BASE MODULES FOR PARAMETRIZED ITERATIVITY

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Abstract. The concept of a base, that is a parametrized finitary monad, which we introduced earlier, followed the footsteps of Tarmo Uustalu in his attempt to formalize parametrized recursion. We proved that for every base free iterative algebras exist, and we called the corresponding monad the rational monad of the base. Here we introduce modules for a base, and we prove that the rational monad of a base gives rise to a canonical module, that is characterized as the free iterative module on the given base.

This generalizes the classical, nonparametric case of iterative Σ-algebras whose rational monad is the monad of rational Σ-trees and that was characterized by Calvin Elgot *et al.* as the free iterative monad on Σ. A basic parametrized example is the base assigning to every parameter set X the monad $A \longmapsto X^* \times A$ whose rational monad is the monad of all right-wellfounded rational binary trees; the rational module for this base is the natural transformation $(X^* \times X) \times A \longrightarrow X^* \times A$ given by parametrized concatenation.

One for the money
Two for the show
Three to get ready
And four to go.

Nursery rhyme

1. Introduction

In our previous work we used bases on a category as a means of studying parametrized iterativity in the sense of Tarmo Uustalu [U]. His idea was to generalize the iterative theories, or iterative monads, of Calvin Elgot [E] to situations in which a set of variables is chosen as a parameter. Thus a monad is called iterative (in the parametrized sense) if every guarded system of recursive equations using only the chosen variables for recursion has a unique solution. This kind of iterativity can, as demonstrated in [AMV$_2$, AMV$_3$], be formalized by studying bases on a category $\mathscr{A}$: a base $\Box$ is a parametrized finitary monad. More precisely, it is a finitary functor assigning to every object X of $\mathscr{A}$ a finitary monad $A \longmapsto X \Box A$ on $\mathscr{A}$. We proved in [AMV$_3$] that every object X of $\mathscr{A}$ generates a free iterative base algebra RX for $\Box$, and the resulting monad $\mathbb{R}$ is called the *rational monad* of the base.

The aim of the present paper is to characterize the rational monad by a universal property. In the non-paramatrized world of iterative monads of Elgot [E] the rational monad was characterized as the free iterative monad on a given finitary endofunctor [AMV$_1$]; we recall this in Section 4.A. In order to formulate the concept of iterativity of a given monad $\mathbb{S} = (S, \eta, \mu)$ Elgot introduced the concept of an ideal: this is a subfunctor S' of S with $S = S' + Id$, where η is the right-hand coproduct injection, and μ restricts to a right S-module $\mu' : S'S \longrightarrow S'$. Then iterativity means that every *guarded* equation morphism for $\mathbb{S}$ has a unique solution, where the concept of *guard* is relative to the ideal S'. Analogously, for bases we introduce the concept of a module: just as a base is a parametrized collection of monads a base module is a base equipped with a parametrized collection of module structures. We then obtain a concept of a guarded equation morphism (relative to a base module) and call the module iterative iff every guarded equation morphism has a unique solution.

With every base $\Box$ we associate the endofunctor $SX = X \Box X$, and it turns out that each base module provides the structure of a finitary monad on S; we call this the *induced monad* of the module. Our main result is the following

Theorem 1.1. *The rational monad of a base is induced by the free base module.*

More precisely, let $\Box$ be a base, and let $\mathbb{R}$ be the rational monad of $\Box$. Then the base given by $RX \Box A$ carries the free iterative module on $\Box$ with the induced monad (isomorphic to) $\mathbb{R}$.

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Example 1.2. ([AMV$_2$]) Consider algebras in Set on one binary operation $*$. We have three choices of a parameter for parametrized iterativity:

(1) If iteration is allowed in both arguments of $*$, then an iterative algebra is a binary algebra $(A, *)$ in which recursive equations such as

$$x \approx x * (a * x), \qquad \text{where } a \in A$$

have a unique solution. Observe that every recursive equation can be flattened so that the right-hand sides are just pairs of variables or single elements of A. For example, introducing new variables y and z the above equation yields the system

$$x \approx x * y \qquad y \approx z * x \qquad z \approx a.$$

Therefore, the base that expresses this type of iterativity is

$$X \mathbin{\square} A = X \times X + A.$$

Its rational monad is given by

$$R_2(X) = \text{all rational binary trees with leaves labelled in } X;$$

"rational tree" means a tree having up to isomorphism finitely many subtrees.

(2) If iteration is retricted to only one argument of $*$, say the left-hand one, then an iterative algebra still has unique solutions of equation systems such as

$$x \approx x * (y * a) \qquad y \approx y * (x * b) \qquad \text{where } a, b \in A.$$

The base for this restricted iterativity is

$$X \mathbin{\square} A = X^* \times A$$

and the rational monad is the submonad of R_2 given by

$$R_1 X = \text{all right-wellfounded trees in } R_2 X$$

meaning that the right-most path from any node is finite.

(3) Finally, iterativity is trivial if no argument of $*$ may be used for iteration: every binary algebra is iterative. The corresponding base is independent of X:

$$X \mathbin{\square} A = \Phi A,$$

where ΦA denotes the free binary algebra on A, which is carried by the set of all finite binary trees on A.

The corresponding rational monad is

$$R_0 X = \text{all finite trees in } R_2 X.$$

In each of the three cases above the free iterative modules are given by $R_i X \mathbin{\square} A$ for $i = 0, 1, 2$.

Contents of the paper. We begin with a summary of the predecessor [AMV$_3$] of our paper in Section 2. In Section 3 we prove that in iterative algebras all rational equation morphisms have a unique solution. In Section 4 we introduce modules for a base and after recalling Elgot's ideal and iterative monads in Subsection 4.A we show in Subsection 4.B how to obtain the rational module for a given base from its rational monad. Our main result in this section is that the category of modules for a base is monadic over the category of bases. In Subsection 4.C we turn our attention to iterative modules. Here we prove that the rational module of a base is the free iterative module on that base. Conclusions form a short Section 5.

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2. A Primer on Bases and Base Algebras

Our paper is a direct continuation of [AMV$_3$]. The aim of this section is to recall the necessary concepts and the basic examples from that paper.

We start by a concrete example demonstrating the concepts introduced below:

Example 2.1. Consider binary algebras in Set which are iterative in the sense that in the formal term $x * y$ (here $*$ is the binary operation) only the variable x can be used for iteration, see Example 1.2(2) above. Thus, an algebra A is iterative iff every system of equations of the form

$$x \approx (\dots(x_1 * x_2) * x_3) * \dots * x_n) * a$$

where a lies in A and x and x_i's are (recursion) variables, has a unique solution in A. Every such system of equations using recursive variables from a finite set X can be expressed by a morphism

$$e : X \longrightarrow X^* \times A.$$

A solution of the system of equations is a substitution

$$e^\dagger : X \longrightarrow A$$

of the recursion variables for the elements of A making the formal equations true identities in A. This means that the square

$$\begin{tikzcd} X \arrow[rr, "e^\dagger"] \arrow[d, "e"'] & & A \\ X^* \times A \arrow[rr, "(e^\dagger)^* \times id"'] & & A^* \times A \arrow[u, "\alpha"'] \end{tikzcd}$$

commutes, where α is the computation of nonempty words in A.

Observe that the codomain $X^* \times A$ of e can be viewed as a functor $\Box$ from Set to the category $\mathsf{FM}(\mathsf{Set})$ of finitary monads on Set: to every X it assigns the monad

$$A \longmapsto X^* \times A$$

whose unit u^X is given by the empty word:

$$u_A^X : A \longrightarrow X^* \times A, \quad a \longmapsto (\varepsilon, a)$$

and multiplication m^X by concatenation:

$$m_A^X : X^* \times (X^* \times A) \longrightarrow X^* \times A, \quad (v, (w, a)) \longmapsto (vw, a).$$

It is rather easy to see that every set X generates a free iterative algebra: its elements are right-wellfounded rational binary trees with leaves labelled in X, its binary operation is the tree-tupling, and the universal arrow assigns to every element x of X the singleton tree labelled by x. This is the rational monad R_1 of Example 1.2(2).

Remark 2.2. In the rest of this section we assume that a *locally finitely presentable category* $\mathscr{A}$ is given, see [GU] or [AR], that is, a cocomplete category $\mathscr{A}$ in which a set $\mathscr{A}_{fp}$ of objects is given such that

(1) every object is a filtered colimit of objects of $\mathscr{A}_{fp}$, and
(2) objects A in $\mathscr{A}_{fp}$ are *finitely presentable*, i.e., the hom-functor $\mathscr{A}(A, -)$ preserves filtered colimits.

Examples: categories of sets, graphs, posets; where finitely presentable means finite. Presheaves $[\mathscr{C}^{op}, \mathsf{Set}]$; where finitely presentable means: a presentation by finitely many generators and finitely many equations exists.

A *monad* on $\mathscr{A}$ is an endofunctor $S : \mathscr{A} \longrightarrow \mathscr{A}$ together with natural transformations "unit" $\eta : Id \longrightarrow S$ and "multiplication" $\mu : S \cdot S \longrightarrow S$ satisfying $\mu \cdot S\eta = id = \mu \cdot \eta S$ and $\mu \cdot S\mu = \mu \cdot \mu S$. It is called *finitary* if S preserves filtered colimits.

A *monad morphism* from (S, η, μ) to $(\overline{S}, \overline{\eta}, \overline{\mu})$ is a natural transformation $h : S \longrightarrow \overline{S}$ such that $h \cdot \eta = \overline{\eta}$ and $\overline{\mu} \cdot (h * h) = h \cdot \mu$ where $h * h : S \cdot S \longrightarrow \overline{S} \cdot \overline{S}$ is the horizontal composition.

Notation 2.3. $\mathsf{FM}(\mathscr{A})$ denotes the category of all finitary monads on $\mathscr{A}$ and monad morphisms.

Example 2.4.

(1) Every finitary endofunctor H of $\mathscr{A}$ generates a free finitary monad: the monad of free H-algebras, see [Ba]. Recall that an H-algebra is a pair (A, a), where A is an object of $\mathscr{A}$ and $a : HA \longrightarrow A$ is a morphism called the *structure* of the algebra. A homomorphism h from (A, a) to (B, b) is a morphism $h : A \longrightarrow B$ of $\mathscr{A}$ satisfying $h \cdot a = b \cdot Hh$.

The classical Σ-algebras in $\mathscr{A} = \mathsf{Set}$ are given by the *polynomial functor*

$$H_\Sigma X = \coprod_{\sigma \in \Sigma} X^{\mathrm{ar}(\sigma)}$$

where $\mathrm{ar}(\sigma)$ denotes the arity (a natural number). The free monad F_Σ assigns to every set A the set $F_\Sigma A$ of all finite Σ-trees on A (with leaves labelled in $A + \Sigma_0$ and n-ary operations labelling nodes with $n > 0$ children).

(2) Every set X yields a finitary monad on Set:

$$A \longmapsto X^* \times A$$

with unit $\eta_A \longrightarrow X^* \times A$ given by $a \longmapsto (\varepsilon, a)$ and multiplication $\mu_A : X^* \times X^* \times A \longrightarrow X^* \times A$ given by concatenation.

Remark 2.5. The last example is parametrized by sets X. Notice also that every morphism $f : X \longrightarrow Y$ in Set yields a monad morphism

$$f^* \times id_A : X^* \times A \longrightarrow Y^* \times A$$

between the corresponding monads. Thus we obtain a *parametrized monad* in the sense of Tarmo Uustalu: this is a finitary functor from Set to $\mathsf{FM}(\mathsf{Set})$. Since we restrict ourselves to finitary monads, we decided for a shorter name, base, in place of "parametrized finitary monad".

Definition 2.6. A *base* is a finitary functor $\square$ from $\mathscr{A}$ to $\mathsf{FM}(\mathscr{A})$. We use the uncurried notation $X \square -$ for the monad on $\mathscr{A}$ assigned to X.

Example 2.7. $X \square A = X^* \times A$ is the base of Example 2.4(2).

Remark 2.8. Thus to give a base means to give

(1) A functor $\mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{A}$, finitary in both variables — notation $X \square A$ for objects and $x \square a : X \square A \longrightarrow X' \square A'$ for morphisms,

(2) the monad units (parametrized by X)

$$u_A^X : A \longrightarrow X \square A,$$

and

(3) the monad multiplications (parametrized by X)

$$m_A^X : X \square (X \square A) \longrightarrow X \square A,$$

satisfying a number of axioms expressing the fact that $\square$ is a functor, each $(X \square -, u_-^X, m_-^X)$ is a monad, and for every morphism $f : X \longrightarrow Y$ in the base category $\mathscr{A}$ we have a monad morphism $f \square - : X \square - \longrightarrow Y \square -$. Explicitly, the diagrams

$$\begin{tikzcd}
X \square A \arrow[r, "X \square u_A^X"] \arrow[dr, equal] & X \square (X \square A) \arrow[d, "m_A^X"] & X \square A \arrow[l, "u_{X \square A}^X"'] \arrow[dl, equal] \\
& X \square A &
\end{tikzcd} \tag{2.1}$$

and

$$\begin{tikzcd}
X \square (X \square (X \square A)) \arrow[r, "X \square m_A^X"] \arrow[d, "m_{X \square A}^X"'] & X \square (X \square A) \arrow[d, "m_A^X"] \\
X \square (X \square A) \arrow[r, "m_A^X"'] & X \square A
\end{tikzcd} \tag{2.2}$$

commute for every X and A (and express the monad axioms for $X \square -$). The diagrams

$$\begin{tikzcd}
A \arrow[r, "u_A^X"] \arrow[d, "f"'] & X \square A \arrow[d, "h \square f"] \\
B \arrow[r, "u_B^Y"'] & Y \square B
\end{tikzcd} \tag{2.3}$$

and

$$\begin{tikzcd}
X \square (X \square A) \arrow[r, "m_A^X"] \arrow[d, "h \square (h \square f)"'] & X \square A \arrow[d, "h \square f"] \\
Y \square (Y \square B) \arrow[r, "m_B^Y"'] & Y \square B
\end{tikzcd} \tag{2.4}$$

express the naturality of u^X and m^X and the fact that for every morphism $h : X \longrightarrow Y$ we have a monad morphism $h \,\square\, (-) : X \,\square\, (-) \longrightarrow Y \,\square\, (-)$.

