On Finitary Functors and Their Presentations

Jiří Adámek[a], Stefan Milius[b,*], Lawrence S. Moss[c,1], Henning Urbat[a]

[a] *Institut für Theoretische Informatik, Technische Universität Braunschweig, Germany*
[b] *Lehrstuhl für Theoretische Informatik, Friedrich-Alexander-Universität Erlangen-Nürnberg, Germany*
[c] *Department of Mathematics, Indiana University, Bloomington, IN, USA*

Abstract

Finitary endofunctors of locally presentable categories are proved to have equational presentations. Special attention is paid to the category of complete metric spaces and two endofunctors: the Hausdorff functor of all compact subsets and the Kantorovich functor of all tight measures.

Keywords: finitary functors, Hausdorff functor, Kantorovich functor, presentation of functors

1. Introduction

Finitary endofunctors, i.e., those preserving filtered colimits, play an important role in algebra and coalgebra. One indication of this is the sufficient conditions for the existence of initial algebras and final coalgebras: the initial algebra for a finitary functor F always exists and it is the colimit

$$\mu F = \operatorname*{colim}_{n<\omega} F^n 0$$

of the initial ω-chain

$$0 \xrightarrow{!} F0 \xrightarrow{F!} FF0 \xrightarrow{FF!} \cdots$$

*Corresponding author.
Email addresses: adamek@iti.cs.tu-bs.de (Jiří Adámek), mail@stefan-milius.eu (Stefan Milius), lsm@cs.indiana.edu (Lawrence S. Moss), urbat@iti.cs.tu-bs.de (Henning Urbat)

[1] This work was partially supported by a grant from the Simons Foundation (#245591 to Lawrence S. Moss).

see [2]. The final coalgebra exists whenever the base category is locally presentable, as proved by M. Makkai and R. Paré [16], see a shorter argument for the category of sets in [6]. Moreover, if F preserves monomorphisms, the final coalgebra has, for some ordinal α, the form

$$\nu F = \lim_{n<\alpha} F^n 1$$

of the limit of the dual (op-)chain of length α, see [5].

This paper presents some new results on finitary endofunctors. First, every finitary endofunctor of a locally finitely presentable category has a presentation by operations and equations. This is based on the idea of a (finitary) signature in a category due to M. Kelly and J. Power [13]. We then turn to two endofunctors of the category **CMS** of complete metric spaces of special interest: the Hausdorff functor $\mathscr{H}$ assigning to every space X the space $\mathscr{H}X$ of all non-empty compact subsets of X with the Hausdorff metric, and the Kantorovich functor $\mathscr{K}X$ of all tight measures on X. F. van Breugel et al. [8] proved that $\mathscr{H}$ and $\mathscr{K}$ are λ-accessible functors for some cardinal λ. We sharpen their result here by proving that both functors are finitary.

Related Work. Section 5 on presentation of functors is closely related to the recent paper of A. Kurz and J. Velebil [19]. When the preliminary version of Section 5 was presented at the PSSL Workshop in Braunschweig in April 2010, J. Velebil told us about his parallel joint work and sent us a preliminary version of the above paper. Since our presentation is quite different, we decided not to change our section. We are grateful to J. Velebil for his comments on the formulation of that section.

We are also very grateful to J. Worrell for a fruitful discussion about details concerning the Kantorovich functor. It appeared that one proof in [8] needed more details, and based on the mentioned discussion H. Urbat provided these details in his Master's Thesis [18]. This is the source of our Appendix. Independently, J. Worrell et al. [9] also provided these details applying much the same method.

2. Locally Presentable Categories

This section presents our preliminaries: we recall the concept of a locally finitely presentable and locally countably presentable category and mention examples we use throughout the paper.

Recall that a category is *filtered* if every finite subcategory has a cocone in it, and *filtered colimits* are colimits of diagrams with filtered domains. A *finitary functor* is a functor $F\colon \mathscr{A} \to \mathscr{B}$ such that $\mathscr{A}$ has filtered colimits and F preserves them. An object A of $\mathscr{A}$ is *finitely presentable* if its hom-functor $\mathscr{A}(A, -)$ is finitary.

