrow[ur, "e"'] \arrow[rrr, equal] & & {} & X \arrow[uu, "e^\sharp"']
\end{tikzcd} \tag{4.24}$$

(2) The proof of $i_Y \cdot j_Y = id_{RY \square Y}$. By Remark 3.12, it is sufficient to prove that for every morphism $q : Q \longrightarrow RY \,\square\, Y$ with Q finitely presentable we have $i_Y \cdot j_Y \cdot q = q$, i.e.,

$$i_Y \cdot \rho_Y \cdot (RY \,\square\, \eta_Y) \cdot q = q. \tag{4.25}$$

Since $\square$ is finitary in each variable, $RY \,\square\, Y$ is a colimit of $\mathrm{Eq}_Y \,\square\, Y$, therefore, q factors through the colimit injection $e^\sharp \,\square\, Y$ for some $e : X \longrightarrow X \,\square\, Y$ in EQ_Y:

$$\begin{tikzcd}
Q \arrow[r, "q"] \arrow[dr, "q_0"'] & RY \,\square\, Y \\
 & X \,\square\, Y \arrow[u, "e^\sharp\square Y"']
\end{tikzcd} \tag{4.26}$$

We use again the coproduct $f : Z \longrightarrow Z \,\square\, Y$ in EQ_Y of e and u_Y^Y, see (4.19), and define

$$r \equiv Q \xrightarrow{q_0} X \,\square\, Y \xrightarrow{\mathsf{inl}\,\square\,\mathsf{inr}} Z \,\square\, Z \tag{4.27}$$

to obtain the commutative diagram

$$
\begin{tikzcd}
Q \arrow[rr, "r"] \arrow[dr, "q_0"] \arrow[dd, "q"'] & (4.27) & Z \,\square\, Z \arrow[dd, "f^\sharp \square f^\sharp"] \\
(4.26) & X \,\square\, Y \arrow[ur, "\mathsf{inl}\,\square\mathsf{inr}"'] \arrow[dl, "e^\sharp \square Y"'] \arrow[dr, "e^\sharp \square \eta_Y"] & (4.19) \\
RY \,\square\, Y \arrow[rr, "RY \square \eta_Y"'] & & RY \,\square\, RY
\end{tikzcd}
\tag{4.28}
$$

It is easy to verify that $\mathsf{inl} : Z \longrightarrow Z + Q$ is a morphism of equations from f to f_r (see Notation 4.6), thus

$$f^\sharp = {f_r}^\sharp \cdot \mathsf{inl}\,. \tag{4.29}$$

The definition of f_r yields

$$f_r \cdot \mathsf{inr} = (\mathsf{inl}\,\square Y)\cdot m_Y^Z \cdot (Z \square f)\cdot r = (\mathsf{inl}\,\square Y)\cdot m_Y^Z \cdot (Z \square f)\cdot (\mathsf{inl}\,\square \mathsf{inr}\,)\cdot q_0. \tag{4.30}$$

Consequently, the diagram

$$
\begin{tikzcd}
RY\square Y \arrow[r, "RY\square\eta_Y"] & RY\square RY \arrow[r, "\rho_Y"] & RY \arrow[rr, "i_Y"] & & RY\square Y \\
(4.28) & Z\square Z \arrow[u, "f^\sharp\square f^\sharp"'] & (4.3) & (4.1) & \\
Q \arrow[u, "q"] \arrow[ur, "r"'] \arrow[rr, "\mathsf{inr}"'] \arrow[d, "q_0"'] & & Z+Q \arrow[uu, "{f_r}^\sharp"'] \arrow[r, "f_r"'] & (Z+Q)\square Y \arrow[uur, "{f_r}^\sharp\square Y"] & (4.29) \\
X\square Y \arrow[rr, "\mathsf{inl}\,\square\mathsf{inr}"] \arrow[dd, "\mathsf{inl}\,\square Y"'] & & Z\square Z \arrow[d, "Z\square f"] & (4.30) & \\
 & (4.22) & Z\square(Z\square Y) \arrow[drr, "m_Y^Z"] & & \\
Z\square Y \arrow[urr, "Z\square u_Y^Z"'] \arrow[rrrr, equal] & & (2.1) & & Z\square Y \arrow[uuuuu, "f^\sharp\square Y"'] \arrow[uuul, "\mathsf{inl}\,\square Y"']
\end{tikzcd}
$$

commutes, which proves (4.25) due to (4.20) and (4.26):

$$i_Y \cdot \rho_Y \cdot (RY \,\square\, \eta_Y) \cdot q = (f^\sharp \,\square\, Y) \cdot (\mathsf{inl}\ \square\, Y) \cdot q_0 = (e^\sharp \,\square\, Y) \cdot q_0 = q.$$

$\square$

Definition 4.10. The isomorphisms

$$j_Y \equiv RY \,\square\, Y \xrightarrow{RY \square \eta_Y} RY \,\square\, RY \xrightarrow{\rho_Y} RY$$

are called *rational constructors* of the base $\square$.

Theorem 4.11. *For every finitely presentable object Y the algebra (RY, ρ_Y) is a free iterative algebra on Y w.r.t. the universal arrow $\eta_Y : Y \longrightarrow RY$.*

Proof. (1) (RY, ρ_Y) is iterative. In fact, every equation morphism

$$e : X \longrightarrow X \,\square\, RY$$

has a unique solution obtained as follows. Since $X \,\square\, RY = \operatorname{colim} X \,\square\, \mathrm{Eq}_Y$, see Remark 4.8, e factors through the colimit injection $X \,\square\, f^\sharp$ for some $f : V \longrightarrow V \,\square\, Y$ in EQ_Y. Thus, we have a commutative triangle

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, "e_0"'] & X \,\square\, RY \\ & X \,\square\, V \arrow[u, "X \square f^\sharp"'] \end{tikzcd} \tag{4.31}$$

We form an equation $\widetilde{e} : X + V \longrightarrow (X + V) \,\square\, Y$ as follows:

$$\begin{tikzcd} X \arrow[r, "e_0"] \arrow[d, "\mathsf{inl}"'] & X \square V \arrow[r, "X \square f"] & X \square (V \square Y) \arrow[r, "\mathsf{inl} \,\square (\mathsf{inr} \,\square Y)"] & (X+V) \square ((X+V) \square Y) \arrow[d, "m_Y^{X+V}"] \\ X+V \arrow[rrr, "\widetilde{e}"] & & & (X+V) \square Y \\ V \arrow[u, "\mathsf{inr}"] \arrow[rrr, "f"'] & & & V \square Y \arrow[u, "\mathsf{inr} \,\square Y"'] \end{tikzcd} \tag{4.32}$$