Remark 2.9. The purpose of the concept of a base is to enable the study of parametrized iterativity of algebras. We thus first introduce the concept of base algebra and then iterative base algebra.

Definition 2.10. A *base algebra* is a pair (A, α) where A is an object of $\mathscr{A}$ and $\alpha : A \,\square\, A \longrightarrow A$ is a morphism making the following two diagrams

$$\begin{tikzcd} A \arrow[r, "u_A^A"] \arrow[dr, equal] & A \,\square\, A \arrow[d, "\alpha"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} A \,\square\, (A \,\square\, A) \arrow[r, "A\square\alpha"] \arrow[d, "m_A^A"'] & A \,\square\, A \arrow[d, "\alpha"] \\ A \,\square\, A \arrow[r, "\alpha"'] & A \end{tikzcd} \tag{2.5}$$

commutative. That is, (A, α) is an Eilenberg-Moore algebra for the monad $A \,\square\, -$.

Base algebras form a category

$$\mathsf{Alg}\,\square$$

where a *homomorphism* from (A, α) to (B, β) is a morphism $h : A \longrightarrow B$ making the square

$$\begin{tikzcd} A \,\square\, A \arrow[r, "\alpha"] \arrow[d, "h\square h"'] & A \arrow[d, "h"] \\ B \,\square\, B \arrow[r, "\beta"'] & B \end{tikzcd} \tag{2.6}$$

commutative.

Example 2.11.

(1) A base algebra for the above base $X \,\square\, A = X^* \times A$ is a set A with a binary operation $*$. In fact, given $\alpha : A^* \times A \longrightarrow A$ satisfying (2.5), then α is determined by its domain restriction $*$ to $A \times A$. Conversely, every binary operation $*$ yields a base algebra with

$$\begin{aligned} \alpha(\varepsilon, x) &= x \\ \alpha(y, x) &= y * x \\ \alpha(y_1 y_2, x) &= y_1 * (y_2 * x)) \\ &\text{etc.} \end{aligned}$$

(2) For every endofunctor H of $\mathscr{A}$ the following base yields H-algebras as base algebras:

$$X \,\square\, A = HX + A.$$

In particular, binary algebras in Set are also given by the base $X \times X + A$.

(3) If Φ denotes the free monad on $HX = X \times X$ then the base

$$X \,\square\, A = \Phi(A)$$

also has binary algebras as base algebras.

(4) For every base $\square$ and every finitary endofunctor $H : \mathscr{A} \longrightarrow \mathscr{A}$ we obtain a new base as the composite

$$\mathscr{A} \xrightarrow{H} \mathscr{A} \xrightarrow{\square} \mathsf{FM}(\mathscr{A}).$$

In uncurried form we have

$$X \; (\square \cdot H) \; A = HX \,\square\, A$$

with the unit given by $u_A^{HX} : A \longrightarrow HX \,\square\, A$ and the multiplication given by $m_A^{HX} : HX \,\square\, (HX \,\square\, A) \longrightarrow HX \,\square\, A$.

Remark 2.12. In [AMV$_3$, Proposition 2.10] it has been proved that the obvious forgetful functor

$$U_\square : \mathsf{Alg}\,\square \longrightarrow \mathscr{A}$$

has a left adjoint (it is even monadic, see [AMV$_3$, Proposition 2.13]) and a free base algebra on X is precisely an initial algebra for the finitary endofunctor $- \,\square\, X : \mathscr{A} \longrightarrow \mathscr{A}$ (see [AMV$_3$, Theorem 2.18]).

Notation 2.13. (1) For every object X of $\mathscr{A}$ we write

$$\varphi_X : FX \,\square\, FX \longrightarrow FX \qquad \text{and} \qquad \eta_X : X \longrightarrow FX$$

for the structure and universal arrow, respectively, of a free base algebra on X.

(2) Now F carries the structure of the monad $\mathbb{F}$ of free base algebras. The monad multiplication of $\mathbb{F}$ is given by the unique base-algebra homomorphisms

$$\mu_X : (FFX, \varphi_{FX}) \longrightarrow (FX, \varphi_X)$$

with $\mu_X \cdot \eta_{FX} = id_{FX}$ (see [AMV$_3$, Notation 2.12]).

(3) We write

$$j'_X : FX \,\Box\, X \longrightarrow FX$$

for the initial algebra for $- \,\Box\, X$. By Lambek's Lemma we have an inverse morphism

$$i'_X : FX \longrightarrow FX \,\Box\, X.$$

Remark 2.14. (1) From [AMV$_3$, Proposition 2.14] we know that

$$j'_X \equiv FX \,\Box\, X \xrightarrow{FX \Box \eta_X} FX \,\Box\, FX \xrightarrow{\varphi_X} FX.$$

It follows that

$$\varphi_X \equiv FX \,\Box\, FX \xrightarrow{\eta_{FX \Box FX}} FFX \,\Box\, FX \xrightarrow{j'_{FX}} FFX \xrightarrow{\mu_X} FX. \tag{2.7}$$

To see this, plug in the previous identity for j'_{FX} and simplify using one unit law for the monad F and the definition of μ as a morphism of base algebras.

(2) Since F carries the structure of the monad $\mathbb{F}$ above, it turns out that also $SX = FX \,\Box\, X$ carries the structure of a monad $\mathbb{S}$ with the unit

$$\eta^S_X \equiv X \xrightarrow{u^{FX}_X} FX \,\Box\, X = SX$$

and the multiplication

$$\mu^S_X \equiv \begin{array}{c} SSX = F(FX \,\Box\, X) \,\Box\, (FX \,\Box\, X) \\ \Big\downarrow {\scriptstyle Fj'_X \Box id} \\ FFX \,\Box\, (FX \,\Box\, X) \\ \Big\downarrow {\scriptstyle \mu_X \Box id} \\ FX \,\Box\, (FX \,\Box\, X) \\ \Big\downarrow {\scriptstyle m^{FX}_X} \\ FX \,\Box\, X = SX \end{array}$$

for which the above isomorphisms form monad isomorphisms

$$i' : \mathbb{F} \longrightarrow \mathbb{S} \qquad \text{and} \qquad j' : \mathbb{S} \longrightarrow \mathbb{F},$$

see [AMV$_3$, Corollary 2.16].

Definition 2.15. ([AMV$_3$]) Suppose (A, α) is a base algebra. Then a (finitary, flat) *equation morphism* is a morphism

$$e : X \longrightarrow X \,\Box\, A, \quad X \text{ finitely presentable.}$$

A *solution* of e is a morphism $e^\dagger : X \longrightarrow A$ making the square

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & A \\ X \,\Box\, A \arrow[r, "e^\dagger \Box A"'] & A \,\Box\, A \arrow[u, "\alpha"'] \end{tikzcd} \tag{2.8}$$

commutative. A base algebra (A, α) is called *iterative*, if every equation morphism $e : X \longrightarrow X \,\Box\, A$ has a unique solution.

Example 2.16.

(1) For classical Σ-algebras the concept of iterativity was introduced by Evelyn Nelson [N] (see a related paper by Jerzy Tiuryn [T]). A Σ-algebra A is iterative if any recursive system of guarded equations

$$x_i \approx t_i(x_1, \dots, x_n, a_1, \dots, a_k), \quad \text{for } i = 1, \dots, n$$

has a unique solution in A; here each t_i is a term in $n + k$ variables and guardedness means that t_i is not a single variable $x_1, \dots, x_n$ (for every $i = 1, \dots, n$). The parameters $a_1, \dots, a_k$ are taken from A.

Considering the algebra A as an H_Σ-algebra

$$a : H_\Sigma A \longrightarrow A$$

iterativity is equivalent to stating that for every "flat" equation morphism $e : X \longrightarrow H_\Sigma X + A$, where X is a finite set (of variables) there exists a unique solution $e^\dagger : X \longrightarrow A$; this means that the square

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & A \\ H_\Sigma X + A \arrow[r, "H_\Sigma e^\dagger + A"'] & H_\Sigma A + A \arrow[u, "{[a,A]}"'] \end{tikzcd}$$

commutes. This is precisely the above concept for the base in Set given by

$$X \,\Box\, A = H_\Sigma X + A$$

see Example 2.11(2).

(2) In particular, a binary algebra A is iterative iff every guarded system of equations $x_i \approx t_i$ whose right-hand sides are binary trees with leaves labelled in $\{x_1, \dots, x_n, a_1, \dots, a_k\}$ has a unique solution. The corresponding base is $X \,\Box\, A = X \times X + A$.

In contrast, the base $X \,\Box\, A = X^* \times A$ yields the same algebras, but a different concept of iterativity: here we only solve equations of the form $x_i \approx x_j * a_l$ (for $j = 1, \dots, n$) or $x_i \approx a_l$.

Finally, for the base $X \,\Box\, A = \Phi(A)$ we again get the binary algebras, but with the trivial concept of iterativity: every algebra is iterative.

Notation 2.17. Morphisms between iterative base algebras are homomorphisms of base algebras, thus the category

$$\mathsf{Alg}_{\mathrm{it}}\, \Box$$

of iterative base algebras is a full subcategory of $\mathsf{Alg}\,\Box$. In fact, this is a full reflective subcategory, thus, free iterative algebras exist, see Proposition 3.13 of [AMV$_3$].

Definition 2.18. The monad $\mathbb{R}$ on $\mathscr{A}$ of free iterative algebras for $\Box$ is called a *rational monad* of the base $\Box$. The universal arrow is denoted by $\widehat{\eta}_A : A \longrightarrow RA$.

Example 2.19. Given a finitary endofunctor H the base $HX + A$ of Example 2.11(2) yields the rational monad $\mathbb{R}$ of H as studied in [AMV$_1$]: it assigns to every object of $\mathscr{A}$ its free iterative H-algebra.

Example 2.20. Rational monads for bases on Set (see [AMV$_3$]).

(1) For the base

$$X \,\Box\, A = X + A$$

of unary algebras we have

$$RY = \mathbb{N} \times Y + 1$$

where the unary operation is given by $(n, y) \longmapsto (n + 1, y)$ and by a fixed point in 1.

(2) For the base

$$X \,\Box\, A = X \times X + A$$

of binary algebras with full iterativity we have

$$RY = \text{rational binary trees on } Y$$

(cf. Example 1.2(1)).

(3) For the base

$$X \,\Box\, A = X^* \times A$$

of binary algebras with restricted iterativity we have

$$RY = \text{right-wellfounded rational binary trees on } Y$$

(cf. Example 1.2(2)).

(4) For the base

$$X \,\square\, A = \Phi(A) = \text{free binary algebra on } A$$

of binary algebras without iterativity

$$RY = \Phi(Y).$$

Notation 2.21. If $\rho_A : RY \,\square\, RY \longrightarrow RY$ is the algebra structure on RY then we put

$$j_Y \equiv RY \,\square\, Y \xrightarrow{RY \square \widehat{\eta}_Y} RY \,\square\, RY \xrightarrow{\rho_Y} RY.$$

Remark 2.22. (1) The main result of [AMV$_3$] is that the rational monad can be constructed coalgebraically as a filtered colimit of flat equations. More specifically, one defines the category

$$\mathsf{EQ}_Y$$

of all equation morphisms as the full subcategory of the category $\mathsf{Coalg}\,(- \,\square\, Y)$ of coalgebras for the endofunctor $- \,\square\, Y$ on all finitely presentable objects Y. Since $\mathsf{Coalg}\,(- \,\square\, Y)$ is cocomplete, with colimits formed on the level of $\mathscr{A}$, it follows that EQ_Y is closed under finite colimits. In particular, EQ_Y is a filtered category. We denote by

$$\mathrm{Eq}_Y : \mathsf{EQ}_Y \longrightarrow \mathscr{A}, \quad (e : X \longrightarrow X \,\square\, Y) \longmapsto X$$

the forgetful functor. This defines a small filtered diagram in $\mathscr{A}$ having a colimit cocone

$$e^\sharp : X \longrightarrow RY.$$

For every object Y we denote by

$$i_Y : RY \longrightarrow RY \,\square\, Y$$

the unique morphism for which the squares

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "e^\sharp"'] & X \,\square\, Y \arrow[d, "e^\sharp \square Y"] \\ RY \arrow[r, "i_Y"'] & RY \,\square\, Y \end{tikzcd} \tag{2.9}$$

commute for all e in EQ_Y. Then we proved in Lemma 4.5 of [AMV$_3$] that $e^\sharp$ are the unique morphisms making the above square commutative, and in Lemma 4.9 that

$$i_Y \text{ is an isomorphism with the inverse } j_Y.$$

For every iterative algebra $a : A \,\square\, A \longrightarrow A$ and every morphism $h_0 : Y \longrightarrow A$ the unique homomorphism $h : RY \longrightarrow A$ extending h_0 is characterized as follows: given an equation morphism $g : X \longrightarrow X \,\square\, Y$, we form

$$h_0 \bullet g \equiv X \xrightarrow{g} X \,\square\, Y \xrightarrow{X \square h_0} X \,\square\, A$$

and h is the unique morphism such that the triangle

$$\begin{tikzcd} RY \arrow[r, "h"] & A \\ X \arrow[u, "e^\sharp"] \arrow[ur, "(h_0 \bullet g)^\dagger"'] & \end{tikzcd} \tag{2.10}$$

commutes for every g. See [AMV$_3$], (4.41).