Definition 2.1. A category $\mathscr{A}$ is **locally finitely presentable** if it has colimits and a set $\mathscr{F}$ of finitely presentable objects such that every object is a filtered colimit of objects from $\mathscr{F}$. We consider $\mathscr{F}$ as a full subcategory of $\mathscr{A}$.

Examples 2.2. (1) **Set** is locally finitely presentable; finite sets are precisely the finitely presentable objects. For $\mathscr{F}$ we can choose the set $\mathbb{N}$ of all natural numbers $n = \{0, 1, \dots, n-1\}$.

(2) **Pos**, the category of posets and order-preserving functions, is locally finitely presentable. Here $\mathscr{F}$ is a set of representatives of all finite (= finitely presentable) posets up to isomorphism.

(3) K-**Vec**, the category of vector spaces over the field K, is locally finitely presentable. Finitely presentable objects are the finite-dimensional spaces. We can put $\mathscr{F} = \{K^n; n \in \mathbb{N}\}$.

(4) If $\mathscr{A}$ is a locally finitely presentable category, then every functor category $\mathscr{A}^{\mathscr{C}}$ ($\mathscr{C}$ a small category) is also locally finitely presentable (see [3]).

(5) The category

MS

of metric spaces with distances in $[0, 1]$ and nonexpanding functions is not locally finitely presentable. In fact, the only finitely presentable objects are the finite discrete spaces (with all distances 0 or 1). The argument that the finitely presentable objects in **Set** are the finite sets shows that finitely presentable objects in **MS** must be finite spaces. Let (A, d) be finitely presentable. For $n \geq 1$, denote by d_n the metric defined by $d_n(x, y) = \min(1, d(x, y) + \frac{1}{n})$ for $x \neq y$. The ω-chain of spaces $A_n = (A, d_n)$ with connecting maps id_A has the filtered colimit (A, d). Since the hom-functor of (A, d) preserves this filtered colimit, for some n, $\mathrm{id}_A\colon (A, d) \to (A, d)$ factorizes through the colimit map $\mathrm{id}_A\colon (A, d_n) \to (A, d)$. But distances in A which are strictly between 0 and 1 are increased in A_n. So A must be discrete.

Remark 2.3. (1) Let $\mathscr{A}$ be a locally finitely presentable category. A finitary endofunctor F is determined by its values on the full subcategory $J\colon \mathscr{F} \hookrightarrow$

$\mathscr{A}$. In fact, F is a left Kan extension of its domain restriction to $\mathscr{F}$: $F = \mathrm{Lan}_J(F \cdot J)$. Consequently, the category $\mathscr{A}^{\mathscr{F}}$ of all functors from $\mathscr{F}$ to $\mathscr{A}$ is equivalent to the category of all finitary endofunctors on $\mathscr{A}$. Thus, the category of finitary endofunctors on a locally finitely presentable category is locally finitely presentable.

(2) For $\mathscr{A} = \mathbf{Set}$ a functor is finitary iff for every set X, every element of FX lies in the image of Fm for some finite subset $m\colon Y \hookrightarrow X$. For example, the finite power-set functor $\mathscr{P}_f X = \{M; M \subseteq X, M \text{ finite}\}$ is finitary. Given a set A, the functor $FX = X^A$ is finitary iff A is finite.

Remark 2.4. Let λ be a regular infinite cardinal (i.e., one that is not cofinal to any smaller cardinal). A category is called λ-*filtered* (*countably filtered* in the case $\lambda = \aleph_1$) if every subcategory of less than λ morphisms (countable, in the case $\lambda = \aleph_1$) has a cocone in it. For example, the (ordered) category ω is filtered, but not countably filtered. The first uncountable ordinal ω_1 is countably filtered. A functor F is called λ-*accessible* if its domain has λ-filtered colimits and F preserves them. An object A of $\mathscr{A}$ is λ-*presentable* if its hom-functor $\mathscr{A}(A, -)$ is λ-accessible.

Definition 2.5. A category $\mathscr{A}$ is called **locally λ-presentable** if it has colimits and a set $\mathscr{A}_\lambda$ of λ-presentable objects such that every object is a λ-filtered colimit of objects of $\mathscr{A}_\lambda$. In the case $\lambda = \aleph_1$ we speak about **locally countably presentable category**, and write $\mathscr{C}$ instead of $\mathscr{A}_{\aleph_1}$.