We will prove that the given equation morphism e has the unique solution

$$e^\dagger \equiv X \xrightarrow{\mathsf{inl}} X + V \xrightarrow{\widetilde{e}^\sharp} RY \,. \tag{4.33}$$

From (4.17) and (4.1) we have

$$j_Y \cdot (f^\sharp \,\square\, Y) \cdot f = i_Y^{-1} \cdot (f^\sharp \,\square\, Y) \cdot f = f^\sharp. \tag{4.34}$$

Furthermore, $\mathsf{inr} : V \longrightarrow X + V$ is a morphism of equations from f to $\widetilde{e}$ (see the lower square of (4.32)), thus,

$$\widetilde{e}^\sharp \cdot \mathsf{inr} = f^\sharp. \tag{4.35}$$

This proves that the diagram

$$
\begin{tikzcd}
X\Box V \arrow[r, "X\Box f"] \arrow[dd, "\mathsf{inl}\,\Box V"] \arrow[ddd, bend right=60, "e^\dagger\Box f^\sharp"'] & X\Box(V\Box Y) \arrow[r, "\mathsf{inl}\,\Box(\mathsf{inr}\,\Box Y)"] \arrow[d, "\mathsf{inl}\,\Box(V\Box Y)"'] & (X+V)\Box((X+V)\Box Y) \arrow[rr, "m_Y^{X+V}"] \arrow[d, "\widetilde{e}^\sharp\Box(\widetilde{e}^\sharp\Box Y)"] \arrow[d, phantom, "(4.35)"', shift right=8] \arrow[dr, phantom, "(2.4)"] & & (X+V)\Box Y \arrow[dl, "\widetilde{e}^\sharp\Box Y"] \arrow[dd, "\widetilde{e}^\sharp\Box\eta_Y"] \\
\arrow[r, phantom, "(4.34)"] & (X+V)\Box(V\Box Y) \arrow[r, "\widetilde{e}^\sharp\Box(f^\sharp\Box Y)"] & RY\Box(RY\Box Y) \arrow[r, "m_Y^{RY}"] \arrow[d, "RY\Box(RY\Box\eta_Y)"] \arrow[ddll, "RY\Box j_Y"'] \arrow[dr, phantom, "(2.4)"] \arrow[dl, phantom, "(4.17)"] & RY\Box Y \arrow[dr, "RY\Box\eta_Y"] & \\
(X+V)\Box V \arrow[d, "\widetilde{e}^\sharp\Box f^\sharp"] & & RY\Box(RY\Box RY) \arrow[rr, "m_{RY}^{RY}"'] \arrow[dll, "RY\Box\rho_Y"] \arrow[d, phantom, "(2.5)"] & & RY\Box RY \arrow[d, "\rho_Y"] \\
RY\Box RY \arrow[rrrr, "\rho_Y"'] & & & & RY
\end{tikzcd}
\tag{4.36}
$$

commutes. Denote by

$$
q : X \Box V \longrightarrow (X+V) \Box Y
$$

the upper horizontal morphism of (4.36).

The proof that (4.33) is a solution of e follows from the fact that the outward square of the diagram

$$
\begin{tikzcd}
X \arrow[rr, "\mathsf{inl}"] \arrow[dr, "e_0"] \arrow[dd, "e"'] & & X+V \arrow[rr, "\widetilde{e}^\sharp"] \arrow[d, "\widetilde{e}"] \arrow[dl, phantom, "(4.32)"] \arrow[dr, phantom, "(4.1)"] & & RY \arrow[dl, shift right, "i_Y"'] \\
\arrow[r, phantom, "(4.31)"] & X\Box V \arrow[r, "q"] \arrow[dl, "X\Box f^\sharp"'] \arrow[drrr, bend right=15, "e^\dagger\Box f^\sharp"'] & (X+V)\Box Y \arrow[r, "\widetilde{e}^\sharp\Box Y"] \arrow[drr, "\widetilde{e}^\sharp\Box\eta_Y"'] \arrow[d, phantom, "(*)"] & RY\Box Y \arrow[ur, shift right, "j_Y"'] \arrow[dr, "RY\Box\eta_Y"] \arrow[r, phantom, "(4.17)"] & \\
X\Box RY \arrow[rrrr, "e^\dagger\Box RY"'] & & & & RY\Box RY \arrow[uu, "\rho_Y"']
\end{tikzcd}
$$

commutes — in fact, all inner parts except $(*)$ commute, and by (4.36) the triangle $(*)$ commutes when postcomposed by ρ_Y, the right-hand vertical arrow.

It remains to prove that the solution $e^\dagger$ is unique. Given another solution

$$
\begin{tikzcd}
X \arrow[r, "s"] \arrow[d, "e"'] & RY \\
X \Box RY \arrow[r, "s\Box RY"'] & RY \Box RY \arrow[u, "\rho_Y"']
\end{tikzcd}
\tag{4.37}
$$

we prove that the square

$$\begin{tikzcd}
X + V \arrow[r, "\widetilde{e}"] \arrow[d, "{[s,f^\sharp]}"'] & (X + V) \square Y \arrow[d, "{[s,f^\sharp] \square Y}"] \\
RY \arrow[r, "i_Y"'] & RY \square Y
\end{tikzcd} \tag{4.38}$$

commutes. By Lemma 4.5, it follows that $\widetilde{e}^\sharp = [s, f^\sharp]$, thus

$$e^\dagger = \widetilde{e}^\sharp \cdot \mathsf{inl} = [s, f^\sharp] \cdot \mathsf{inl} = s\,.$$

The right-hand component of (4.38) with domain V commutes:

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "\mathsf{inr}"'] \arrow[dd, bend right=60, "f^\sharp"'] \arrow[dr, phantom, "(4.32)"] & V \square Y \arrow[d, "\mathsf{inr} \square Y"] \arrow[dd, bend left=60, "f^\sharp \square Y"] \\
X + V \arrow[r, "\widetilde{e}"] \arrow[d, "{[s,f^\sharp]}"'] \arrow[dr, phantom, "(4.38)"] & (X + V) \square Y \arrow[d, "{[s,f^\sharp] \square Y}"] \\
RY \arrow[r, "i_Y"'] & RY \square Y
\end{tikzcd}$$