(2) Similarly as in Remark 2.14(2), it turns out that $SX = RX \,\square\, X$ carries the structure of a monad $\mathbb{S}$ for which the above isomorphisms form monad isomorphisms

$$i : \mathbb{R} \longrightarrow \mathbb{S} \qquad \text{and} \qquad j : \mathbb{S} \longrightarrow \mathbb{R}.$$

3. Rational Equation Morphisms

In the "non-parametrized world" of algebras $a : HA \longrightarrow A$ for an endofunctor the definition of iterativity is based on equation morphisms of the form $e : X \longrightarrow HX + A$ (where X is a finitely presentable object "of variables"). Iterative algebras are requested to have a unique solution $e^\dagger : X \longrightarrow A$ characterized by $e^\dagger = [a, A] \cdot (He^\dagger + A) \cdot e$. However iterative algebras have a much stronger iteration property, see [AMV$_1$]: every guarded rational equation morphism $e : X \longrightarrow R(X + A)$ has a unique solution $e^\dagger : X \longrightarrow A$ (compare Example 2.16(1)). Here $\mathbb{R}$ is the rational monad (of free iterative algebras) which has the form $R \cong HR + Id$, and e is called guarded if it factorizes through the summand $HR(X + A) + A$ of $R(X + A)$:

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[ddr, "e_0"'] & R(X+A) \\
& HR(X+A)+X+A \arrow[u, "\cong"'] \\
& HR(X+A)+A \arrow[u, "\mathsf{incl}"']
\end{tikzcd} \tag{3.1}$$

The aim of the present section is to prove the corresponding strengthening to parametrized iterativity w.r.t. a given base.

Recall that $\mathscr{A}$ denotes a locally finitely presentable category, and $\Box$ is a base on it with the rational monad $\mathbb{R}$, see Definition 2.18.

Definition 3.1. A *rational equation morphism* is a morphism $e : X \longrightarrow R(X + A)$ with X finitely presentable. It is called *guarded*, if it factors through $j_{X+A} \cdot (R(X + A) \,\Box\, \mathsf{inr})$, see Notation 2.21:

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[ddr, "e_0"'] & R(X+A) \\
& R(X+A) \,\Box\, (X+A) \arrow[u, "j_{X+A}"'] \\
& R(X+A) \,\Box\, A \arrow[u, "R(X+A)\Box\mathsf{inr}"']
\end{tikzcd} \tag{3.2}$$

Notice that in the above definition we do not require that the morphism e_0 need not be unqiue. But this does not matter for our results.

Example 3.2. For the base $X \,\Box\, A = X \times X + A$ a rational equation morphism

$$e : X \longrightarrow R(X + A), \quad X = \{x_1, \ldots, x_n\}$$

represents equations $x_1 \approx t_1, \ldots, x_n \approx t_n$ where t_i are rational trees on $X + A$. Guardedness means that e factors through the right-hand summand of $R(X + A) = X + [A + R(X + A) \times R(X + A)]$, i.e., t_i does not lie in X, for $i = 1, \ldots, n$.

Analogously for the base $X \,\Box\, A = X^* \times A$: here the trees $t_1, \ldots, t_n$ are right-wellfounded, and guardedness, again, means $t_i \notin X$, for all $i = 1, \ldots, n$.

Notation 3.3. For any iterative base algebra $\alpha : A \,\Box\, A \longrightarrow A$ on $\Box$, we denote by $\alpha^* : RA \longrightarrow A$ the unique homomorphism with

$$\alpha^* \cdot \widehat{\eta}_A = id_A$$

see Definition 2.18. We get the commutative diagram

$$\begin{tikzcd}
RA \,\Box\, A \arrow[r, "RA\Box\widehat{\eta}_A"] \arrow[rr, bend left, "j_A"] \arrow[dr, "\alpha^*\Box A"'] & RA \,\Box\, RA \arrow[r, "\rho_A"] \arrow[d, "\alpha^*\Box\alpha^*"] & RA \arrow[d, "\alpha^*"] & A \arrow[l, "\widehat{\eta}_A"'] \arrow[dl, equal] \\
& A \,\Box\, A \arrow[r, "\alpha"'] & A &
\end{tikzcd} \tag{3.3}$$

see Definition 2.18. We get the commutative diagram

Lemma 3.4. *In every iterative base algebra $\alpha : A \,\square\, A \longrightarrow A$ the solutions of equation morphisms $e : X \longrightarrow X \,\square\, A$ in EQ_A are determined by α^* as follows:*

$$e^\dagger \equiv X \xrightarrow{e^\sharp} RA \xrightarrow{\alpha^*} A$$

Proof. For $h_0 = id_A$ we have $h = \alpha^*$ in Notation 3.3, see Remark 2.22. Thus, diagram (2.10) with $g = e$ is our lemma. ☐

Theorem 3.5. *Every iterative base algebra has unique solutions of guarded rational equation morphisms.*

Proof. Let (A, α) be an iterative base algebra. Suppose that a guarded rational equation morphism $e : X \longrightarrow R(X + A)$ is given. The codomain of $e_0 : X \longrightarrow R(X + A) \,\square\, A$ from (3.2) is a filtered colimit

$$R(X + A) \,\square\, A \cong \mathrm{colim}(\mathrm{Eq}_{X+A} \,\square\, A)$$

because $- \,\square\, A$ is a finitary functor. So since X is finitely presentable we know that e_0 factors through the colimit morphism $g^\sharp \,\square\, A$ of some object

$$g : W \longrightarrow W \,\square\, (X + A) \quad \text{in } \mathsf{EQ}_{X+A}.$$

That is, there exists $w : X \longrightarrow W \,\square\, A$ for which the diagram

$$\begin{tikzcd}
X \arrow[rr, "e"] \arrow[rrdd, "e_0"] \arrow[rrddd, "w"'] & & R(X+A) \\
& & R(X+A) \,\square\, (X+A) \arrow[u, "j_{X+A}"'] \\
& & R(X+A) \,\square\, A \arrow[u, "R(X+A)\square\mathsf{inr}"'] \\
& & W \,\square\, A \arrow[u, "g^\sharp\square A"']
\end{tikzcd} \tag{3.4}$$

commutes. Define a flat equation morphism

$$f : W + X \longrightarrow (W + X) \,\square\, A \quad \text{in } \mathsf{EQ}_A$$

componentwise as follows:

$$\begin{tikzcd}
W \arrow[r, "g"] \arrow[d, "\mathsf{inl}"'] & W \,\square\, (X+A) \arrow[r, "W\square[w,u_A^W]"] & W \,\square\, (W \,\square\, A) \arrow[r, "m_A^W"] & W \,\square\, A \arrow[d, "\mathsf{inl}\,\square A"] \\
W + X \arrow[rrr, "f"] & & & (W+X) \,\square\, A \\
X \arrow[u, "\mathsf{inr}"] \arrow[rrr, "w"'] & & & W \,\square\, A \arrow[u, "\mathsf{inl}\,\square A"']
\end{tikzcd} \tag{3.5}$$

We denote the components of

$$f^\dagger \equiv W + X \xrightarrow{f^\sharp} RA \xrightarrow{\alpha^*} A$$

(see Lemma 3.4) by $f_r^\dagger$ and $f_l^\dagger$, respectively:

$$\begin{tikzcd}
W \arrow[r, "\mathsf{inl}"] \arrow[rdd, bend right, "f_l^\dagger"'] & W + X \arrow[d, "f^\sharp"] & X \arrow[l, "\mathsf{inr}"'] \arrow[ldd, bend left, "f_r^\dagger"] \\
& RA \arrow[d, "\alpha^*"] & \\
& A &
\end{tikzcd} \tag{3.6}$$

(1) Existence of solution. We prove that $f_r^\dagger : X \longrightarrow A$ is a solution of e. Before doing so, we establish the equality

$$f_l^\dagger = \alpha^* \cdot R[f_r^\dagger, A] \cdot g^\sharp : X \longrightarrow A \tag{3.7}$$

by proving that both sides are solutions of the flat equation morphism

$$[f_r^\dagger, A] \bullet g \equiv W \xrightarrow{\;g\;} W \,\Box\, (X + A) \xrightarrow{W\Box[f_r^\dagger,A]} W \,\Box\, A.$$

To see that $\alpha^* \cdot R[f_r^\dagger, A] \cdot g^\sharp$ is the solution of $[f_r^\dagger, A] \bullet g$ use diagram (2.10) with the homomorphism $h = \alpha^* \cdot R[f_r^\dagger, A]$ and $h_0 = [f_r^\dagger, A]$. To prove that $f_l^\dagger$ is a solution of $[f_r^\dagger, A] \bullet g$ we shall show that the perimeter of the diagram below commutes:

$$\begin{tikzcd}
W \arrow[rrr, "\mathsf{inl}"] \arrow[d, "g"'] \arrow[rrrrr, bend left, "f_l^\dagger"] \arrow[drr, phantom, "(3.5)"] & & & W+X \arrow[r, "f^\sharp"] \arrow[d, "f"] \arrow[dr, phantom, "(2.9)"] & RA \arrow[r, "\alpha^*"] \arrow[d, "i_A"] \arrow[dr, phantom, "(3.3)"] & A \\
W \,\Box\, (X+A) \arrow[r, "W\Box[w,u_A^W]"] \arrow[dd, "W\Box[f_r^\dagger,A]"'] & W \,\Box\, (W \,\Box\, A) \arrow[r, "m_A^W"] \arrow[d, "f_l^\dagger\Box(f_l^\dagger\Box A)"'] \arrow[dr, phantom, "(2.4)"] & W \,\Box\, A \arrow[r, "\mathsf{inl}\,\Box A"] \arrow[ddrrr, "f_l^\dagger\Box A"] & (W+X) \,\Box\, A \arrow[r, "f^\sharp\Box A"] \arrow[dr, phantom, "(3.6)"] & RA \,\Box\, A \arrow[ddr, "\alpha^*\Box A"] & \\
& A \,\Box\, (A \,\Box\, A) \arrow[drrrr, "m_A^A"] & & & & \\
W \,\Box\, A \arrow[rrrrr, "f_l^\dagger\Box A"'] & & & & & A \,\Box\, A \arrow[uuu, "\alpha"']
\end{tikzcd}$$

Indeed, since all inner parts except for the lower left-hand one commute as indicated, it is sufficient to prove that α, the right-hand vertical side, merges the two sides of the lower left-hand part. From (2.5) we know that α merges m_A^A and $A \,\Box\, \alpha$. Consequently all we have to do is to verify that the square

$$\begin{tikzcd}
W \,\Box\, (X+A) \arrow[r, "W\Box[w,u_A^W]"] \arrow[dd, "W\Box[f_r^\dagger,A]"'] & W \,\Box\, (W \,\Box\, A) \arrow[d, "f_l^\dagger\Box(f_l^\dagger\Box A)"] \\
& A \,\Box\, (A \,\Box\, A) \arrow[d, "A\Box\alpha"] \\
W \,\Box\, A \arrow[r, "f_l^\dagger\Box A"'] & A \,\Box\, A
\end{tikzcd}$$

commutes. The left hand $\Box$-component, with domain W, commutes, yielding $f_l^\dagger : W \longrightarrow A$. The right-hand component, with domain $X + A$, has obviously its right-hand component (with domain A) commutative, due to (2.3) and (2.5). The left-hand component, with domain X, also commutes: recall (3.6) and use the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\mathsf{inr}"] \arrow[d, "w"'] \arrow[rrr, bend left, "f_r^\dagger"] \arrow[dr, phantom, "(3.5)"] & W+X \arrow[r, "f^\sharp"] \arrow[d, "f"] \arrow[dr, phantom, "(2.9)"] & RA \arrow[r, "\alpha^*"] \arrow[d, "i_A"] \arrow[dr, phantom, "(3.3)"] & A \\
W \,\Box\, A \arrow[r, "\mathsf{inl}\,\Box A"'] \arrow[rrr, bend right, "f_l^\dagger\Box A"'] & (W+X) \,\Box\, A \arrow[r, "f^\sharp\Box A"'] & RA \,\Box\, A \arrow[r, "\alpha^*\Box A"'] & A \,\Box\, A \arrow[u, "\alpha"']
\end{tikzcd}$$

to conclude $f_r^\dagger = \alpha \cdot (f_l^\dagger \,\Box\, A) \cdot w$. This establishes (3.7), i.e., both $f_l^\dagger$ and $\alpha^* \cdot R[f_r^\dagger, A] \cdot g^\sharp$ are solutions of the same equation.

We are prepared to prove that $f_r^\dagger$ is a solution of e, i.e., that the perimeter of the diagram[1]

$$
\begin{tikzcd}
X \arrow[rr, "\mathsf{inr}"] \arrow[dr, "w"'] \arrow[dddd, "e"'] \arrow[drr, phantom, "(3.5)"] & & W+X \arrow[r, "f^\sharp"] \arrow[d, "f"] \arrow[dr, phantom, "(2.9)"] & RA \arrow[r, "\alpha^*"] \arrow[ddddr, equal] & A \\
& W \Box A \arrow[r, "\mathsf{inl}\, \Box A"] \arrow[d, "g^\sharp \Box A"'] \arrow[dr, phantom, "(*)"] & (W+X) \Box A \arrow[d, "f^\sharp \Box A"] & & \\
\arrow[r, phantom, "(3.4)"] & R(X+A) \Box A \arrow[r, "{R[f_r^\dagger, A] \Box A}"'] \arrow[d, "R(X+A)\Box \mathsf{inr}"'] & RA \Box A \arrow[uur, "j_A"'] \arrow[ddrr, "j_A"] \arrow[d, phantom, "(\mathrm{N})"] & & \\
& R(X+A) \Box (X+A) \arrow[ur, "{R[f_r^\dagger, A] \Box [f_r^\dagger, A]}"'] \arrow[dl, "j_{X+A}"] & & & \\
R(X+A) \arrow[rrrr, "{R[f_r^\dagger, A]}"'] & & & & RA \arrow[uuuu, "\alpha^*"']
\end{tikzcd}
$$

commutes. All inner parts, except for the middle square $(*)$, are seen to commute as indicated. It remains to show that the morphism $\alpha^* \cdot j_A$ merges the two sides of that square. By (3.3), we have $\alpha^* \cdot j_A = \alpha \cdot (\alpha^* \Box A)$, and so it suffices to prove that $\alpha^* \Box A$ merges the two sides of $(*)$. The upper passage composed with $\alpha^* \Box A$ yields $f_l^\dagger \Box A$, see (3.6). The lower passage yields $(\alpha^* \cdot R[f_r^\dagger, A] \cdot g^\sharp) \Box A$ which, by (3.7), is also $f_l^\dagger \Box A$.