Examples 2.6. (1) **Set**, **Pos** and K-**Vec** are locally countably presentable (since this is weaker than locally finitely presentable). In **Set** and **Pos** the countably presentable objects are precisely the countable ones, in K-**Vec** precisely the countably dimensional spaces.

(2) The category $\omega\,$**CPO** of all posets with joins of increasing ω-chains (and all ω-continuous functions) is not locally finitely presentable: no non-trivial object is finitely presentable. However, it is locally countably presentable.

(3) The category **MS** is locally countably presentable. So is the larger category

PMS

of all pseudometric spaces (where distinct elements may have distance 0) with distances in $[0, 1]$ and nonexpanding functions. Indeed, **PMS** is obviously cocomplete with colimits computed on the level of underlying sets (and endowed with the supremum of all pseudometrics making all colimit maps nonexpanding). Consequently, **MS** is cocomplete, since this full subcategory is reflective in **PMS**: a reflection of a pseudometric space (X, d) is its quotient modulo the equivalence $x \sim y$ iff $d(x, y) = 0$.

Every (pseudo)metric space is a countably filtered colimit of its countable subspaces. In Corollary 2.8 we will see that these spaces are countably presentable. This proves that **MS** and **PMS** are locally countably presentable categories. In both cases $\mathscr{C}$ is a choice set of all countable spaces up to isometry.

(4) The full subcategory

$$\mathbf{CMS}$$

of **MS** formed by all complete spaces (in which every Cauchy sequence has a limit point) is also locally countably presentable. Indeed, **CMS** is a reflective subcategory of **MS**, where the reflection of a space (X, d) is its Cauchy completion $e\colon (X, d) \to (X^*, d^*)$. Recall that e is an isometry such that every element of X^* is a limit of a Cauchy sequence in X. For every nonexpanding function $f\colon (X, d) \to (Y, \bar{d})$ where $(Y, \bar{d})$ is complete we have the unique extension $f^*\colon X^* \to Y$ defined in $x \in X^*$ by

$$f^*(x) = \lim_{n\to\infty} f(x_n)$$

for an arbitrary Cauchy sequence (x_n) converging to x. To see that f^* is nonexpanding, use that f is, and that the distance of two elements $x = \lim x_n$ and $y = \lim y_n$ in X^* is simply $\lim_{n\to\infty} d(x_n, y_n)$. Thus, **CMS** is cocomplete.

Now choose a set $\mathscr{C}$ of representatives of all *separable* complete metric spaces (which means complete spaces with a countable dense subset). We prove in Corollary 2.9 below that every separable space is countably presentable in **CMS**. And every complete space X is a countably filtered colimit of separable spaces: this follows from the fact that given $M \subseteq X$ countable, the closure of M in X is separable.

Thus, **CMS** is a locally countably presentable category.

Lemma 2.7. *Given a countably filtered diagram in* **PMS** *with a colimit cocone*

$$c_t\colon C_t \to C \qquad (t \in T),$$

then for every countable subset $M \subseteq C$ there exists $t \in T$ and a countable subset $M' \subseteq C_t$ such that c_t restricts to an isometry from M' to M.

Proof. Let d_t denote the pseudometric of C_t and d that of C. Since our diagram is filtered, it follows from Example 2.6(3) that for every pair $x, y \in C$ we have

$$d(x,y) = \inf d_t(x', y'),$$

where t ranges through T and $x' \in c_t^{-1}(x)$ and $y' \in c_t^{-1}(y)$.

(a) Assume first that M consists of precisely two elements, $M = \{x, y\}$. For every $n \in \mathbb{N}$ choose $t(n) \in T$ and elements $x_n \in c_{t(n)}^{-1}(x)$ and $y_n \in c_{t(n)}^{-1}(y)$ with

$$d_{t(n)}(x_n, y_n) \leq d(x,y) + \frac{1}{n}. \tag{2.1}$$

Since our diagram is countably filtered, there exists $s \in T$ and connecting morphisms $f_n\colon C_{t_n} \to C_s$ for all $n \in \mathbb{N}$.