For the left-hand component we prove first that the equation

$$m_Y^{RY} \cdot (RY \square i_Y) = i_Y \cdot \rho_Y \tag{4.39}$$

holds. In fact, in the diagram

$$\begin{tikzcd}
RY \square RY \arrow[r, "RY \square i_Y"] \arrow[d, "\rho_Y"'] & RY \square (RY \square Y) \arrow[r, "RY \square (RY \square \eta_Y)"] \arrow[d, "m_Y^{RY}"'] \arrow[dr, phantom, "(2.4)"] & RY \square (RY \square RY) \arrow[r, "RY \square \rho_Y"] \arrow[d, "m_{RY}^{RY}"'] \arrow[dr, phantom, "(2.5)"] & RY \square RY \arrow[d, "\rho_Y"] \\
RY \arrow[r, "i_Y"'] & RY \square Y \arrow[r, "RY \square \eta_Y"'] & RY \square RY \arrow[r, "\rho_Y"'] & RY
\end{tikzcd}$$

the horizontal morphisms are both identity morphisms (see (4.17)), which proves that the outward square commutes. Consequently, the left-hand square, which is (4.39), commutes since $j_Y = \rho_Y \cdot (RY \square \eta_Y)$ is an isomorphism, see (4.17).

The left-hand component of (4.38) is the outward square of the commutative diagram

$$
\begin{tikzcd}
X \arrow[rrr, "s"] \arrow[d, "e_0"'] \arrow[dr, "e"] \arrow[dddd, bend right=60, "\widetilde{e}\cdot\mathsf{inl}"'] & & & RY \arrow[dddd, "i_Y"] \\
X \Box V \arrow[r, "X \Box f^\sharp"'] \arrow[d, "X \Box f"'] \arrow[ur, phantom, "(4.31)"] & X \Box RY \arrow[r, "s \Box RY"] \arrow[r, phantom, "(4.37)", shift left=4] & RY \Box RY \arrow[ur, "\rho_Y"] \arrow[d, "RY \Box i_Y"] \arrow[dr, phantom, "(4.39)"] & \\
X \Box (V \Box Y) \arrow[rr, "s \Box (f^\sharp \Box Y)"] \arrow[d, "\mathsf{inl} \Box (\mathsf{inr} \Box Y)"'] \arrow[ur, phantom, "(4.1)"] & & RY \Box (RY \Box Y) \arrow[ddr, "m_Y^{RY}"] & \\
(X+V) \Box ((X+V) \Box Y) \arrow[urr, "{[s,f^\sharp] \Box ([s,f^\sharp] \Box Y)}"'] \arrow[d, "m_Y^{X+V}"'] \arrow[drrr, phantom, "(2.4)"] \arrow[l, phantom, "(4.32)"] & & & \\
(X+V) \Box Y \arrow[rrr, "{[s,f^\sharp] \Box Y}"'] & & & RY \Box Y
\end{tikzcd}
$$

(2) RY is free, i.e., for every iterative algebra $\alpha : A \Box A \longrightarrow A$ and for every morphism $h_0 : Y \longrightarrow A$ there exist a unique homomorphism $h : RY \longrightarrow A$ with $h \cdot \eta_Y = h_0$.

(2a) Existence of h. For every equation morphism $g : X \longrightarrow X \Box Y$ in EQ_Y we form $h_0 \bullet g : X \longrightarrow X \Box A$, see Notation 3.9, and obtain the unique solution $(h_0 \bullet g)^\dagger : X \longrightarrow A$. This is a cocone of the diagram Eq_Y. In fact, given a morphism

$$
\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "p"'] & X \Box Y \arrow[d, "p \Box Y"] \\
X' \arrow[r, "g'"'] & X' \Box Y
\end{tikzcd}
\tag{4.40}
$$

in EQ_Y, then $(h_0 \bullet g')^\dagger \cdot p$ is a solution of $h_0 \bullet g$:

$$
\begin{tikzcd}
X \arrow[r, "p"] \arrow[d, "g"'] \arrow[dr, phantom, "(4.40)"] & X' \arrow[r, "(h_0 \bullet g')^\dagger"] \arrow[d, "g'"] \arrow[dr, phantom, "(3.1)"] & A \\
X \Box Y \arrow[r, "p \Box Y"] \arrow[dd, "X \Box h_0"'] & X' \Box Y \arrow[d, "X' \Box h_0"] & \\
& X' \Box A \arrow[dr, "(h_0 \bullet g')^\dagger \Box A"] & \\
X \Box A \arrow[ur, "p \Box A"] \arrow[rr, "((h_0 \bullet g)^\dagger \cdot p) \Box A"'] & & A \Box A \arrow[uuu, "\alpha"']
\end{tikzcd}
$$

Thus, $(h_0 \bullet g')^\dagger = (h_0 \bullet g)^\dagger \cdot p$, as desired. Consequently, we can define h by the commutativity of the triangles

$$
\begin{tikzcd}
RY \arrow[r, "h"] & A \\
X \arrow[u, "g^\sharp"] \arrow[ur, "(h_0 \bullet g)^\dagger"'] &
\end{tikzcd}
\tag{4.41}
$$

for all g in EQ_Y. We will show that this is the unique algebra homomorphism extending h_0.