This completes the proof that $f_l^\dagger$ is a solution of e.

(2) Uniqueness. Let a solution of e be given:

$$
\begin{tikzcd}
X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & A \\
R(X+A) \arrow[r, "{R[e^\dagger, A]}"'] & RA \arrow[u, "\alpha^*"']
\end{tikzcd}
$$

Then we will prove that

$$e^\dagger = f_r^\dagger = f^\dagger \cdot \mathsf{inr}$$

by establishing that the morphism $\widehat{f} : W + X \longrightarrow A$ with the following components

$$
\begin{tikzcd}
W \arrow[r, "g^\sharp"] \arrow[d, "\mathsf{inl}"'] & R(X+A) \arrow[r, "{R[e^\dagger, A]}"] & RA \arrow[d, "\alpha^*"] \\
W+X \arrow[rr, "\widehat{f}"] & & A \\
X \arrow[u, "\mathsf{inr}"] \arrow[urr, "e^\dagger"'] & &
\end{tikzcd}
\tag{3.8}
$$

is a solution of f. Then from $e^\dagger = \widehat{f} \cdot \mathsf{inr}$ we get $e^\dagger = f_r^\dagger$.

It is our task to prove that the square

$$
\begin{tikzcd}
W+X \arrow[r, "\widehat{f}"] \arrow[d, "f"'] & A \\
(W+X) \Box A \arrow[r, "\widehat{f} \Box A"'] & A \Box A \arrow[u, "\alpha"']
\end{tikzcd}
\tag{3.9}
$$

[1]Whenever a natural transformation is used in diagrams, the naturality squares are denoted by (N).

commutes. The right-hand component, with domain X, does:[2]

$$
\begin{tikzcd}
X \arrow[rrrr, "e^\dagger"] \arrow[drr, "e"] \arrow[dd, "w"'] & & (\dagger) & & A \\
& (3.4) & R(X+A) \arrow[r, "{R[e^\dagger,A]}"] & RA \arrow[ur, "\alpha^*"] & \\
W\square A \arrow[r, "g^\sharp\square A"] \arrow[dd, "\mathsf{inl}\,\square A"'] & R(X+A)\square A \arrow[r, "R(X+A)\square\mathsf{inr}"] \arrow[drr, "{R[e^\dagger,A]\square A}"'] & R(X+A)\square(X+A) \arrow[u, "j_{X+A}"'] \arrow[ur, phantom, "(\mathrm{N})"] \arrow[dr, "{R[e^\dagger,A]\square[e^\dagger,A]}"] & & \\
& (3.8) & & RA\square A \arrow[uu, "j_A"] \arrow[dr, "\alpha^*\square A"] \arrow[uur, phantom, "(3.3)"] & \\
(W+X)\square A \arrow[rrrr, "\widehat{f}\square A"'] & & & & A\square A \arrow[uuuu, "\alpha"']
\end{tikzcd}
\tag{3.10}
$$

We now prove that the left-hand component of (3.9) also commutes:

$$
\begin{tikzcd}
W \arrow[r, "g^\sharp"] \arrow[d, "g"'] \arrow[dr, phantom, "(2.9)"] & R(X+A) \arrow[r, "{R[e^\dagger,A]}"] \arrow[dr, phantom, "(\mathrm{N})"] & RA \arrow[r, "\alpha^*"] & A \\
W\square(X+A) \arrow[r, "g^\sharp\square(X+A)"] \arrow[d, "{W\square[w,u_A^W]}"'] \arrow[drr, "{(\widehat{f}\cdot\mathsf{inl}\,)\square[e^\dagger,A]}"] & R(X+A)\square(X+A) \arrow[u, "j_{X+A}"] \arrow[r, "{R[e^\dagger,A]\square[e^\dagger,A]}"] \arrow[dr, phantom, "(3.8)"] & RA\square A \arrow[u, "j_A"'] \arrow[d, "\alpha^*\square A"] & (3.3) \\
W\square(W\square A) \arrow[d, "m_A^W"'] \arrow[drr, "(\widehat{f}\cdot\mathsf{inl}\,)\square((\widehat{f}\cdot\mathsf{inl}\,)\square A)"] & (**) & A\square A \arrow[ddr, equal] & \\
W\square A \arrow[d, "\mathsf{inl}\,\square A"'] \arrow[drrr, "(\widehat{f}\cdot\mathsf{inl}\,)\square A"] & (2.4) & A\square(A\square A) \arrow[dr, "m_A^A"'] & \\
(W+X)\square A \arrow[rrr, "\widehat{f}\square A"'] & & & A\square A \arrow[uuuu, "\alpha"']
\end{tikzcd}
\tag{3.11}
$$

Indeed, all inner parts except for that denoted by $(**)$ commute, and for $(**)$ we will prove that the right-hand vertical arrow α merges both sides. Since α merges m_A^A and $A \square \alpha$, see (2.5), it is sufficient to show the commutativity of the square

$$
\begin{tikzcd}
W\square(X+A) \arrow[r, "{(\widehat{f}\cdot\mathsf{inl}\,)\square[e^\dagger,A]}"] \arrow[d, "{W\square[w,u_A^W]}"'] & A\square A \arrow[dd, equal] \\
W\square(W\square A) \arrow[d, "(\widehat{f}\cdot\mathsf{inl}\,)\square((\widehat{f}\cdot\mathsf{inl}\,)\square A)"'] & \\
A\square(A\square A) \arrow[r, "A\square\alpha"'] & A\square A
\end{tikzcd}
\tag{3.12}
$$

The left-hand $\square$-component with domain W commutes, yielding $\widehat{f}\cdot\mathsf{inl}$. For the right-hand $\square$-component with domain $X+A$, the component with domain A is equal to id_A, see (2.3) and (2.5). It remains to verify

[2]Whenever definition of solution is used in our diagrams, the corresponding square is denoted by $(\dagger)$.

the left-hand component with domain X: it commutes due to the commutative diagram

$$\begin{tikzcd}
X \arrow[rrrr, "e^\dagger"] \arrow[drr, "e"] \arrow[d, "w"'] & & & & A \arrow[dddd, equal] \\
X \,\square\, A \arrow[dr, "g^\sharp \square A"] \arrow[ddd, "(\widehat{f}\cdot \mathsf{inl})\square A"'] \arrow[r, phantom, "(3.4)"] & & R(X+A) \arrow[r, "{R[e^\dagger, A]}"] \arrow[u, phantom, "(\dagger)"] & RA \arrow[ur, "\alpha^*"] & \\
& R(X+A)\,\square\, A \arrow[r, "R(X+A)\square \mathsf{inr}"] \arrow[drr, "{R[e^\dagger,A]\square A}"'] \arrow[d, phantom, "(3.8)"] & R(X+A)\,\square\,(X+A) \arrow[u, "j_{X+A}"'] \arrow[dr, "{R[e^\dagger,A]\square[e^\dagger,A]}"] \arrow[ur, phantom, "(\mathrm{N})"] & & \\
& & & RA \,\square\, A \arrow[uu, "j_A"'] \arrow[dlll, "\alpha^* \square A"'] \arrow[dr, phantom, "(3.3)"] & \\
A \,\square\, A \arrow[rrrr, "\alpha"'] & & & & A
\end{tikzcd}$$

Consequently, (3.12) commutes — thus the outward square of (3.11) commutes. This proves that (3.9) commutes, concluding the proof. ☐

4. Modules for a Base

The aim of the present section is to characterize the rational monad $\mathbb{R}$ of a given base $\square$ by a universal property. For Elgot's iterative theories we proved in [AMV$_1$] that the rational monad is a free iterative monad on the given finitary endofunctor, we recall this briefly in Subsection 4.A. We then introduce modules (generalizing Elgot's ideal monads to the world of bases) in Subsection 4.B, and iterative modules (generalizing iterative monads) in Subsection 4.C. Our main result is that $\mathbb{R}$ is induced by the free iterative module $\square \cdot R$ on the base $\square$.

4.A. **Elgot's Ideal and Iterative Monads.** Let us start with the "classical" example of algebras in Set for a given finitary signature Σ: let $\mathbb{F}_\Sigma$ denote the monad of finite Σ-trees. Then Σ-algebras are precisely the Eilenberg-Moore algebras for $\mathbb{F}_\Sigma$. The monad $\mathbb{F}_\Sigma$ is ideal in the sense of Definition 4.1 below because the underlying endofunctor is a coproduct

$$F_\Sigma = F'_\Sigma + Id$$

where $F'_\Sigma X$ are the nontrivial Σ-trees, which means that at least one node is labelled in Σ.

We want to solve systems of recursive equations

$$\begin{aligned} x_1 &= t_1(x_1, \ldots, x_n, y_1, \ldots, y_k) \\ &\ \vdots \\ x_n &= t_n(x_1, \ldots, x_n, y_1, \ldots, y_k) \end{aligned}$$

where t_i are polynomials, i.e., elements of $F_\Sigma(X+Y)$ for $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_k\}$ (compare with Example 2.16(1)). We can express the above system by a morphism

$$e : X \longrightarrow F_\Sigma(X+Y), \quad x_i \longmapsto t_i.$$

If the system is *guarded*, i.e., if none of the right-hand sides is a single variable x_i, then a unique solution is obtained in the algebra $R_\Sigma Y$ of all rational Σ-trees on Y: this solution $e^\dagger : X \longrightarrow R_\Sigma Y$ is given by an obvious tree expansion of the given equations. To say that $e^\dagger$ solves e means precisely that if $\eta_Y : Y \longrightarrow R_\Sigma Y$ is the inclusion of variables and $\mu_Y : R_\Sigma R_\Sigma Y \longrightarrow R_\Sigma Y$ is the canonical morphism interpreting a "tree of trees" as a tree, then the square

$$\begin{tikzcd}
X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & R_\Sigma Y \\
F_\Sigma(X+Y) \arrow[d] & \\
R_\Sigma(X+Y) \arrow[r, "{R_\Sigma[e^\dagger, \eta_Y]}"'] & R_\Sigma R_\Sigma Y \arrow[uu, "\mu_Y"']
\end{tikzcd}$$

commutes (the unnamed arrow is the inclusion).

Definition 4.1. (C. Elgot [E])

(1) A monad $\mathbb{S} = (S, \eta, \mu)$ on Set is called *ideal* if the complements of $\eta_X[X]$ form a subfunctor of S, i.e., if there exists a subfunctor

$$\sigma : S' \hookrightarrow S$$

such that

$$S = S' + Id$$

with injections σ and η, for which μ has a domain-codomain restriction

$$s : S'S \longrightarrow S'.$$

(2) By an *equation morphism* is meant a morphism

$$e : X \longrightarrow S(X + Y), \quad X \text{ finite},$$

and e is called *guarded* provided that e factors as follows:

$$\begin{tikzcd} X \arrow[rr, "e"] \arrow[dr, dotted, "e_0"'] & & S(X+Y) \\ & S'(X+Y)+Y \arrow[ur, "{[\sigma_{X+Y}, \eta_{X+Y} \cdot \mathsf{inr}]}"'] & \end{tikzcd} \tag{4.1}$$

(3) An *iterative monad* is an ideal monad $\mathbb{S}$ such that every guarded equation morphism $e : X \longrightarrow S(X+Y)$ has a unique *solution*, i.e., a unique morphism

$$e^\dagger : X \longrightarrow SY$$

exists for which the square below commutes:

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & SY \\ S(X+Y) \arrow[r, "{[e^\dagger, \eta_Y]}"'] & SSY \arrow[u, "\mu_Y"'] \end{tikzcd} \tag{4.2}$$

Remark 4.2.

(1) The notion of an ideal monad was defined less categorically by C. Elgot, however, the above formulation is equivalent to his, see [AAMV].
(2) The equation morphisms of Elgot are not related to an explicitly given $\mathbb{S}$-algebra, however, the free $\mathbb{S}$-algebra (SY, μ_Y) plays here a similar rôle to A in Section 3.
(3) The domain-codomain restriction s of μ is simply a natural transformation $s : S'S \longrightarrow S'$ such that the square

$$\begin{tikzcd} S'S \arrow[r, "s"] \arrow[d, "\sigma S"'] & S' \arrow[d, "\sigma"] \\ SS \arrow[r, "\mu"'] & S \end{tikzcd} \tag{4.3}$$

commutes.

Example 4.3. The monad $\mathbb{R}_\Sigma$ of rational Σ-trees is ideal with $S' = H_\Sigma R_\Sigma$. Indeed, every equation morphism $e : X \longrightarrow R_\Sigma(X + Y)$ which is guarded in the sense of Definition 4.1 yields, by embedding Y to $R_\Sigma Y$ (and $R_\Sigma(X + Y)$ to $R_\Sigma(X + A)$ for the free $\mathbb{R}_\Sigma$-algebra $A = R_\Sigma Y$) an equation morphism in $A = R_\Sigma Y$ which is guarded in the sense of Definition 3.1. The unique solution guaranteed by Theorem 3.5 is, then, a unique solution in the sense of Definition 4.1.