The countable set $\{f_n(x_n); n \in \mathbb{N}\}$ is mapped by c_s to the single element x, since $c_s \cdot f_n = c_{t(n)}$. Thus, since our diagram is countably filtered, there exists $t \in T$ and a connecting morphism $g\colon C_s \to C_t$ also mapping all elements $f_n(x_n)$ to a single element x' of C_t; in symbols: $g \cdot f_n(x_n) = x'$. Thus,

$$c_t(x') = c_t \cdot g \cdot f_n(x_n) = c_s \cdot f_n(x_n) = c_{t(n)}(x_n) = x.$$

Analogously for the countable set $\{f_n(y_n); n \in \mathbb{N}\}$; we can assume without loss of generality, using that our diagram is filtered, that the choice of t and g is the same for the latter set. Thus we have $y' \in C_t$ with $g \cdot f_n(y_n) = y'$ and $c_t(y') = y$. We now prove that the set $M' = \{x', y'\}$ has the desired property:

$$d_t(x', y') = d(x, y).$$

In fact, since $g \cdot f_n$ is nonexpanding, we derive from (2.1)

$$d_t(x', y') \leq d_{t(n)}(x_n, y_n) \leq d(x,y) + \frac{1}{n}$$

and since c_t is nonexpanding, $d(x,y) \leq d_t(x', y')$.

(b) Let $M \subseteq C$ be a countable set. Then so is $M \times M$. For every pair (x, y) in $M \times M$ find $t_{x,y} \in T$ and $x', y' \in C_{t_{x,y}}$ as in (a). Since our diagram is countably filtered, we can choose t independent of the given pair. Given $x \in M$, all the chosen elements x' in C_t (for all $y \in M$) form a countable set that c_t maps to x. This implies, since our diagram is countably filtered, that there exists a connecting map $f\colon C_t \to C_{\bar{t}}$ which also maps all these elements x' to one element, say $\bar{x}$, of $C_{\bar{t}}$. Moreover, since M is countable, we can assume that t and $\bar{t}$ are chosen to be the same for all $x \in M$. It follows that the set $\overline{M} = \{\bar{x}; x \in M\}$ is mapped by $c_{\bar{t}}$ isometrically to M: for every pair $x, y \in M$ we have unique $\bar{x}, \bar{y} \in M$ with $c_{\bar{t}}(\bar{x}) = x$ and $c_{\bar{t}}(\bar{y}) = y$, and since f is nonexpanding

$$d(x, y) = d_{t_{x,y}}(x', y') \geq d_{\bar{t}}(\bar{x}, \bar{y}).$$

Since $c_{\bar{t}}$ is nonexpanding, $d(x, y) \leq d_{\bar{t}}(\bar{x}, \bar{y})$. □

Corollary 2.8. *Every countable space in* **MS** *or in* **PMS** *is a countably presentable object.*

Proof. Let A be a countable space in **PMS**. Consider a colimit as in Lemma 2.7. The hom-functor of A preserves it because for every morphism $f\colon A \to C$ there exists an essentially unique factorization through some c_t: apply the lemma to $M = f[A]$.

The argument for **MS** is analogous: **MS** is clearly closed under (countably) filtered colimits in **PMS**. □

Corollary 2.9. *Every separable space in* **CMS** *is a countably presentable object.*

Proof. Let A be a complete metric space with a countable dense set $M \subseteq A$. To verify that $\mathbf{CMS}(A, -)$ preserves countably filtered colimits, we first observe that **CMS** is closed under such colimits in **PMS**. (To see this, consider a colimit as in Lemma 2.7 and take a Cauchy sequence x_n in C. There exists, for $M = \{x_n; n \in \mathbb{N}\}$, an index $t \in T$ and a Cauchy sequence x'_n in C_t with $x_n = c_t(x'_n)$ for every n. Then x'_n has a limit y' in C_t, yielding a limit $y = f(y')$ of x_n in C.) For every nonexpanding map $f\colon A \to C = \operatorname{colim} C_t$ there exists t and $M' \subseteq C_t$ such that the colimit map c_t is an isometry between M' and $f[M]$. It follows easily that, since M is dense in A, there exists a factorization $f = c_t \cdot f'$ where $f'\colon A \to C_t$ is nonexpanding. Consequently, $\mathbf{CMS}(A, C)$ is a colimit of $\mathbf{CMS}(A, C_t)$ in **Set**, as required. □