We first prove that

$$h_0 = h \cdot \eta_Y = h \cdot (u_Y^Y)^\sharp.$$

In fact, the commutative diagram

$$
\begin{tikzcd}
Y \arrow[rr, "h_0"] \arrow[d, "u_Y^Y"'] \arrow[drr, phantom, "(2.3)"] & & A \arrow[r, equal] \arrow[ddr, "u_A^A"'] \arrow[dr, phantom, "(2.5)"] & A \\
Y \,\Box\, Y \arrow[d, "Y \Box h_0"'] \arrow[drrr, "h_0 \Box h_0"] & & & \\
Y \,\Box\, A \arrow[rrr, "h_0 \Box A"'] & & & A \,\Box\, A \arrow[uu, "\alpha"']
\end{tikzcd}
$$

shows that h_0 is a solution of $h_0 \bullet u_Y^Y$ in A, thus, by (4.41) we have

$$h_0 = (h_0 \bullet u_Y^Y)^\dagger = h \cdot (u_Y^Y)^\sharp.$$

Next we will prove that h is a homomorphism. Due to Remark 3.11, it is sufficient to prove that h preserves solutions: for every equation morphism $e : X \longrightarrow X \,\Box\, RY$ we form

$$\overline{e} = h \bullet e \equiv X \xrightarrow{e} X \,\Box\, RY \xrightarrow{X \Box h} X \,\Box\, A \tag{4.42}$$

and we prove that

$$\overline{e}^\dagger = h \cdot e^\dagger. \tag{4.43}$$

Recall from (4.31) and (4.32) the equation morphisms $f : V \longrightarrow V \,\Box\, Y$ and $\widetilde{e} : X + V \longrightarrow (X + V) \,\Box\, Y$, and put

$$\overline{f} = h_0 \bullet f \equiv V \xrightarrow{f} V \,\Box\, Y \xrightarrow{V \Box h_0} V \,\Box\, A\,. \tag{4.44}$$

Let us prove that

$$[\overline{e}^\dagger, \overline{f}^\dagger] = (h_0 \bullet \widetilde{e})^\dagger : X + V \longrightarrow A. \tag{4.45}$$

Then (4.43) follows: due to (4.33) and (4.41) applied to $g = \widetilde{e}$ we get the equation

$$h \cdot e^\dagger = h \cdot \widetilde{e}^\sharp \cdot \mathsf{inl} = (h_0 \bullet \widetilde{e})^\dagger \cdot \mathsf{inl} = \overline{e}^\dagger.$$

Thus, the proof of (2a) will be complete by proving that the square

$$\begin{tikzcd}
X+V \arrow[rr, "{[\overline{e}^\dagger,\overline{f}^\dagger]}"] \arrow[d, "\widetilde{e}"'] & & A \\
(X+V)\,\Box\,Y \arrow[d, "(X+V)\Box h_0"'] & & \\
(X+V)\,\Box\,A \arrow[rr, "{[\overline{e}^\dagger,\overline{f}^\dagger]\Box A}"'] & & A\,\Box\,A \arrow[uu, "\alpha"']
\end{tikzcd} \tag{4.46}$$

commutes.

For the right-hand component with domain V this follows from the commutative diagram

$$\begin{tikzcd}
V \arrow[rr, "\overline{f}^\dagger"] \arrow[d, "f"'] \arrow[dr, bend left, "\overline{f}"] \arrow[dd, bend right=90, "\widetilde{e}\cdot\mathsf{inr}"'] \arrow[d, phantom, shift right=30, "(4.32)"] \arrow[dr, phantom, shift right=3, "(4.44)"] \arrow[drr, phantom, "(3.1)"] & & A \\
V\,\Box\,Y \arrow[d, "\mathsf{inr}\,\Box Y"'] \arrow[dr, "V\Box h_0"] & & \\
(X+V)\,\Box\,Y \arrow[d, "(X+V)\Box h_0"'] & V\,\Box\,A \arrow[dl, "\mathsf{inr}\,\Box A"'] \arrow[dr, "\overline{f}^\dagger\Box A"] & \\
(X+V)\,\Box\,A \arrow[rr, "{[\overline{e}^\dagger,\overline{f}^\dagger]\Box A}"'] & & A\,\Box\,A \arrow[uuu, "\alpha"']
\end{tikzcd}$$

For the left-hand component with domain X, first observe that the definition of solution (3.1) yields

$$\overline{e}^\dagger = \alpha\cdot(\overline{e}^\dagger\,\Box\,A)\cdot(X\,\Box\,h)\cdot e = \alpha\cdot(\overline{e}^\dagger\,\Box\,h)\cdot e\,, \tag{4.47}$$

and

$$\overline{f}^\dagger = \alpha\cdot(\overline{f}^\dagger\,\Box\,A)\cdot(V\,\Box\,h_0)\cdot f = \alpha\cdot(\overline{f}^\dagger\,\Box\,h_0)\cdot f. \tag{4.48}$$

Therefore, we get a commutative diagram

$$
\begin{tikzcd}
X \arrow[rrr, "\overline{e}^\dagger"] \arrow[d, "e_0"'] \arrow[dr, "e"] \arrow[dddd, bend right=60, "\widetilde{e}\cdot\mathsf{inl}"'] \arrow[ddd, phantom, "(4.32)"'] & \arrow[d, phantom, "(4.31)"'] & \arrow[d, phantom, "(4.47)"] & A \\
X\Box V \arrow[r, "X\Box f^\sharp"'] \arrow[d, "X\Box f"'] & X\Box RY \arrow[r, "\overline{e}^\dagger\Box h"'] \arrow[d, phantom, "(4.48)\&(4.41)"] & A\Box A \arrow[ur, "\alpha"] & \\
X\Box(V\Box Y) \arrow[rr, "\overline{e}^\dagger\Box(\overline{f}^\dagger\Box h_0)"] \arrow[d, "\mathsf{inl}\Box(\mathsf{inr}\Box Y)"'] & & A\Box(A\Box A) \arrow[u, "A\Box\alpha"'] \arrow[dddr, "m_A^A"] \arrow[r, phantom, "(2.5)"] & \, \\
(X+V)\Box((X+V)\Box Y) \arrow[urr, "{[\overline{e}^\dagger,\overline{f}^\dagger]\Box([\overline{e}^\dagger,\overline{f}^\dagger]\Box h_0)}"'] \arrow[d, "m_Y^{X+V}"'] & & \arrow[d, phantom, "(2.4)"] & \\
(X+V)\Box Y \arrow[drrr, "{[\overline{e}^\dagger,\overline{f}^\dagger]\Box h_0}"] \arrow[d, "(X+V)\Box h_0"'] & & \, & \\
(X+V)\Box A \arrow[rrr, "{[\overline{e}^\dagger,\overline{f}^\dagger]\Box A}"'] & & & A\Box A \arrow[uuuuu, "\alpha"']
\end{tikzcd}
$$

which shows that the left-hand component of (4.46) commutes.