If we work with arbitrary locally finitely presentable categories, the concept of an ideal monad has to be changed from a property to a structure:

Definition 4.4. ([AMV$_1$]) By an *ideal monad* is meant a quadruple

$$(S, S', \sigma, s)$$

where S is a monad (with a unit η and a multiplication μ), S' is an endofunctor called the *ideal*, and $\sigma : S' \longrightarrow S$ and $s : S'S \longrightarrow S'$ are natural transformations such that

(1) $S = S' + Id$ with injections σ and η
(2) (S', s) is a right S-module, i. e., the following two diagrams commute:

$$\begin{tikzcd} S' \arrow[r, "S'\eta"] \arrow[dr, equal] & S'S \arrow[d, "s"] \\ & S' \end{tikzcd} \qquad \begin{tikzcd} S'SS \arrow[r, "S'\mu"] \arrow[d, "sS"'] & S'S \arrow[d, "s"] \\ S'S \arrow[r, "s"'] & S' \end{tikzcd} \tag{4.4}$$

(3) the square (4.3) commutes.

The ideal monad is said to be *iterative* provided that for every equation morphism

$$e : X \longrightarrow S(X + Y) \quad X \text{ finitely presentable}$$

which is guarded, i.e., has a factorization (4.1), there exists a unique solution, defined by (4.2).

Example 4.5. For every finitary endofunctor H the rational monad $\mathbb{R}$ (see Example 2.19) is iterative. In fact, $\mathbb{R}$ can be characterized as a free iterative monad on H, see [AMV$_1$].

4.B. **Modules for a Base.** Recall that bases can be understood as generalizing the family of parametrized endofunctors $H(-) + A$ (cf. Example 2.11(2)) occuring on right-hand sides of the equation morphisms $e : X \longrightarrow HX + A$ (see the beginning of Section 3). The notion of module we shall now introduce is intended to generalize the notion of ideal monad (*not* mere monads) we saw in Definition 4.1 to the world of bases. So given a base $\Box$ we introduce modules as a canonical generalization of the structure

$$s : S' \cdot (S' + Id) \longrightarrow S'$$

of an ideal monad. Here the rôle of S is played by $X \Box X$ and that of s by a natural transformation

$$(X \Box X) \Box A \longrightarrow X \Box A$$

This is somehow "complementary" to $m^X_A : X \Box (X \Box A) \longrightarrow X \Box A$. The subfunctor $S' \hookrightarrow S$ will not have an analogue—it is implicit in the notion of module.

Notation 4.6. For a base $\Box$ we denote by S the endofunctor

$$SX = X \Box X.$$

We will see in Theorem 4.11 below that given a module for $\Box$, the functor S gets a monad structure.

Remark 4.7. Recall that a monad (S, η, μ) is precisely a monoid in the category of all endofunctors of $\mathscr{A}$, and a right module for this monoid is given by an endofunctor M and a morphism (natural transformation)

$$s : M \cdot S \longrightarrow M$$

such that the diagrams

$$\begin{tikzcd} M \arrow[r, "M\eta"] \arrow[dr, equal] & MS \arrow[d, "s"] \\ & M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} MSS \arrow[r, "M\mu"] \arrow[d, "sS"'] & MS \arrow[d, "s"] \\ MS \arrow[r, "s"'] & M \end{tikzcd} \tag{4.5}$$

commute. Analogously, given a base $\Box$, that is a parametrized monad, a module for $\Box$ in the following definition is a natural family M_A of right S-modules indexed by objects of $\mathscr{A}$ where $M_A X = X \Box A$.

Definition 4.8. By a *module for a base* $\Box$ is meant natural transformation $s : \Box \cdot S \longrightarrow \Box$ such that the following two diagrams commute:

$$\begin{tikzcd} \Box \arrow[r, "\Box\eta"] \arrow[dr, equal] & \Box \cdot S \arrow[d, "s"] \\ & \Box \end{tikzcd} \qquad \begin{tikzcd} \Box \cdot S \cdot S \arrow[r, "\Box\mu"] \arrow[d, "sS"'] & \Box \cdot S \arrow[d, "s"] \\ \Box \cdot S \arrow[r, "s"'] & \Box \end{tikzcd} \tag{4.6}$$

where $\eta : Id \longrightarrow S$ and $\mu : SS \longrightarrow S$ are the following natural transformations:

$$\eta_X \equiv X \xrightarrow{u^X_X} X \Box X \tag{4.7}$$

and

$$\mu_X \equiv SSX = SX \Box SX \xrightarrow{s^X_{SX}} X \Box SX = X \Box (X \Box X) \xrightarrow{m^X_X} SX. \tag{4.8}$$

Remark 4.9. More explicitly, a module for $\square$ is formed by the components $s_A^X : SX \square A \longrightarrow X \square A$, which form a family that is natural in X and A and for every object X the morphisms s_A^X (with variable A) form a monad morphism from the monad $(X \square X) \square -$ to the monad $X \square -$. In other words, the diagrams below commute:

$$\begin{tikzcd} SX \square A \arrow[rr, "s_A^X"] & & X \square A \\ & A \arrow[ul, "u_A^{X\square X}"] \arrow[ur, "u_A^X"'] & \end{tikzcd} \tag{4.9}$$

and

$$\begin{tikzcd} SX \square (SX \square A) \arrow[r, "s^X_{SX\square A}"] \arrow[d, "m_A^{X\square X}"'] & X \square (SX \square A) \arrow[r, "X\square s_A^X"] & X \square (X \square A) \arrow[d, "m_A^X"] \\ SX \square A \arrow[rr, "s_A^X"'] & & X \square A \end{tikzcd} \tag{4.10}$$

In addition the two diagrams in (4.6) commute, which can be written objectwise as:

$$\begin{tikzcd} X \square A \arrow[r, "u_X^X\square A"] \arrow[dr, equal] & SX \square A \arrow[d, "s_A^X"] \\ & X \square A \end{tikzcd} \qquad \begin{tikzcd} SSX \square A \arrow[r, "s^X_{SX}\square A"] \arrow[d, "s_A^{SX}"'] & (X \square SX) \square A \arrow[r, "m_X^X\square A"] & SX \square A \arrow[d, "s_A^X"] \\ SX \square A \arrow[rr, "s_A^X"'] & & X \square A \end{tikzcd}$$

Example 4.10.

(1) Consider the base
$$X \square A = X + A.$$
A module
$$s_A^X : X + X + A \longrightarrow X + A$$
can be defined via codiagonals.

(2) For the base
$$X \square A = X^* \times A$$
with $SX = X^* \times X = X^+$ we have the module
$$s_A^X : (X^+)^* \times A \longrightarrow X^* \times A$$
given by the concatenation.

(3) Let $\mathbb{S}$ be an ideal monad (see Definition 4.1) with $SX = S'X + X$. Then the base $+ \cdot S'$ given by
$$X\,(+\cdot S')\,A = S'X + A$$
has a canonical module structure (with S the original functor) given by
$$S'SX + A \xrightarrow{s_X + A} S'X + A$$
where $s : S'S \longrightarrow S'$ is the restriction of μ turning S' into a right $\mathbb{S}$-module (see (4.4)).

Theorem 4.11. *For every module the functor $SX = X \square X$ has the monad structure with the unit η and multiplication μ given by* (4.7) *and* (4.8).

Proof. We prove that (S, η, μ) is a monad:

(1) For the two unit laws consider the diagrams

$$\begin{tikzcd} X \square X \arrow[r, "X\square u_X^X"] \arrow[rr, bend left, "S\eta_X"] \arrow[ddrr, equal] \arrow[drr, phantom, "(2.1)"] & X \square SX \arrow[r, "u_X^X\square SX"] \arrow[dr, equal] \arrow[r, phantom, shift right=3, "(4.6)"] & SX \square SX \arrow[d, "s^X_{SX}"] \arrow[dd, bend left=60, "\mu_X"] \\ & & X \square SX \arrow[d, "m_X^X"] \\ & & X \square X \end{tikzcd}$$

and

$$
\begin{tikzcd}
X \,\square\, X \arrow[r, "\eta_{SX} = u^{SX}_{SX}"] \arrow[dr, "u^X_{SX}"] \arrow[ddr, equal] \arrow[r, phantom, shift right=6, "(4.9)"] \arrow[dr, phantom, shift right=6, "(2.1)"] & SX \,\square\, SX \arrow[d, "s^X_{SX}"] \arrow[dd, bend left=60, "\mu_X"] \\
& X \,\square\, SX \arrow[d, "m^X_X"] \\
& X \,\square\, X
\end{tikzcd}
$$

(2) For the multiplication law $\mu_X \cdot \mu_{SX} = \mu_X \cdot S\mu_X$, consider the diagram

$$
\begin{tikzcd}
SSSX \arrow[rrr, "s^X_{SX} \square s^X_{SX}"] \arrow[dr, "s^X_{SX} \square SSX"] \arrow[dd, "s^{SX}_{SSX}"'] \arrow[ddd, bend right=60, "\mu_{SX}"'] \arrow[rrrrr, bend left=30, "S\mu_X"] & & & (X\square SX)\square(X\square SX) \arrow[rr, "m^X_X \square m^X_X"] & & SSX \arrow[dd, "s^X_{SX}"] \arrow[ddd, bend left=60, "\mu_X"] \\
& (X\square SX)\square SSX \arrow[r, "m^X_X \square SSX"] \arrow[d, phantom, "(4.6)"] & SX\square SSX \arrow[rr, "SX \square s^X_{SX}"] \arrow[dl, "s^X_{SSX}"] \arrow[dr, phantom, "(\mathrm{N})"] & & SX\square(X\square SX) \arrow[ur, "SX\square m^X_X"'] \arrow[dl, "s^X_{X\square SX}"] \arrow[dr, phantom, "(\mathrm{N})"] & \\
SX\square SSX \arrow[r, "s^X_{SSX}"] \arrow[d, "m^{SX}_{SX}"'] \arrow[drrr, phantom, "(4.10)"] & X\square SSX \arrow[rr, "X\square s^X_{SX}"] & & X\square(X\square SX) \arrow[rr, "X\square m^X_X"] \arrow[d, "m^X_{SX}"] \arrow[drr, phantom, "(\mathrm{N})"] & & X\square SX \arrow[d, "m^X_X"] \\
SSX \arrow[rrr, "s^X_{SX}"'] \arrow[rrrrr, bend right=30, "\mu_X"'] \arrow[rrrrr, phantom, shift right=4, "(4.8)"] & & & X\square SX \arrow[rr, "m^X_X"'] & & X\square X
\end{tikzcd}
$$

The upper part commutes since $S\mu_X = \mu_X \,\square\, \mu_X$ and then we apply (4.8) twice.

□

Definition 4.12. The above monad (S, η, μ) is called the *induced monad* of the module s.

Example 4.13. The codiagonal module of Example 4.10(1) induces the monad $X \longmapsto X + X$ with unit $\mathsf{inr} : X \longrightarrow X + X$ and multiplication given by $\nabla : (X + X) + (X + X) \longrightarrow X + X$.

Remark 4.14. The previous theorem shows how to obtain μ from the module s. We shall need in Theorem 4.24 that s^X_X can be reconstructed from μ_X:

$$s^X_X \equiv SX \,\square\, X \xrightarrow{SX \square \eta_X} SX \,\square\, SX = SSX \xrightarrow{\mu_X} SX = X \,\square\, X. \tag{4.11}$$

To see this consider the following commutative diagram:

$$
\begin{tikzcd}
SX \,\square\, X \arrow[r, "SX\square\eta_X"] \arrow[dd, "s^X_X"'] & SX \,\square\, SX \arrow[d, "s^{SX}_X"] \\
& X \,\square\, SX \arrow[d, "m^X_X"] \\
X \,\square\, X \arrow[ur, "X\square\eta_X"] \arrow[r, equal] & SX
\end{tikzcd}
$$

The upper part commutes by the naturality of s and for the lower triangle use $\eta_X = u^X_X$ and (2.1); finally notice that the right-hand edge is μ_X.

Example 4.15. (1) Let $\mathbb{F} = (F, \eta, \mu)$ be the monad of free algebras for the base $\square$ (see Notation 2.13). This is isomorphic to the induced monad of the following module of the base $\square \cdot F$. Firstly, for this base we have

$$SX = FX \,\square\, X.$$

Recall the isomorphism

$$j' : \mathbb{S} \longrightarrow \mathbb{F}$$

of Notation 2.13. Let us define the module structure s by

$$s \equiv (\Box \cdot F) \cdot S \xrightarrow{(\Box \cdot F)j'} \Box \cdot F \cdot F \xrightarrow{\Box\mu} \Box \cdot F.$$

Then the monad structure on S induced by the isomorphism j' is precisely the one given from s by (4.7) and (4.8). So in order to see that s is a module structure we just observe the commutativity of the two diagrams below:

$$\begin{tikzcd} \Box \cdot F \cdot S \arrow[r, "\Box F j'"] & \Box \cdot F \cdot F \arrow[r, "\Box\mu"] & \Box \cdot F \\ \Box \cdot F \arrow[u, "\Box F \eta^S"] \arrow[ur, "\Box F \eta"'] \arrow[urr, equal] & & \end{tikzcd} \qquad \begin{tikzcd} \Box \cdot F \cdot S \cdot S \arrow[rr, "\Box F \mu^S"] \arrow[d, "\Box F j' S"'] & & \Box \cdot F \cdot S \arrow[d, "\Box F j'"] \\ \Box \cdot F \cdot F \cdot S \arrow[r, "\Box F F j'"] \arrow[d, "\Box \mu S"'] & \Box \cdot F \cdot F \cdot F \arrow[r, "\Box F \mu"] \arrow[d, "\Box \mu F"'] & \Box \cdot F \cdot F \arrow[d, "\Box\mu"] \\ \Box \cdot F \cdot S \arrow[r, "\Box F j'"'] & \Box \cdot F \cdot F \arrow[r, "\Box\mu"'] & \Box \cdot F \end{tikzcd}$$

To see that both diagrams commute use that j' is a monad morphism and apply the monad laws for $\mathbb{F}$.