Remark 2.10. No non-empty space is finitely presentable in **CMS**. To see this, express the real interval $[0, 1]$ as a filtered colimit in **CMS** of $[\frac{1}{n}, 1]$ for $n = 1, 2, 3, \ldots$ If $A \neq \emptyset$, then the constant map $f\colon A \to [0, 1]$ with value 0 does not factorize through any of the colimit maps.

Fact 2.11. Remark 2.3(1) generalizes as follows: Let $\mathscr{A}$ be a locally λ-presentable category. Every λ-accessible endofunctor F is determined by its values on $\mathscr{A}_\lambda$: we have $F = \mathrm{Lan}_J(F \cdot J)$ for the full embedding $J\colon \mathscr{A}_\lambda \to \mathscr{A}$.

Consequently, the category of all λ-accessible endofunctors on $\mathscr{A}$ is locally λ-presentable.

3. The Hausdorff Functor

The aim of this section is to prove that on the category **CMS** of complete metric spaces the Hausdorff functor $\mathscr{H}$ introduced in Example 3.13 below is finitary.

It was proved by F. van Breugel et al. [8] that $\mathscr{H}X$ is the free semilattice on X in **CMS**. Thus, this functor is a special case of the monad $M^{\mathscr{T}}$ on **CMS** induced by free $\mathscr{T}$-algebras for a Lawvere algebraic theory $\mathscr{T}$. We start by proving that $M^{\mathscr{T}}$ is a finitary functor for every algebraic theory $\mathscr{T}$, then we turn to the special case.

Recall from [15] that an *algebraic theory* $(\mathscr{T}, T)$ is a category $\mathscr{T}$ whose objects are natural numbers, together with a functor $T\colon \mathbb{N}^{\mathrm{op}} \to \mathscr{T}$ (cf. Example 2.2(1)) which is identity on objects and preserves finite products. This means that in $\mathscr{T}$ the object n is a product $n = 1 \times \cdots \times 1$ with projections

$$Tp_0, \ldots, Tp_{n-1}\colon n \to 1$$

corresponding to the canonical injections $p_i\colon 1 \to n$ in $\mathbb{N}$.

Notation 3.1. Let $\mathscr{A}$ be a category with finite products. A *$\mathscr{T}$-algebra in $\mathscr{A}$* is a functor $A\colon \mathscr{T} \to \mathscr{A}$ preserving finite products. The category of $\mathscr{T}$-algebras,

$$\mathbf{Alg}_{\mathscr{A}}\, \mathscr{T}$$

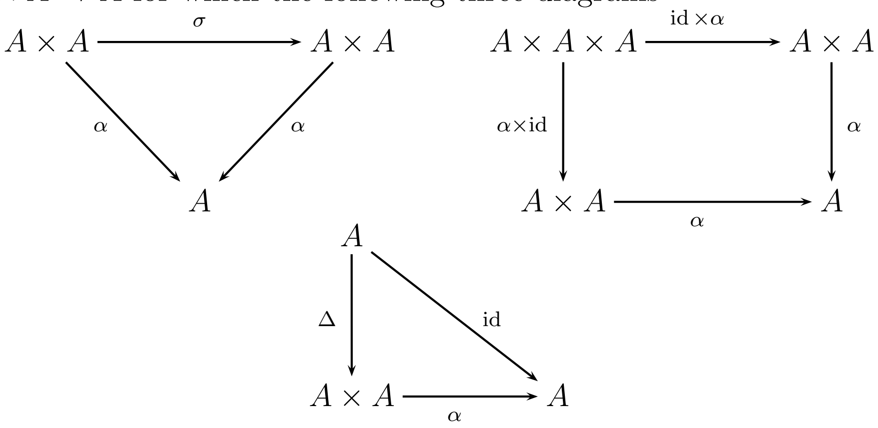
is the full subcategory of the functor category $\mathscr{A}^{\mathscr{T}}$. We denote by

$$U^{\mathscr{T}}_{\mathscr{A}}\colon\ \mathbf{Alg}_{\mathscr{A}}\, \mathscr{T} \to \mathscr{A}$$

the forgetful functor defined by $A \mapsto A(1)$ for all algebras $A\colon \mathscr{T} \to \mathscr{A}$.