(2b) Uniqueness. Let $k : RY \longrightarrow A$ be a homomorphism, i.e.

$$k \cdot \rho_Y = \alpha \cdot (k \Box k) \tag{4.49}$$

with

$$h_0 = k \cdot \eta_Y . \tag{4.50}$$

Then for every equation morphism $g : X \longrightarrow X \Box Y$ of EQ_Y the diagram

$$
\begin{tikzcd}
X \arrow[rr, "g^\sharp"] \arrow[dd, "g"] \arrow[ddd, bend right=60, "h_0\bullet g"'] \arrow[dd, phantom, "(3.2)"', shift right=8] & & RY \arrow[rr, "k"] \arrow[dl, shift right, "i_Y"'] & & A \\
\arrow[r, phantom, "(4.1)"] & RY\Box Y \arrow[ur, shift right, "j_Y"'] \arrow[dr, equals] & \arrow[r, phantom, "(4.17)"] & RY\Box RY \arrow[ul, "\rho_Y"'] \arrow[ddr, "k\Box k"] \arrow[ur, phantom, "(4.49)"] & \\
X\Box Y \arrow[rr, "g^\sharp\Box Y"'] \arrow[d, "X\Box h_0"'] & & RY\Box Y \arrow[ur, "RY\Box\eta_Y"'] \arrow[drr, "k\Box h_0"'] \arrow[r, phantom, "(4.50)"'] & \, & \\
X\Box A \arrow[rrrr, "(k\cdot g^\sharp)\Box A"'] & & & & A\Box A \arrow[uuu, "\alpha"']
\end{tikzcd}
$$

commutes, proving

$$k \cdot g^\sharp = (h_0 \bullet g)^\dagger = h \cdot g^\sharp ,$$

see (4.41). Thus, $k = h$, since the morphisms $g^\sharp$ form a colimit cocone. $\Box$

Remark 4.12. We now extend the above definitions of j_Y, ρ_Y, η_Y and μ_Y to all objects Y of $\mathcal{A}$ by systematically using filtered colimits:

(a) Definition of $\rho_Y : RY \,\square\, RY \longrightarrow RY$: Express Y as a filtered colimit of finitely presentable objects Y_t with a colimit cocone $y_t : Y_t \longrightarrow Y$, $(t \in T)$. Then

$$RY = \operatorname*{colim}_{t\in T} RY_t = \operatorname*{colim}_{t\in T} \operatorname{colim} \mathrm{Eq}_{Y_t}.$$

Now observe that every object $e : X \longrightarrow X \,\square\, Y = \operatorname*{colim}_{t\in T} X \,\square\, Y_t$ can be expressed as

$$\begin{tikzcd} X \arrow[r, "\overline{e}"] \arrow[d, equal] & X \,\square\, Y_t \arrow[d, "X \square y_t"] \\ X \arrow[r, "e"'] & X \,\square\, Y \end{tikzcd}$$

where $\overline{e}$ is an object of EQ_{Y_t}. From that it easily follows that $\operatorname*{colim}_{t\in T} \operatorname{colim} \mathrm{Eq}_{Y_t} = \operatorname{colim} \mathrm{Eq}_Y$.

Since $\square$ is finitary in both variables, from $RY = \operatorname*{colim}_{t\in T} RY_t$ conclude $RY \,\square\, RY = \operatorname*{colim}_{t\in T} RY_t \,\square\, RY_t$. Define

$$\rho_Y = \operatorname*{colim}_{t\in T} \rho_{Y_t} : RY \,\square\, RY \longrightarrow RY.$$

Observe that

$$\rho_Y \cdot u_{RY}^{RY} = id_{RY} \tag{4.51}$$

because for finitely presentable objects this is true by (4.5), and we have $u_{RY}^{RY} = \operatorname{colim} u_{RY_t}^{RY_t}$, see Remark 4.2. Analogously, the equality (4.7), applied to each RY_t, yields

$$\rho_Y \cdot (RY \,\square\, \rho_Y) = \rho_Y \cdot m_{RY}^{RY}. \tag{4.52}$$

(b) Definition of $j_Y = i_Y^{-1} : RY \longrightarrow RY \,\square\, Y$: Observe that the definition of i_Y in (4.1) clearly implies that $i_Y = \operatorname*{colim}_{t\in T} i_{Y_t}$. Since each i_{Y_t} is an isomorphism, see (4.17), so is i_Y. Moreover, the inverse is, by (4.17), equal to

$$j_Y = \rho_Y \cdot (RY \,\square\, \eta_Y). \tag{4.53}$$

To prove this we just observe that $\eta_Y = \operatorname{colim} \eta_{Y_t}$, since the monad R is finitary.

Corollary 4.13. *A left adjoint*

$$R : \mathcal{A} \longrightarrow \mathsf{Alg}_{\mathrm{it}}\, \square$$

of the forgetful functor U_{it}, see Proposition 3.13, is given pointwise by

$$RY = \operatorname{colim} \mathrm{Eq}_Y.$$

In fact, for finitely presentable Y's the above has been established in Theorem 4.11. Since R preserves (filtered) colimits, the argument for Y arbitrary is simple: express $Y = \operatorname*{colim}_t Y_t$ as in Remark 4.12.

5 Polynomial Functors

This section is devoted to a general type of bases: those freely generated by parametrized signatures. In $\mathcal{A} = \mathsf{Set}$, parametrized signatures and their bases were studied in [AMV$_3$] in some detail. The step from Set to $\mathcal{A}$, a general locally finitely presentable category, follows the ideas of Max Kelly and John Power [KP] where signatures are introduced as collections

$$\Sigma = (\Sigma(n))_{n \in \mathcal{A}_{fp}}$$

of objects $\Sigma(n)$ of $\mathcal{A}$ indexed by objects n of $\mathcal{A}_{fp}$ (the small subcategory of $\mathcal{A}$ representing finitely presentable objects).

We first recall this non-parametrized concept of signature and illustrate it on two examples in $\mathcal{A} = \mathsf{Pos}$, the category of posets and monotone maps. Then we turn to parametrized signatures and continue our poset example there.