(2) Let $\mathbb{R} = (R, \widehat{\eta}, \widehat{\mu})$ be the rational monad of the base $\Box$ (see Definition 2.18). This is isomorphic to the induced monad of the following module for the base $\Box \cdot R$. Firstly, for this base we have

$$SX = RX \,\Box\, X.$$

Recall the isomorphism

$$j : \mathbb{S} \longrightarrow \mathbb{R}.$$

from Remark 2.22(2). The module structure is given by

$$s \equiv \Box \cdot R \cdot S \xrightarrow{\Box R j} \Box \cdot R \cdot R \xrightarrow{\Box \widehat{\mu}} \Box \cdot R \tag{4.12}$$

The two diagrams verifying the module axioms are completely analogous to the previous point.

Definition 4.16. The *rational module* of a base $\Box$ is the above module s for the base $\Box \cdot R$.

Definition 4.17. ([AMV$_3$]) We denote by $\mathsf{Base}(\mathscr{A})$ the category of bases and their morphisms; a *morphism of bases* is just a natural transformation $p : \Box \longrightarrow \dot{\Box}$. In more elementary terms, we are given morphisms

$$p_A^X : X \,\Box\, A \longrightarrow X \,\dot{\Box}\, A$$

which

(1) make the diagrams

$$\begin{tikzcd} X \,\Box\, A \arrow[r, "p_A^X"] \arrow[d, "h \Box f"'] & X \,\dot{\Box}\, A \arrow[d, "h \dot{\Box} f"] \\ Y \,\Box\, B \arrow[r, "p_B^Y"'] & Y \,\dot{\Box}\, B \end{tikzcd} \tag{4.13}$$

commutative for every $h : X \longrightarrow Y$ and $f : A \longrightarrow B$ (expressing naturality of p), and

(2) the diagrams

$$\begin{tikzcd} X \,\Box\, A \arrow[rr, "p_A^X"] & & X \,\dot{\Box}\, A \\ & A \arrow[ul, "u_A^X"] \arrow[ur, "\dot{u}_A^X"'] & \end{tikzcd} \tag{4.14}$$

and

$$\begin{tikzcd} X \,\Box\, (X \,\Box\, A) \arrow[r, "p_{X\Box A}^X"] \arrow[d, "m_A^X"'] & X \,\dot{\Box}\, (X \,\Box\, A) \arrow[r, "X \dot{\Box} p_A^X"] & X \,\dot{\Box}\, (X \,\dot{\Box}\, A) \arrow[d, "\dot{m}_A^X"] \\ X \,\Box\, A \arrow[rr, "p_A^X"'] & & X \,\dot{\Box}\, A \end{tikzcd} \tag{4.15}$$

commutative (which express the fact that every p_-^X is a monad morphism from $X \,\Box\, -$ to $X \,\dot{\Box}\, -$).

Lemma 4.18. *Let* $p : \Box \longrightarrow \dot\Box$ *be a morphism of bases. Then precomposition by* p *yields a functor* $p^* : \mathsf{Alg}\,\dot\Box \longrightarrow \mathsf{Alg}\,\Box$ *commuting with the underlying functors. Moreover,* p^* *restricts to the corresponding full subcategories of iterative algebras.*

Proof. The first assertion is straightforward. To prove the latter one, consider an iterative $\dot\Box$-algebra (A, α) and an equation morphism $e : X \longrightarrow X \,\Box\, A$. Define $\overline{e} = p_A^X \cdot e$. Then there exists a unique $\overline{e}^\dagger : X \longrightarrow A$ such that the outside of the following diagram

$$\begin{tikzcd}
X \arrow[r, "\overline{e}^\dagger"] \arrow[d, "e"'] \arrow[dd, bend right=60, "\overline{e}"'] \arrow[dr, phantom, "(2.8)"] & A \\
X \,\Box\, A \arrow[r, "\overline{e}^\dagger \Box A"] \arrow[d, "p_A^X"'] \arrow[dr, phantom, "(\mathrm{N})"] & A \,\Box\, A \arrow[u, "\alpha \cdot p_A^A"'] \arrow[d, "p_A^A"] \\
X \,\dot\Box\, A \arrow[r, "\overline{e}^\dagger \dot\Box A"'] & A \,\dot\Box\, A \arrow[uu, bend right=60, "\alpha"']
\end{tikzcd}$$

commutes. This implies that the upper part commutes, and so $\overline{e}^\dagger$ is a solution of e in the algebra $p^*(A, \alpha) = (A, \alpha \cdot p_A^A)$. The above diagram also shows that solutions of e and $\overline{e}$ are in 1-1-correspondence. Hence, since $\overline{e}$ has a unique solution so does e and therefore the algebra $p^*(A, \alpha)$ is iterative. $\square$

Remark 4.19. We are now considering the category of modules over variable bases. That is, the objects are all pairs $(\Box, s)$, where $\Box$ is a base and s is a module of it. What is the natural concept of a morphism from a module $(\Box, s)$ to a module $(\dot\Box, \dot s)$?

If $\mathbb{S}$ and $\dot{\mathbb{S}}$ denote the induced monads of Theorem 4.11, we expect that a module morphism consists of a base morphism $p : \Box \longrightarrow \dot\Box$ and a monad morphism $h : \mathbb{S} \longrightarrow \dot{\mathbb{S}}$ "naturally" related to each other. For example, since $SX = X \,\Box\, X$ and $\dot SX = X \,\dot\Box\, X$, we expect h_X to be just $p_X^X : X \,\Box\, X \longrightarrow X \,\dot\Box\, X$, and we expect the square

$$\begin{tikzcd}
\Box \cdot S \arrow[r, "p * h"] \arrow[d, "s"'] & \dot\Box \cdot \dot S \arrow[d, "\dot s"] \\
\Box \arrow[r, "p"'] & \dot\Box
\end{tikzcd}$$

to commute.

It turns out that we do not need to consider h as given in the definition of morphisms since as we show immediately, the monad morphism from $\mathbb{S}$ to $\dot{\mathbb{S}}$ comes automatically:

Definition 4.20. Let $(\Box, s)$ and $(\dot\Box, \dot s)$ be modules with the induced monads $\mathbb{S}$ and $\dot{\mathbb{S}}$. By a *module morphism* is meant a base morphism $p : \Box \longrightarrow \dot\Box$ such that for the natural transformation h with components $p_X^X : SX \longrightarrow \dot SX$ the square

$$\begin{tikzcd}
\Box \cdot S \arrow[r, "p * h"] \arrow[d, "s"'] & \dot\Box \cdot \dot S \arrow[d, "\dot s"] \\
\Box \arrow[r, "p"'] & \dot\Box
\end{tikzcd} \tag{4.16}$$

commutes.

As previously mentioned, in the above definition we did *not* require $p_X^X : SX \longrightarrow \dot SX$ to be a monad morphism because this is a consequence:

Lemma 4.21. *For every module morphism* $p : \Box \longrightarrow \dot\Box$*, putting* $h_X := p_X^X$ *yields a monad morphism* $h : \mathbb{S} \longrightarrow \dot{\mathbb{S}}$.

Proof. The triangle

$$\begin{tikzcd}
X \,\Box\, X \arrow[rr, "p_X^X"] & & X \,\dot\Box\, X \\
& X \arrow[ul, "u_X^X"] \arrow[ur, "\dot u_X^X"'] &
\end{tikzcd}$$

commutes by (4.14). The following diagram

$$
\begin{tikzcd}
SX \mathbin{\Box} SX \arrow[r, "p^{SX}_{SX}"] \arrow[d, "s^X_{SX}"'] \arrow[rrr, bend left, "(h*h)_X"] \arrow[dd, bend right=60, "\mu_X"'] \arrow[dr, phantom, "(4.16)"] & SX \mathbin{\dot{\Box}} SX \arrow[r, "p^X_X \dot{\Box} SX"] & \dot{S}X \mathbin{\dot{\Box}} SX \arrow[r, "\dot{S}X \dot{\Box} p^X_X"] \arrow[d, "\dot{s}^X_{SX}"] \arrow[dr, phantom, "(N)"] & \dot{S}X \mathbin{\dot{\Box}} \dot{S}X \arrow[d, "\dot{s}^X_{\dot{S}X}"] \arrow[dd, bend left=60, "\dot{\mu}_X"] \\
X \mathbin{\Box} SX \arrow[rr, "p^X_{SX}"] \arrow[d, "m^X_X"'] \arrow[drrr, phantom, "(4.15)"] & & X \mathbin{\dot{\Box}} SX \arrow[r, "X \dot{\Box} p^X_X"] & X \mathbin{\dot{\Box}} \dot{S}X \arrow[d, "\dot{m}^X_X"] \\
X \mathbin{\Box} X \arrow[rrr, "p^X_X"'] \arrow[rrr, bend right, "h_X"'] & & & X \mathbin{\dot{\Box}} X
\end{tikzcd}
$$

commutes and this finishes the proof that h is a monad morphism. $\square$

Notation 4.22. We denote by

$$\mathsf{Mod}(\mathscr{A})$$

the category of modules and their morphisms.

Before we proceed to proving the monadicity of the obvious forgetful functor

$$U : \mathsf{Mod}(\mathscr{A}) \longrightarrow \mathsf{Base}(\mathscr{A}),$$

we will establish an auxiliary lemma.

Lemma 4.23. *Given a module for a base, the induced monad $\mathbb{S}$ has its Eilenberg-Moore category $\mathscr{A}^{\mathbb{S}}$ as a full subcategory of* $\mathsf{Alg}\,\Box$.

Proof. (1) We prove first that every $\mathbb{S}$-algebra $\alpha : A \mathbin{\Box} A \longrightarrow A$ is a base algebra for $\Box$.

(a) The triangle equality in (2.5) for α. Since $\eta_A = u^A_A$ holds, the triangle

$$
\begin{tikzcd}
A \arrow[r, "u^A_A"] \arrow[dr, equal] & A \mathbin{\Box} A \arrow[d, "\alpha"] \\
& A
\end{tikzcd}
$$

commutes.

(b) The square in (2.5) commutes for α. We make use of the fact that μ_A is the composite

$$(A \mathbin{\Box} A) \mathbin{\Box} (A \mathbin{\Box} A) \xrightarrow{s^A_{A\Box A}} A \mathbin{\Box} (A \mathbin{\Box} A) \xrightarrow{m^A_A} A \mathbin{\Box} A$$

and therefore the diagram

$$
\begin{tikzcd}
(A \mathbin{\Box} A) \mathbin{\Box} (A \mathbin{\Box} A) \arrow[r, "\alpha\Box(A\Box A)"] \arrow[d, "s^A_{A\Box A}"'] & A \mathbin{\Box} (A \mathbin{\Box} A) \arrow[r, "A\Box\alpha"] & A \mathbin{\Box} A \arrow[dd, "\alpha"] \\
A \mathbin{\Box} (A \mathbin{\Box} A) \arrow[d, "m^A_A"'] & & \\
A \mathbin{\Box} A \arrow[rr, "\alpha"'] & & A
\end{tikzcd}
$$

commutes, since α is associative as an $\mathbb{S}$-algebra. Precompose both passages with

$$u^A_A \mathbin{\Box} (A \mathbin{\Box} A) : A \mathbin{\Box} (A \mathbin{\Box} A) \longrightarrow (A \mathbin{\Box} A) \mathbin{\Box} (A \mathbin{\Box} A)$$

Due to the triangle axiom in (4.6) we get

$$\alpha \cdot m^A_A \cdot s^A_{A\Box A} \cdot (u^A_A \mathbin{\Box} (A \mathbin{\Box} A)) = \alpha \cdot m^A_A \tag{4.17}$$

The second passage now gives us

$$\alpha \cdot (A \mathbin{\Box} \alpha) \cdot (\alpha \mathbin{\Box} (A \mathbin{\Box} A)) \cdot (u^A_A \mathbin{\Box} (A \mathbin{\Box} A)) = \alpha \cdot (A \mathbin{\Box} \alpha) \tag{4.18}$$

by the triangle equality $\alpha \cdot u^A_A = \alpha \cdot \eta_A = id_A$. Now (4.17) and (4.18) yield the desired equality

$$\alpha \cdot m^A_A = \alpha \cdot (A \Box \alpha).$$

(2) Since for every morphism $f : A \longrightarrow B$ the square

$$\begin{tikzcd}
SA \arrow[r, "Sf"] \arrow[d, "\alpha"'] & SB \arrow[d, "\beta"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

commutes if and only if the square

$$\begin{tikzcd}
A \Box A \arrow[r, "f \Box f"] \arrow[d, "\alpha"'] & B \Box B \arrow[d, "\beta"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

does, the proof is concluded. $\Box$

Theorem 4.24. *Modules are monadic over bases. That is, the forgetful functor*

$$U : \mathsf{Mod}(\mathscr{A}) \longrightarrow \mathsf{Base}(\mathscr{A}), \quad (\Box, s) \longmapsto \Box$$

is monadic.

Proof. (1) We prove first that U has a left adjoint, i.e., that a free module exists on every base $\Box$. Let $\mathbb{F} = (F, \eta, \mu)$ be the monad of free base algebras on $\Box$. We prove that

$$\kappa \equiv \Box\eta : \Box \longrightarrow \Box \cdot F$$

is a universal arrow w. r. t. the canonical structure s of a module (see Example 4.15). We now prove that for every morphism of bases $\lambda : \Box \longrightarrow \dot{\Box}$ and for every module $(\dot{\Box}, \dot{s})$ there exists a unique morphism $\overline{\lambda} : (\Box \cdot F, s) \longrightarrow (\dot{\Box}, \dot{s})$ of modules with $\lambda = \overline{\lambda} \cdot \kappa$.