Example 3.2. Semilattices. These are algebras on one binary operation which is commutative, associative, and idempotent. Let $\mathscr{T}_s$ be the corresponding algebraic theory, i.e., its morphisms from n to 1 are the semilattice-terms on n variables, modulo the semilattice laws mentioned above.

A $\mathscr{T}_s$-algebra in a category $\mathscr{A}$ is an object A together with a morphism $\alpha\colon A \times A \to A$ for which the following three diagrams



commute (where σ is the swapping isomorphism).

For example, $\mathbf{Alg_{CMS}}\,\mathscr{T}_s$ is the category of complete metric spaces with a nonexpanding semilattice operation.

Proposition 3.3. *Let $(\mathscr{T}, T)$ be an algebraic theory. For every locally finitely presentable category $\mathscr{A}$ the category $\mathbf{Alg}_{\mathscr{A}}\,\mathscr{T}$ is also locally finitely presentable, and $U^{\mathscr{T}}_{\mathscr{A}}$ is a finitary functor having a left adjoint.*

Proof. $\mathscr{A}^{\mathscr{T}}$ is locally finitely presentable (see Example 2.2(4)), and it is clear that the category of algebras is closed under limits and filtered colimits in it. Thus, the latter is also a locally finitely presentable category, see [3], Theorem 2.48. It is also clear that $U^{\mathscr{T}}_{\mathscr{A}}$ preserves limits and is finitary, thus, it has a left adjoint by [3], Theorem 1.66. $\square$

Remark 3.4. Analogously, if $\mathscr{A}$ is a locally countably presentable category, then so is $\mathbf{Alg}_{\mathscr{A}}\,\mathscr{T}$, and $U^{\mathscr{T}}$ is countably accessible.

Notation 3.5. We denote by $F^{\mathscr{T}}_{\mathscr{A}}$ the left adjoint of $U^{\mathscr{T}}_{\mathscr{A}}$, and by

$$M^{\mathscr{T}}_{\mathscr{A}} = U^{\mathscr{T}}_{\mathscr{A}} \cdot F^{\mathscr{T}}_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}$$

the corresponding monad on $\mathscr{A}$.

Example 3.6. For $\mathscr{A} = \mathbf{PMS}$ we conclude that $U^{\mathscr{T}}_{\mathbf{PMS}}$ is countably accessible for every algebraic theory $\mathscr{T}$. But here we can do better: $U^{\mathscr{T}}_{\mathbf{PMS}}$ is always finitary. In fact, the functor from $\mathbf{PMS}^{\mathscr{T}}$ to $\mathbf{PMS}$ given by evaluation at 1 preserves colimits, and $U^{\mathscr{T}}_{\mathbf{PMS}}$ is its composite with the full embedding $\mathbf{Alg_{PMS}}\,\mathscr{T} \hookrightarrow \mathbf{PMS}^{\mathscr{T}}$. Thus, it is sufficient to show that $\mathscr{T}$-algebras are closed under filtered colimits in $\mathbf{PMS}^{\mathscr{T}}$. This follows from the next

Lemma 3.7. *Filtered colimits commute with finite products in* **PMS**.

Proof. Let $(A, d) = \mathrm{colim}_{i\in I}(A_i, d_i)$ and $(A', d') = \mathrm{colim}_{j\in J}(A'_j, d'_j)$ be two filtered colimits. Let (c_i) and (c'_j) be the colimit cocones. Then the product $(A, d) \times (A', d')$ carries the maximum pseudometric d_0:

$$d_0\big((a, a'), (b, b')\big) = \max\big\{d(a, b), d'(a', b')\big\},$$

where

$$d(a, b) = \inf\big\{d_i(a_i, b_i); i \in I, a_i \in c_i^{-1}(a) \text{ and } b_i \in c_i^{-1}(b)\big\},$$

and analogously $d'(a', b')$.