Definition 5.1. (see [KP]) A *signature* Σ is a family $\Sigma(n)$, $n \in \mathcal{A}_{fp}$, of objects. A Σ-*algebra* is an object A of $\mathcal{A}$ together with functions

$$\widehat{(-)} : \mathcal{A}(n, A) \longrightarrow \mathcal{A}(\Sigma(n), A), \quad n \in \mathcal{A}_{fp}$$

assigning to every morphism $f : n \longrightarrow A$ a morphism $\widehat{f} : \Sigma(n) \longrightarrow A$. A *homomorphism* from $(A, \widehat{(-)}_A)$ to $(B, \widehat{(-)}_B)$ is a morphism $h : A \longrightarrow B$ such that

$$h \cdot \widehat{(f)}_A = \widehat{(h \cdot f)}_B$$

holds for every $f : n \longrightarrow A$, i.e., the triangle

$$\begin{tikzcd} A \arrow[rr, "h"] & & B \\ & \Sigma(n) \arrow[ul, "\widehat{f}_A"] \arrow[ur, "\widehat{(h\cdot f)}_B"'] & \end{tikzcd}$$

commutes.

Notation 5.2. We denote by $\mathsf{Alg}\,\Sigma$ the category of Σ-algebras and their homomorphisms.

Remark 5.3. A more compact way of expressing the structure of a Σ-algebra is a morphism

$$\alpha : \coprod_{n \in \mathcal{A}_{fp}} \mathcal{A}(n, A) \bullet \Sigma(n) \longrightarrow A$$

where $M \bullet \Sigma(n)$ denotes the coproduct of M copies of $\Sigma(n)$.

Thus, for the endofunctor

$$H_\Sigma : \mathcal{A} \longrightarrow \mathcal{A}, \quad A \longmapsto \coprod_{n \in \mathcal{A}_{fp}} \mathcal{A}(n, A) \bullet \Sigma(n)$$

we see that Σ-algebras are precisely the usual algebras for H_Σ (and the same holds true for homomorphisms).

Example 5.4.

(1) If $\mathcal{A} = \mathsf{Set}$ (and objects of $\mathcal{A}_{fp}$ are natural numbers), then algebras have their classical meaning. In fact, given a Σ-algebra A, then for every n-tuple $f : n \longrightarrow A$ and every n-ary operation symbol $\sigma \in \Sigma(n)$ we obtain the corresponding element of A, denoted by $\widehat{f}(\sigma)$. This defines the desired $\widehat{f} : \Sigma(n) \longrightarrow A$.

(2) In Pos there are situations when a binary operation is supposed to be defined only for pairs (x, y) satisfying $x \geq y$ (example: the subtraction operation on natural numbers). This corresponds to the signature Σ assigning to the two-chain $\mathbf{2}$ the value 1 (the terminal object) and to all other finite posets the value 0.

(3) If q denotes the coproduct of $\mathbf{2}$ and 1 and Σ is the signature

$$\Sigma(q) = \mathbf{2} \text{ and } \Sigma(n) = \emptyset \text{ for all } n \neq q,$$

then Σ-algebras have two partial ternary operations σ and τ with the above domain of definition and with $\sigma(a_1, a_2, a_3) \leq \tau(a_1, a_2, a_3)$ whenever $a_1 \leq a_2$.

Remark 5.5.

(1) For later reference, given a poset A, we consider A^* (the set of all finite words on A) as a poset with the least ordering respecting that of A:

$$v_1 \dots v_n \leq w_1 \dots w_m \text{ if and only if } n = m \text{ and } v_i \leq w_i \text{ in } A \ (i = 1, \dots, n)$$

(2) Analogously, the set of all ternary trees with leaves labelled in A is considered as a poset: we write $t \leq s$ if t and s have the same shape (i. e., they are isomorphic if we forget the labelling) and for every leaf of t the t-labelling is smaller or equal to the s-labelling (in A).

(3) Observe that a free Σ-algebra on A, for the signature of Example 5.4(3), is the algebra of all finite ternary trees t with leaves labelled in A such that

$$\text{whenever a node has branches } t_1,\ t_2,\ t_3, \text{ then } t_1 \leq t_2 \tag{5.1}$$

in the sense of the ordering in (2) above.

Remark 5.6. In [AMV$_3$] we defined, for $\mathcal{A} = \mathsf{Set}$, *parametrized signatures* as signatures $\Sigma = (\Sigma(n))_{n\in\mathbb{N}}$ endowed with a function, called *iterativity*, assigning to every operation symbol $\sigma \in \Sigma(n)$ a number $\mathrm{it}(\sigma) = 0, 1, \dots, n$. The corresponding *parametrized polynomial functor* $H_\Sigma : \mathsf{Set} \times \mathsf{Set} \longrightarrow \mathsf{Set}$ is defined by

$$H_\Sigma(X, A) = \coprod_{i,p\in\mathbb{N}} \Sigma(i, p) \times X^i \times A^p$$

where

$$\Sigma(i, p) \text{ are all symbols of } \Sigma \text{ of iterativity } i \text{ and arity } n = i + p.$$

We then introduced Σ-algebras as a concept not related to the parametrization at all: they are just the classical Σ-algebras A, here expressed via

$$\alpha : H(A, A) \longrightarrow A.$$

And we introduced the concept of an *iterative* Σ*-algebra* as an algebra with unique solutions of flat recursive equations. This very much depends on the parametrization: in the recursive equations for every n-ary operation symbol $\sigma \in \Sigma(n)$ with $\mathrm{it}(\sigma) = i$ only the first i variables are allowed to be used for iteration.

Example: let Σ consist of a ternary operation symbol σ of iterativity 2, i.e., $H(X, A) = X^2 \times A$. A Σ-algebra is a set A with a ternary operation $\alpha : H_\Sigma(A, A) = A^3 \longrightarrow A$. It is iterative if and only if every system of equations

$$x_i \approx t_i(x_1, \dots, x_m, a_1, \dots, a_k) \quad (i = 1, \dots, m)$$

where $a_1, \dots, a_k$ are elements of A has a unique solution provided that the right-hand sides t_i are either elements of A or they are finite Σ-trees where each left- and middle-child of σ is a variable and the only right-hand leaf (lying at the end of the "right-most path") is an element $a \in A$.

We now generalize the above concepts to locally finitely presentable categories:

Definition 5.7.

(1) By a *parametrized signature* Σ is meant a collection

$$\Sigma(i, p) \quad (i \text{ and } p \text{ are in } \mathcal{A}_{fp})$$

of objects of $\mathcal{A}$.