(1a) Existence. Recall that λ yields a functor $\lambda^* : \mathsf{Alg}\,\dot{\Box} \longrightarrow \mathsf{Alg}\,\Box$ (by precomposition), see Lemma 4.18. Recall further from Notation 2.13 that the forgetful functor $U_\Box : \mathsf{Alg}\,\Box \longrightarrow \mathscr{A}$ is monadic, in other words, the comparison functor $\mathsf{Alg}\,\Box \longrightarrow \mathscr{A}^{\mathbb{F}}$ is an isomorphism of categories. Then we obtain a functor

$$\mathscr{A}^{\dot{\mathbb{S}}} \hookrightarrow \mathsf{Alg}\,\dot{\Box} \longrightarrow \mathsf{Alg}\,\Box \simeq \mathscr{A}^{\mathbb{F}}$$

that commutes with the underlying functors, hence, it is induced by a monad morphism $k : \mathbb{F} \longrightarrow \dot{\mathbb{S}}$. Its component $k_X : FX \longrightarrow \dot{S}X$ is the unique homomorphism of base algebras extending $\dot{\eta}_X : X \longrightarrow \dot{S}X$:

$$\begin{tikzcd}
FX \Box FX \arrow[r, "k_X \Box k_X"] \arrow[dd, "\varphi_X"'] & \dot{S}X \Box \dot{S}X \arrow[d, "\lambda^{\dot{S}X}_{\dot{S}X}"] \\
 & \dot{S}X \mathbin{\dot{\Box}} \dot{S}X = \dot{S}\dot{S}X \arrow[d, "\dot{\mu}_X"] \\
FX \arrow[r, "k_X"'] & \dot{S}X
\end{tikzcd} \tag{4.19}$$

Define the base morphism

$$\overline{\lambda} \equiv \Box \cdot F \xrightarrow{\lambda * k} \dot{\Box} \cdot \dot{S} \xrightarrow{\dot{s}} \dot{\Box}$$

Now let $h = k \cdot j'$ for j' from Notation 2.13. We first prove that $h_X = \overline{\lambda}^X_X$:

$$\begin{tikzcd}
FX \,\square\, X \arrow[rr, "k_X \square X"] \arrow[d, "FX\square\eta_X"'] \arrow[ddd, bend right=60, "j'_X"'] & & \dot{S} \,\square\, X \arrow[dl, "\dot{S}X\square\dot{\eta}_X"'] \arrow[d, "\lambda^{\dot{S}X}_X"] \\
FX \,\square\, FX \arrow[r, "k_X\square k_X"] \arrow[dd, "\varphi_X"'] \arrow[dr, phantom, "(4.19)"] & \dot{S}X \,\square\, \dot{S}X \arrow[d, "\lambda^{\dot{S}X}_{\dot{S}X}"] \arrow[r, phantom, "(\mathrm{N})"] & \dot{S}X \,\square\, X \arrow[dl, "\dot{S}X\dot{\square}\dot{\eta}_X"'] \arrow[dd, "\dot{s}^X_X"] \\
\arrow[u, phantom, "(2.7)"] & \dot{S}X \,\dot{\square}\, \dot{S}X \arrow[dr, "\dot{\mu}_X"] \arrow[r, phantom, "(4.11)"] & \\
FX \arrow[rr, "k_X"'] & & \dot{S}X
\end{tikzcd}$$

The left-hand part commutes as shown in Remark 2.14(1). The upper part commutes since k_X extends $\dot{\eta}_X$.

Now we are ready to prove that $\overline{\lambda}$ is a module morphism:

$$\begin{tikzcd}
\square \cdot F \cdot S \arrow[r, "\square F j'"] \arrow[d, "s"'] \arrow[rrr, bend left, "\overline{\lambda} * h"] & \square \cdot F \cdot F \arrow[r, "\lambda * k * k"] \arrow[dl, "\square\mu"'] & \dot{\square} \cdot \dot{S} \cdot \dot{S} \arrow[r, "\dot{s}\dot{S}"] \arrow[d, "\dot{\square}\dot{\mu}"] & \dot{\square} \cdot \dot{S} \arrow[d, "\dot{s}"] \\
\square \cdot F \arrow[rr, "\lambda * k"'] \arrow[rrr, bend right, "\overline{\lambda}"'] & & \dot{\square} \cdot \dot{S} \arrow[r, "\dot{s}"'] & \dot{\square}
\end{tikzcd}$$

where the middle part commutes since k is a monad morphism, that square on the right is the square on the right in (4.5), the left-hand triangle is the definition of the module structure, and the upper and lower parts commute by the definitions of $\overline{\lambda}$ and h.

It remains to prove that $\overline{\lambda}$ extends λ. This is clear from the following diagram

$$\begin{tikzcd}
\square \cdot F \arrow[r, "\square k"] \arrow[rrr, bend left, "\overline{\lambda}"] & \square \cdot \dot{S} \arrow[r, "\lambda \dot{S}"] & \dot{\square} \cdot \dot{S} \arrow[r, "\dot{s}"] \arrow[dr, phantom, "(4.6)"] & \dot{\square} \arrow[d, equal] \\
\square \arrow[u, "\square\eta"] \arrow[ur, "\square\dot{\eta}"'] \arrow[rrr, "\lambda"'] & & & \dot{\square} \arrow[ul, "\dot{\square}\dot{\eta}"]
\end{tikzcd}$$

where the triangle on the left commutes since k is a monad morphism.

(1b) To prove uniqueness of $\overline{\lambda}$, consider any base morphism $\tau : \square \cdot F \longrightarrow \dot{\square}$ with $l_X = \tau^X_X$ and such that the diagram

$$\begin{tikzcd}
\square \cdot F \cdot S \arrow[r, "\tau * l"] \arrow[d, "s"'] & \dot{\square} \cdot \dot{S} \arrow[d, "\dot{s}"] \\
\square \cdot F \arrow[r, "\tau"] & \dot{\square} \\
\square \arrow[u, "\square\eta"] \arrow[ur, "\lambda"'] &
\end{tikzcd} \tag{4.20}$$

commutes. Observe that the morphisms

$$k_X \equiv FX \xrightarrow{i'_X} FX \,\square\, X \xrightarrow{l_X} \dot{S}X \tag{4.21}$$

form a monad morphism; indeed, l_X is a monad morphism by Lemma 4.21, and i'_X is a monad (iso)morphism as recalled in Remark 2.14(2).

We prove that k is the monad morphism from (1a) and that $\tau = \overline{\lambda}$. For the former, it is sufficient to show that $k_X : FX \longrightarrow \dot{S}X$ is a morphism of base algebras for $\square$ with $k_X \cdot \eta_X = \dot{\eta}_X$. The latter equality is clear since k is a monad morphism. Thus all we need is to prove that k_X is a base algebra homomorphism from the free base algebra FX to $\dot{S}X$ with the structure on the right of the diagram below:

$$\begin{tikzcd}
FX \Box FX \arrow[rr, "k_X \Box k_X"] \arrow[d, "\eta_{FX} \Box FX"'] \arrow[dr, phantom, "(N)"] & & \dot{S}X \Box \dot{S}X \arrow[dl, "\eta_{\dot{S}X} \Box SX"'] \arrow[d, "\lambda^{SX}_{SX}"] \\
FFX \Box FX \arrow[r, "Fk_X \Box k_X"] \arrow[d, "i'^{-1}_{FX}"'] \arrow[dr, phantom, "(N)"] & F\dot{S}X \Box \dot{S}X \arrow[r, "\tau^{SX}_{SX}\ (4.20)"] \arrow[d, "i'^{-1}_{\dot{S}X}"] \arrow[dr, phantom, "(4.21)"] & \dot{S}X \mathbin{\dot{\Box}} \dot{S}X \arrow[d, equal] \\
FFX \arrow[r, "Fk_X"] \arrow[d, "\mu_X"'] & F\dot{S}X \arrow[r, "k_{SX}"] & \dot{S}\dot{S}X \arrow[d, "\dot{\mu}_X"] \\
FX \arrow[rr, "k_X"'] & & \dot{S}X
\end{tikzcd}$$

The lower rectangle commutes since k is a monad morphism. Recall from Remark 2.14 that the algebra on the left is the free base algebra $\varphi_X : FX \Box FX \longrightarrow FX$ on X. Then we have the commutative diagram

$$\begin{tikzcd}
FX \Box FX \arrow[r, "\eta_{FX} \Box FX"] \arrow[dr, "\eta_{FX} \Box \eta_{FX}"'] \arrow[ddr, bend right, "id \Box id"'] & FFX \Box FX \arrow[r, "i^{-1}_{FX}"] \arrow[d, "FFX \Box \eta_{FX}"] & FFX \arrow[dd, "\mu_X"] \\
& FFX \Box FFX \arrow[ur, "\varphi_{FX}"'] \arrow[d, "\mu_X \Box \mu_X"] & \\
& FX \Box FX \arrow[r, "\varphi_X"'] & FX
\end{tikzcd}$$

(Notice that the right-hand part commutes since μ_X is a homomorphism, see Notation 2.13.) Thus, we proved that k is the monad morphism from point (1a).

To prove that $\tau = \overline{\lambda}$, consider the following commutative diagram

$$\begin{tikzcd}
\Box \cdot F \arrow[r, "\Box \eta F"] \arrow[rrr, bend left, "\lambda * k"] \arrow[ddrr, equal] & \Box \cdot F \cdot F \arrow[r, "\Box F i'"] \arrow[dr, equal] \arrow[ddr, "\Box\mu"'] & \Box \cdot F \cdot S \arrow[r, "\tau * l"] \arrow[d, "\Box F j'"] & \dot{\Box} \cdot \dot{S} \arrow[dd, "\dot{s}"] \\
& & \Box \cdot F \cdot F \arrow[d, "\Box\mu"] \arrow[r, phantom, "(4.20)"] & \\
& & \Box \cdot F \arrow[r, "\tau"] & \dot{\Box}
\end{tikzcd}$$

It upper part commutes due to the definitions of k in (4.21) and since $\lambda = \tau \cdot \Box\eta$; all other parts clearly commute. Thus we have

$$\tau = \dot{s} \cdot (\lambda * k) = \overline{\lambda}$$

as desired.

(2) To prove monadicity of U, we use Paré's "absolute coequalizer" version [P] of Beck's Theorem. This means that we have to show that

(2a) U reflects isomorphisms, and
(2b) $\mathsf{Mod}(\mathscr{A})$ has and U preserves coequalizers of reflexive pairs that are U-absolute.

(2a) Consider modules $(\Box, s)$ and $(\dot{\Box}, \dot{s})$ and let $p : \Box \longrightarrow \dot{\Box}$ be an isomorphism of bases. Then $p * h : \Box \cdot S \longrightarrow \dot{\Box} \cdot \dot{S}$ is an isomorphism as well (where $h_X = p^X_X : SX \longrightarrow \dot{S}X$) with an inverse $p^{-1} * h^{-1}$. Since the diagram

$$\begin{tikzcd}
\Box \cdot S \arrow[r, "p*h"] \arrow[d, "s"'] \arrow[rr, bend left, "id"] & \dot{\Box} \cdot \dot{S} \arrow[r, "p^{-1}*h^{-1}"] \arrow[d, "\dot{s}"] & \Box \cdot S \arrow[d, "s"] \\
\Box \arrow[r, "p"'] \arrow[rr, bend right, "id"'] & \dot{\Box} \arrow[r, "p^{-1}"'] & \Box
\end{tikzcd}$$

commutes, we proved that $p : (\Box, s) \longrightarrow (\dot{\Box}, \dot{s})$ is an isomorphism of modules.

(2b) Suppose

$$\begin{tikzcd}
(\Box, s) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & (\dot{\Box}, \dot{s}) \arrow[l, bend left=60, "t"]
\end{tikzcd}$$

is a reflexive pair in $\mathsf{Mod}(\mathscr{A})$ such that

$$\begin{tikzcd}
\Box \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \dot{\Box} \arrow[r, "r"] & \ddot{\Box}
\end{tikzcd} \tag{4.22}$$

is an absolute coequalizer in $\mathsf{Base}(\mathscr{A})$. Put $\ddot{S}X = X \mathbin{\ddot{\Box}} X$. We are to provide a module structure on $\ddot{\Box}$ turning r into a module morphism.

First observe that there is a functor from $\mathsf{Base}(\mathscr{A})$ to the category $\mathsf{Fin}[\mathscr{A}, \mathscr{A}]$ of all finitary functors on $\mathscr{A}$ mapping a base $\Box$ to its corresponding functor $SX = X \mathbin{\Box} X$. Applying this functor to (4.22) we obtain an absolute, reflexive coequalizer

$$\begin{tikzcd}
S \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & \dot{S} \arrow[r, "l"] & \ddot{S}
\end{tikzcd}$$

in $\mathsf{Fin}[\mathscr{A}, \mathscr{A}]$.