We now form the filtered diagram of all

$$(A_i, d_i) \times (A'_j, d'_j)$$

indexed by $I \times J$. Its colimit has the same underlying cocone $c_i \times c'_j \colon A_i \times A'_j \to A \times A'$ as above. And its pseudometric is

$$\begin{aligned} &d_1\big((a, a'), (b, b')\big) = \inf\Big\{\max\big\{d_i(a_i, b_i), d'_j(a'_j, b'_j)\big\};\\ &(i, j) \in I \times J, a_i \in c_i^{-1}(a), b_i \in c_i^{-1}(b), a'_j \in (c'_j)^{-1}(a') \text{ and } b'_j \in (c'_j)^{-1}(b')\Big\}. \end{aligned}$$

This is the same pseudometric as d_0. ◻

Corollary 3.8. *The forgetful functor from* $\mathbf{Alg}_{\mathbf{PMS}}\,\mathscr{T}$ *to* **PMS** *is finitary for every algebraic theory* $\mathscr{T}$.

Remark 3.9. We have seen in Example 2.6 that **CMS** is a full reflective subcategory of **PMS**, i.e., the embedding $E\colon$ **CMS** $\to$ **PMS** has a left adjoint $R\colon$ **PMS** $\to$ **CMS**. Indeed, **CMS** is reflective in **MS**, where the reflector $R_1\colon$ **MS** $\to$ **CMS** is given by Cauchy completion, and **MS** is reflective in **PMS**, where the reflector $R_2\colon$ **PMS** $\to$ **MS** is the quotient modulo zero distance (cf. Example 2.6(3)). Thus

$$R = R_1 \cdot R_2 \colon \mathbf{PMS} \to \mathbf{CMS}$$

is a left adjoint to E.

Corollary 3.10. *The reflector* $R : \mathbf{PMS} \to \mathbf{CMS}$ *preserves finite products, thus, also in* **CMS** *filtered colimits commute with finite products.*

Indeed, $R = R_1 \cdot R_2$ and both R_1 and R_2 clearly preserve finite products.

Proposition 3.11. *Let $\mathscr{A}$ be a cocomplete category with finite products and $\mathscr{B}$ a full reflective subcategory whose reflector $R\colon \mathscr{A} \to \mathscr{B}$ preserves finite products. For every algebraic theory $\mathscr{T}$ the forgetful functor $U^{\mathscr{T}}_{\mathscr{B}}$ preserves every type of colimits that $U^{\mathscr{T}}_{\mathscr{A}}$ preserves.*

Proof. Recall that for a full reflective subcategory $E\colon \mathscr{B} \hookrightarrow \mathscr{A}$ we can always choose a reflector $R\colon \mathscr{A} \to \mathscr{B}$ with $R \cdot E = \mathrm{Id}_{\mathscr{B}}$. We use this to prove that the category $\mathbf{Alg}_{\mathscr{B}}\, \mathscr{T}$ is reflective in $\mathbf{Alg}_{\mathscr{A}}\, \mathscr{T}$. In fact, we have the full embedding

$$E^{\mathscr{T}}\colon\ \mathbf{Alg}_{\mathscr{B}}\, \mathscr{T} \hookrightarrow \mathbf{Alg}_{\mathscr{A}}\, \mathscr{T}, \qquad B \mapsto E \cdot B$$

for every $\mathscr{T}$-algebra $B\colon \mathscr{T} \to \mathscr{B}$, and its left adjoint is

$$R^{\mathscr{T}}\colon\ \mathbf{Alg}_{\mathscr{B}}\, \mathscr{T} \to \mathbf{Alg}_{\mathscr{A}}\, \mathscr{T}, \qquad A \mapsto R \cdot A$$

for every $\mathscr{T}$-algebra $A\colon \mathscr{T} \to \mathscr{A}$. Notice that since R preserves products, $R \cdot A$ is indeed an algebra. Obviously, from $R \dashv E$ and $R \cdot E = \mathrm{Id}$ we get $R^{\mathscr{T}} \dashv E^{\mathscr{T}}$ and $R^{\mathscr{T}} \cdot E^{\mathscr{T}} = \mathrm{Id}$. Moreover

$$U^{\mathscr{T}}_{\mathscr{B}} \cdot R^{\mathscr{T}} = R \cdot U^{\mathscr{T}}_{\mathscr{A}}. \tag{3.2}$$