(2) The corresponding *parametrized polynomial functor* $H_\Sigma : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ is defined by

$$H_\Sigma(X, A) = \coprod_{i,\, p \text{ in } \mathcal{A}_{fp}} (\mathcal{A}(i, X) \times \mathcal{A}(p, A)) \bullet \Sigma(i, p)$$

on objects, and analogously on morphisms.

(3) The *derived signature* Σ^0 is defined by

$$\Sigma^0(n) = \coprod_{i+p \cong n} \Sigma(i, p) \quad \text{for all } n \text{ in } \mathcal{A}_{fp}.$$

By Σ-algebras and Σ-homomorphisms are meant the concepts of Definition 5.1 for Σ^0.

Example 5.8. Let Σ be the parametrized signature on Pos with

$$\Sigma(\mathbf{2}, 1) = 1, \text{ else } \Sigma(i, p) = \emptyset$$

(where, recall, 1 is the singleton poset and $\mathbf{2}$ is the two-element chain). A Σ-algebra is a poset A together with a partial ternary operation σ with

$$\sigma(a_1, a_2, a_3) \text{ defined if and only if } a_1 \leq a_2 \text{ in } A.$$

Thus this is precisely the algebra for the non-parametrized signature Σ^0 of Example 5.4(3). The corresponding paramterized polynomial functor is $H_\Sigma(X, A) = X^{\mathbf{2}} \times A$, where $X^{\mathbf{2}}$ is the subposet of $X^2 = X \times X$ on all pairs (x_1, x_2) with $x_1 \leq x_2$ in X.

Remark 5.9. Let a finitary functor

$$H : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$$

be given. Recall from Example 2.15 of [AMV$_3$] that H generates a free base given by

$$X \Box A = \text{free algebra for } H(X, -) \text{ on } A.$$

Then the concept of a base algebra coincides with an algebra for the endofunctor $A \longmapsto H(A, A)$. More precisely, the base algebra $\widehat{\alpha} : A \Box A \longrightarrow A$ corresponding to $\alpha : H(A, A) \longrightarrow A$ is obtained by extending the identity on A along the universal arrow $A \longrightarrow A \Box A$.

Similarly, $\alpha : H(A, A) \longrightarrow A$ is called an iterative algebra for H if $\widehat{\alpha} : A \Box A \longrightarrow A$ is an iterative base algebra in the sense of Definition 3.2.

Example 5.10. Let Γ be the parametrized signature on Pos which corresponds to $H(X, A) = X^2 \times A$. That is, for discrete posets 1 and 2 (on one and two elements, respectively) we have

$$\Gamma(2, 1) = 1, \text{ else } \Gamma(i, p) = \emptyset.$$

Thus, Γ-algebras are posets endowed with a ternary, order-preserving operation. A free Γ-algebra on A is the algebra of all finite ternary trees with leaves labelled in A ordered as in Remark 5.5(2).

It is easy to describe a free iterative Γ-algebra on A (compare with Example 3.16(ii) of [AMV$_3$]): let us call an infinite ternary tree t *right-well-founded* provided that for every node of t the right-most path from that node is finite. A free iterative Γ-algebra

$$R_\Gamma(A)$$

is the algebra of all right-well-founded ternary trees with leaves labelled in A, ordered as in Remark 5.5(2).

This is just completely analogous to the situation of the polynomial functor $H(X, A) = X \times X \times A$ in Set.

Example 5.11. Let us describe the rational monad of the signature Σ^0 of Example 5.4(2). We know that free Σ-algebras are strong subobjects of free Γ-algebras, see Remark 5.5 (where *strong* refers to strong monomorphisms in Pos) and we will prove that, analogously, free iterative Σ-algebras are strong subalgebras of free iterative Γ-algebras formed by precisely those right-well-founded trees which fulfill (5.1) above.

Denote by

$$\mu : H_\Sigma \longrightarrow H_\Gamma$$

the natural transformation whose components are the inclusion maps

$$\mu_{X,A} : X^{\mathbf{2}} \times A \hookrightarrow X^2 \times A.$$

They are, obviously, strong monomorphisms. This natural transformation gives rise to a base morphism which we denote by the same symbol

$$\mu : \Box_\Sigma \longrightarrow \Box_\Gamma$$

because its components are, again, inclusion maps

$$\mu_{X,A} : (X^{\mathbf{2}})^* \times A \hookrightarrow (X^2)^* \times A.$$

In fact, recall that $X \Box_\Gamma A$ is the free algebra on A for the endofunctor $H_\Sigma(X, -) = X^2 \times -$ of unary operations indexed by X^2. It is easy to see that $X \Box_\Gamma A$ can be describes as the poset $(X^2)^* \times A$, see Remark 5.5(1), of all words $(v_1 \dots v_n, a)$ with $v_1, \dots, v_n$ in X^2 and $a \in A$. Analogously, $X \Box_\Sigma A$ is $(X^{\mathbf{2}})^* \times A$, where we recall that $X^{\mathbf{2}}$ is the subposet of X^2 on all compatible pairs.

By applying the coalgebraic construction of Section 4, we see that $R_\Gamma(A)$, the algebra of all right-well-founded ternary trees, is a colimit of the diagram of all

$$e : X \longrightarrow X \,\Box_\Gamma\, A, \quad X \text{ finite.}$$

Analogously, $R_\Sigma(A)$ is a colimit of the diagram of all

$$f : X \longrightarrow X \,\Box_\Sigma\, A, \quad X \text{ finite.}$$

This is a subdiagram of the above one provided that each f is identified with the corresponding composite

$$\overline{f} \equiv X \xrightarrow{f} X \,\Box_\Sigma\, A \xrightarrow{\mu_{X,A}} X \,\Box_\Gamma\, A.$$

Thus, we obtain a natural transformation

$$\lambda : R_\Sigma \longrightarrow R_\Gamma$$

defined by the colimit injections $f^\sharp$, see Notation 4.1, of the above objects f:

$$\begin{tikzcd} & X \arrow[dl, "f^\sharp"'] \arrow[dr, "\overline{f}^\sharp"] & \\ R_\Sigma A \arrow[rr, "\lambda_A"'] & & R_\Gamma A \end{tikzcd}$$