To define $\ddot{s} : \ddot{\Box} \cdot \ddot{S} \longrightarrow \ddot{\Box}$, we use the "$3 \times 3$ Lemma" [J, Lemma 0.17]: we consider the following diagram

$$\begin{tikzcd}
\Box \cdot S \arrow[r, shift left, "\Box*h"] \arrow[r, shift right, "\Box*k"'] \arrow[d, shift right, "q*S"'] \arrow[d, shift left, "p*S"] & \Box \cdot \dot{S} \arrow[r, "\Box*l"] \arrow[d, shift right, "q*\dot{S}"'] \arrow[d, shift left, "p*\dot{S}"] & \Box \cdot \ddot{S} \arrow[d, shift right, "q*\ddot{S}"'] \arrow[d, shift left, "p*\ddot{S}"] \\
\dot{\Box} \cdot S \arrow[r, shift left, "\dot{\Box}*h"] \arrow[r, shift right, "\dot{\Box}*k"'] \arrow[d, "r*S"'] & \dot{\Box} \cdot \dot{S} \arrow[r, "\dot{\Box}*l"] \arrow[d, "r*\dot{S}"] & \dot{\Box} \cdot \ddot{S} \arrow[d, "r*\ddot{S}"] \\
\ddot{\Box} \cdot S \arrow[r, shift left, "\ddot{\Box}*h"] \arrow[r, shift right, "\ddot{\Box}*k"'] & \ddot{\Box} \cdot \dot{S} \arrow[r, "\ddot{\Box}*l"] & \ddot{\Box} \cdot \ddot{S}
\end{tikzcd}$$

and conclude that the diagonal

$$\begin{tikzcd}
\Box \cdot S \arrow[r, shift left, "p*h"] \arrow[r, shift right, "q*k"'] & \dot{\Box} \cdot \dot{S} \arrow[r, "r*l"] & \ddot{\Box} \cdot \ddot{S}
\end{tikzcd}$$

is a coequalizer in $\mathsf{Base}(\mathscr{A})$. So we can define $\ddot{s} : \ddot{\Box} \cdot \ddot{S} \longrightarrow \ddot{\Box}$ via

$$\begin{tikzcd}
\Box \cdot S \arrow[r, shift left, "p*h"] \arrow[r, shift right, "q*k"'] \arrow[d, "s"'] & \dot{\Box} \cdot \dot{S} \arrow[r, "r*l"] \arrow[d, "\dot{s}"] & \ddot{\Box} \cdot \ddot{S} \arrow[d, dotted, "\ddot{s}"] \\
\Box \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \dot{\Box} \arrow[r, "r"'] & \ddot{\Box}
\end{tikzcd}$$

It is then straightforward to verify that $\ddot{S}$ is the structure of a right $\ddot{\mathbb{S}}$-module on $\ddot{\Box}$.

We prove now that r is a coequalizer of p and q in $\mathsf{Mod}(\mathscr{A})$. Suppose that $w : (\dot\Box, \dot s) \longrightarrow (\dddot\Box, \dddot s)$ is a module morphism such that $w \cdot p = w \cdot q$. Since (4.22) is a coequalizer, there is a unique $z : \ddot\Box \longrightarrow \dddot\Box$ such that $z \cdot r = w$. Moreover, $o_X = z_X^X$ is the unique monad morphism $o : \ddot{\mathbb{S}} \longrightarrow \dddot{\mathbb{S}}$ such that $o \cdot l = v$ holds, where $v : \dot{\mathbb{S}} \longrightarrow \dddot{\mathbb{S}}$ is the monad morphism corresponding to $w : (\dot\Box, \dot s) \longrightarrow (\dddot\Box, \dddot s)$. That the diagram

$$\begin{tikzcd}
\ddot\Box \cdot \ddot S \arrow[r, "z*o"] \arrow[d, "\ddot s"'] & \dddot\Box \cdot \dddot S \arrow[d, "\dddot s"] \\
\ddot\Box \arrow[r, "z"'] & \dddot\Box
\end{tikzcd}$$

commutes, follows from the fact that $r * l : \dot\Box \cdot \dot S \longrightarrow \ddot\Box \cdot \ddot S$ is an epimorphism, being a coequalizer. ☐

4.C. **Iterative Modules.** The following concept of iterativity of a module $(\Box, s)$ generalizes canonically iterative monads, see Definition 4.1:

Definition 4.25. Let $\Box$ be a base and s be a module for it.
is a module and let (S, η, μ) be its corresponding monad.

(1) By an *equation morphism* we mean a morphism

$$e : X \longrightarrow S(X+Y) = (X+Y) \Box (X+Y) \qquad X \text{ finitely presentable}$$

(2) An equation morphism e is called *guarded* if it factors through $(X+Y) \Box \mathsf{inr} : (X+Y) \Box Y \longrightarrow (X+Y) \Box (X+Y)$:

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[dr, "e_0"'] & S(X+Y) \\
 & (X+Y) \Box Y \arrow[u, "(X+Y)\Box\mathsf{inr}"']
\end{tikzcd} \tag{4.23}$$

(3) The module is called *iterative* provided that every guarded finitary equation morphism $e : X \longrightarrow S(X+Y)$ has a unique *solution* $e^\dagger : X \longrightarrow SY$ i.e., a unique morphism making the square

$$\begin{tikzcd}
X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & SY \\
S(X+Y) \arrow[r, "{S[e^\dagger, \eta_Y]}"'] & SSY \arrow[u, "\mu_Y"']
\end{tikzcd}$$

commutative; here (S, η, μ) is the induced monad.

Notation 4.26. We denote by

$$\mathsf{Mod}_{it}(\mathscr{A})$$

the full subcategory of $\mathsf{Mod}(\mathscr{A})$ spanned by iterative modules.

Example 4.27. The rational module (see Definition 4.16) is iterative.

Indeed, recall first that the induced monad is isomorphic to $\mathbb{R}$ via $j_Y : RY \Box Y \longrightarrow RY$ (see Notation 2.21). Thus, a guarded equation morphism (4.23) is precisely a rational guarded equation morphism in the sense of Definition 3.1. Now form the rational equation morphism

$$\overline{e} \equiv X \xrightarrow{\ e\ } R(X+Y) \xrightarrow{R(X+\widehat{\eta}_Y)} R(X+RY)$$

w.r.t. the iterative base algebra RY, and observe that it is guarded:

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[dd, "e_0"', bend right] \arrow[ddr, phantom, "(4.23)"] & R(X+Y) \arrow[r, "R(X+\widehat{\eta}_Y)"] \arrow[dr, phantom, "(\mathrm{N})"] & R(X+RY) \\
 & R(X+Y) \Box (X+Y) \arrow[u, "j_{X+Y}"'] \arrow[r, "R(X+\widehat{\eta}_Y)\Box(X+\widehat{\eta}_Y)"'] & R(X+RY) \Box (X+RY) \arrow[u, "j_{X+RY}"'] \\
 {} \arrow[r] & R(X+Y) \Box Y \arrow[u, "(RX+Y)\Box\mathsf{inr}"'] \arrow[r, "R(X+\widehat{\eta}_Y)\Box\widehat{\eta}_Y"'] & R(X+RY) \Box RY \arrow[u, "R(X+RY)\Box\mathsf{inr}"']
\end{tikzcd}$$

Therefore, $\overline{e}$ has a unique solution in the free iterative base algebra RY:

$$
\begin{tikzcd}
X \arrow[rr, "\overline{e}^\dagger"] \arrow[d, "e"'] & & RY \\
R(X+Y) \arrow[d, "R(X+\widehat{\eta}_Y)"'] \arrow[rrd, "{R[\overline{e}^\dagger, \widehat{\eta}_Y]}"] & & \\
R(X+RY) \arrow[rr, "{R[\overline{e}^\dagger, RY]}"'] & & RRY \arrow[uu, "\widehat{\mu}_Y"']
\end{tikzcd}
$$

¿From the last diagram we see that a solution of e is precisely the same as a solution of $\overline{e}$. Therefore, e has a unique solution, since $\overline{e}$ does by Theorem 3.5.

Remark 4.28. We prove below that the rational module is a free iterative module on $\square$. In proving this, we will closely follow the proof of Theorem 4.24, but we will need a technical result first (compare with Lemma 4.23).

Proposition 4.29. *For every iterative module $(\square, s)$ all free Eilenberg-Moore algebras for the induced monad $\mathbb{S}$ are iterative. That is, the full embedding $\mathscr{A}^{\mathbb{S}} \hookrightarrow \mathsf{Alg}\,\square$ of Lemma 4.23 has a restriction to a full embedding $\mathscr{A}_{\mathbb{S}} \hookrightarrow \mathsf{Alg}_{\mathrm{it}}\,\square$.*

Proof. We know by Lemma 4.23 that every free $\mathbb{S}$-algebra (SY, μ_Y) is a base algebra for $\square$. We start with the uniqueness of solutions of equation morphisms

$$e : X \longrightarrow X \,\square\, SY, \quad X \text{ finitely presentable.}$$

Given a solution $e^\dagger : X \longrightarrow SY$ in the algebra SY, then the following diagram

$$
\begin{tikzcd}
X \arrow[rr, "e^\dagger"] \arrow[d, "e"'] \arrow[dr, phantom, "(\dagger)"] & & SY \\
X \,\square\, SY \arrow[r, "e^\dagger \square SY"] \arrow[d, equal] \arrow[ddr, "e^\dagger \square S\eta_Y"] & SY \,\square\, SY = SSY \arrow[ur, "\mu_Y"] & \\
X \,\square\, (Y \,\square\, Y) \arrow[dd, "\mathsf{inl} \,\square\, (\mathsf{inr} \,\square\, \mathsf{inr})"'] \arrow[ddr, "e^\dagger \square (\eta_Y \square \eta_Y)"] & & \\
 & SY \,\square\, SSY \arrow[uu, "SY \square \mu_Y"'] \arrow[d, equal] \arrow[r, phantom, "(*)"] & \\
(X+Y) \,\square\, ((X+Y) \,\square\, (X+Y)) \arrow[r, "{[e^\dagger, \eta_Y] \square ([e^\dagger, \eta_Y] \square [e^\dagger, \eta_Y])}"] \arrow[d, "m^{X+Y}_{X+Y}"'] & SY \,\square\, (SY \,\square\, SY) \arrow[d, "m^{SY}_{SY}"] & \\
(X+Y) \,\square\, (X+Y) \arrow[r, "{[e^\dagger, \eta_Y] \square [e^\dagger, \eta_Y]}"] \arrow[d, equal] & SY \,\square\, SY \arrow[dr, equal] & \\
S(X+Y) \arrow[rr, "{S[e^\dagger, \eta_Y]}"'] & & SSY \arrow[uuuuuu, "\mu_Y"']
\end{tikzcd}
\tag{4.24}
$$

commutes: all parts except the area $(*)$ commute trivially and $(*)$ does due to the diagram

$$\begin{tikzcd}
SY \,\square\, SY \arrow[r, "s^Y_{SY}"] \arrow[rr, bend left=40, "\mu_Y"] & Y \,\square\, SY \arrow[r, "m^Y_Y"] \arrow[u, phantom, "(4.8)"] & Y \,\square\, Y \\
SY \,\square\, (Y \,\square\, SY) \arrow[u, "SY \square m^Y_Y"] \arrow[r, "s^Y_{Y \square SY}"] & Y \,\square\, (Y \,\square\, SY) \arrow[u, "Y \square m^Y_Y"'] \arrow[r, "m^Y_{SY}"] & Y \,\square\, SY \arrow[u, "m^Y_Y"'] \\
SY \,\square\, SSY \arrow[u, "SY \square s^Y_{SY}"] \arrow[rr, "m^{SY}_{SY}"'] \arrow[uu, bend left=80, "SY \square \mu_Y", "(4.8)"'] & (4.10) & SY \,\square\, SY \arrow[u, "s^Y_{SY}"'] \arrow[uu, bend right=80, "\mu_Y"', "(4.8)"]
\end{tikzcd}$$

(N) (2.2)

Thus, if in (4.24) we define $\overline{e} : X \longrightarrow S(X + Y)$ as the left-hand vertical morphism, we obtain a guarded equation morphism w.r.t. the module $(\square, s)$ having $e^\dagger$ as its solution. This proves that $e^\dagger$ is unique.

The diagram (4.24) serves also for proving the existence of a solution of e: the equation morphism $\overline{e}$ has a unique solution, let us call it $e^\dagger$. Then the perimeter of diagram (4.24) and all inner parts, except for $(\dagger)$, commute. This proves that $(\dagger)$ commutes as well. Thus $e^\dagger$ is a solution of e w.r.t. the algebra (SY, μ_Y). $\square$

Theorem 4.30. *Rational modules are free iterative modules. That is, the forgetful functor*

$$U : \mathsf{Mod}_{it}(\mathscr{A}) \longrightarrow \mathsf{Base}(\mathscr{A})$$

has a left adjoint $\square \longmapsto (\square \cdot R, s)$ *for* s *in* (4.12).

Proof. The rational module is iterative, see Example 4.27. We prove that for the unit $\widehat{\eta} : Id \longrightarrow R$ of the rational monad the arrow

$$\widehat{\kappa} \equiv \square \xrightarrow{\square \widehat{\eta}} \square \cdot R$$

is universal.

In fact, the proof of Theorem 4.24 applies: replace systematically $\mathbb{F}$ by $\mathbb{R}$ and in place of diagram (4.19) use the fact that the algebra

$$\dot{S}X \,\square\, \dot{S}X \xrightarrow{\lambda^{SX}_{SX}} \dot{S}X \,\dot{\square}\, \dot{S}X = \dot{S}\dot{S}X \xrightarrow{\mu_X} \dot{S}X$$

is iterative by combining Proposition 4.29 and Lemma 4.18. Then the rest of the proof of part (I) of Theorem 4.24 goes through, proving that $\widehat{\kappa}$ is a universal arrow. $\square$

Open Problem 4.31. Is the forgetful functor U above monadic? We do not think it is, but at this moment we do not see a clear argument either way.

5. Conclusions

The result of Calvin Elgot and his co-authors that for every signature Σ the monad of rational Σ-trees is the free iterative monad on Σ (see [EBT]) was a basic step in the theory of iteration. We later generalized this to every finitary endofunctor H of a locally finitely presentable category: the rational monad of H, which is the monad of free iterative H-algebras, is a free iterative monad on H, see [AMV$_1$].

In the present paper we proved a much more general result: in place of an endofunctor H we work with a base, i.e., a "coherent" interpretation of objects of the base category as finitary monads. In our previous work we proved that free iterative base algebras always exist, and we called the corresponding monad the rational monad of the base. In the present article we introduced modules of a base, and particularly, the rational module. Our main result is the characterization of the rational module as the free iterative module.

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