We prove below that each λ_A is a strong monomorphism. It follows that $R_\Sigma A$ is a strong subalgebra of the algebra $R_\Gamma A$ of all right-well-founded ternary trees. If ρ_Γ denotes the algebra structure of $R_\Gamma A$, and analogously ρ_Σ, then we have a commutative square

$$\begin{tikzcd} H_\Sigma(R_\Sigma A, R_\Sigma A) \arrow[rr, "\rho_\Sigma"] \arrow[d, "H_\Sigma(\lambda_A, \lambda_A)"'] & & R_\Sigma A \arrow[d, "\lambda_A"] \\ H_\Sigma(R_\Gamma A, R_\Gamma A) \arrow[r, "\mu"'] & H_\Gamma(R_\Gamma A, R_\Gamma A) \arrow[r, "\rho_\Gamma"'] & R_\Gamma A \end{tikzcd}$$

This proves that $R_\Sigma A$ is the subposet of all trees of $R_\Gamma A$ satisfying (5.1). Thus, to finish the proof we only need to verify the following:

Lemma 5.12. *The morphism $\lambda_A : R_\Sigma A \longrightarrow R_\Gamma A$ is a strong monomorphism.*

Proof. Denote by EQ_Y^Σ and EQ_Y^Γ the diagrams of Section 4 for H_Σ and H_Γ, respectively.

(a) λ_A is a monomorphism for every A. To prove this, it is sufficient to show that given objects

$$e : X \longrightarrow X \,\Box_\Sigma\, A \quad \text{and} \quad e' : X' \longrightarrow X' \,\Box_\Sigma\, A$$

of EQ_A^Σ and given morphisms u and u' for the corresponding objects $\overline{e}$ and $\overline{e'}$ in the diagram EQ_A^Γ:

$$\begin{tikzcd} \overline{e} \arrow[dr, "u"'] & & \overline{e'} \arrow[dl, "u'"] \\ & f & \end{tikzcd} \qquad \text{for some } f : Z \longrightarrow Z \,\Box_\Gamma\, A$$

then there exists a commutative diagram in EQ_A^Γ

$$\begin{tikzcd} \overline{e} \arrow[r, "v"] \arrow[dr, "u"'] & \overline{g} \arrow[d, "w"] & \overline{e'} \arrow[l, "v'"'] \arrow[dl, "u'"] \\ & f & \end{tikzcd}$$

for some object $g : Y \longrightarrow Y \,\Box_\Sigma\, A$ of EQ_A^Σ.

In fact, since the functor $- \,\Box_\Sigma\, A$ obviously preserves strong monomorphisms, the (epi, strong mono)-factorization system of Pos carries over to EQ_A^Σ. Thus, we can assume without loss of generality that $u : X \longrightarrow Z$ and $u' : X' \longrightarrow Z$ are collectively epimorphic. We conclude that f is actually an object of EQ_A^Σ, i.e., given

$$z \in Z \text{ with } f(z) = ((z_1, z_1'), \dots, (z_n, z_n'), a)$$

then $z_i \le z_i'$ for all $i = 1, \dots, n$. In fact, we can assume that $z = u(x)$ for some $x \in X$ (the case $z = u'(x')$ for some $x' \in X'$ is the same by symmetry). Then we have

$$e(x) = ((x_1, x_1'), \dots, (x_n, x_n'), a)$$

where

$$z_i = u(x_i) \quad \text{and} \quad z_i' = u(x_i') \quad \text{for } i = 1, \dots, n.$$

Since e is an object of EQ_A^Σ, we know that $x_i \le x_i'$, and thus $z_i \le z_i'$.

(b) λ_A is a strong monomorphism, i.e., if $\lambda_A(x) \le \lambda_A(x')$, then $x \le x'$. This follows from the fact that for every colimit map $e^\sharp : X \longrightarrow R_\Sigma A$ for $e : X \longrightarrow X \,\Box_\Sigma\, A$ in EQ_A^Σ we have the corresponding object $\overline{e} : X \longrightarrow X \,\Box_\Gamma\, A$ of EQ_A^Γ with the colimit morphism $\overline{e}^\sharp : X \longrightarrow R_\Gamma A$ satisfying $e^\sharp = \lambda_A \cdot \overline{e}^\sharp$. $\Box$

6 Conclusion and Future Research

Following the idea of Tarmo Uustalu, we studied bases, i.e., finitary functors from $\mathcal{A}$ to the category $\mathsf{FM}(\mathcal{A})$ of all finitary monads on $\mathcal{A}$. Base algebras which model algebras with unique solutions of "certain" iterative equations are called iterative; the specification which iterative equations are considered is a parameter one can choose. This explains our terminology of parametrized iterativity.

In our previous paper [AMV$_3$] we showed how parametrized iterativity works in the category of sets by providing a number of concrete examples. The present paper is devoted to a construction of a free iterative base algebra in an abstract category as a colimit of the filtered diagram of all equations whose solutions are required. This is not surprising in view of the fact that precisely the same result holds in the "non-parametrized world", see [AMV$_1$]. What *is* surprising is the technical difficulty the more general result seems to carry, in spite of the basic idea being essentially the same. On the other hand, it was a relief for us to find out that no side conditions are needed: our proof works for all bases in all locally finitely presentable categories.

In the "non-parametrized world" the ultimate result is an abstract characterization of the rational monad of an endofunctor H as a free iterative theory (in Elgot's sense) on H. The results of the current paper will be used to extend that abstract characterization to the rational monad R over any base $\square$ in a future paper. Recall that a part of the monadic structure of $\square$ is the "multiplication" transformation

$$m_A^X : X \square (X \square A) \longrightarrow X \square A.$$

We are going to introduce the concept of a *module* of $\square$ as a natural transformation

$$s_A^X : (X \square X) \square A \longrightarrow X \square A$$

satisfying certain conditions and then we will show that the canonical module associated to the rational monad has a universal property.

Furthermore, in the "non-parametrized world", although iterativity is introduced by means of unique solvability of just flat equations, iterative algebras also have unique solutions of the (much more flexible) rational equations, see [AMV$_1$]. Also this will be proved later for algebras with parametrized iterativity.

Finally, we intend to provide a description of general bases by means of free bases and "base equations" and show how this projects to the concept of iterative algebras.

Another future project is a categorical description of algebraic trees introduced by Bruno Courcelle, see [C]. We will show how to apply our results in a certain category of endofunctors to obtain this description, i.e., to obtain a monad of (generalized) algebraic trees.

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