Let $\mathscr{D}$ be a small category such that $U^{\mathscr{T}}_{\mathscr{A}}$ preserves colimits of diagrams with domain $\mathscr{D}$. We are to prove that for every diagram $D\colon \mathscr{D} \to \mathbf{Alg}_{\mathscr{B}}\, \mathscr{T}$ the functor $U^{\mathscr{T}}_{\mathscr{B}}$ preserves the colimit of D:

$$\begin{aligned}
U^{\mathscr{T}}_{\mathscr{B}}(\operatorname{colim} D) &= U^{\mathscr{T}}_{\mathscr{B}}(\operatorname{colim} R^{\mathscr{T}} \cdot E^{\mathscr{T}} \cdot D) && R^{\mathscr{T}} \cdot E^{\mathscr{T}} = \mathrm{Id}\\
&\cong U^{\mathscr{T}}_{\mathscr{B}} \cdot R^{\mathscr{T}}(\operatorname{colim} E^{\mathscr{T}} \cdot D) && R^{\mathscr{T}} \dashv E^{\mathscr{T}}\\
&= R \cdot U^{\mathscr{T}}_{\mathscr{A}}(\operatorname{colim} E^{\mathscr{T}} \cdot D) && (3.2)\\
&\cong R(\operatorname{colim} U^{\mathscr{T}}_{\mathscr{A}} \cdot E^{\mathscr{T}} \cdot D) && \text{hypothesis on } \mathscr{D}\\
&\cong \operatorname{colim} R \cdot U^{\mathscr{T}}_{\mathscr{A}} \cdot E^{\mathscr{T}} \cdot D && R \dashv E\\
&\cong \operatorname{colim} U^{\mathscr{T}}_{\mathscr{B}} \cdot D && (3.2) \text{ and } R^{\mathscr{T}} \cdot E^{\mathscr{T}} = \mathrm{Id}
\end{aligned}$$

□

Corollary 3.12. *The forgetful functor of $\mathbf{Alg}_{\mathbf{CMS}}\, \mathscr{T}$ is finitary for every algebraic theory $\mathscr{T}$.*

This follows from Proposition 3.11 applied to $R\colon \mathbf{PMS} \to \mathbf{CMS}$ using Corollaries 3.8 and 3.10.

Example 3.13. The Hausdorff functor $\mathscr{H}$. Recall that for a metric space (X,d), the distance of a point $x \in X$ to a set $M \subseteq X$ is $d(x,M) = \inf\{d(x,m); m \in M\}$. The *Hausdorff distance* of sets M, N in $\mathscr{P}X$ is

$$d^*(M,N) = \max\left\{\sup_{x\in M} d(x,N), \sup_{y\in N} d(y,M)\right\}.$$

The Hausdorff functor is the endofunctor $\mathscr{H}$ of **CMS** defined on objects X by

$$\mathscr{H}(X,d) = \text{all non-empty compact subsets of } X \text{ with the metric } d^*,$$

and on morphisms by direct images. For the semilattice theory $\mathscr{T}_s$ of Example 3.2 this is an algebra with the semilattice operation

$$\alpha(M,N) = M \cup N.$$

As proved in [8], $\mathscr{H}(X,d)$ is the free semilattice in **CMS** on (X,d). In other words, for the monad $M_{\mathscr{A}}^{\mathscr{T}}$ of Notation 3.5 we have

$$\mathscr{H} = M_{\mathbf{CMS}}^{\mathscr{T}_s}.$$

Open Problem 3.14. The Plotkin power-domain is a complete analogy of the Hausdorff functor with **CMS** substituted by the category $\omega\,$CPO of ω-cpo's. Indeed, the Plotkin power-domain can be characterized as a free semilattice on $\omega\,$CPO, see e.g. [1]. Is the corresponding endofunctor of $\omega\,$CPO finitary?

4. The Kantorovich Functor

Based on the result of F. van Breugel et al. [8] (which we present in detail in the appendix) that the Kantorovich functor below is the free mean-value-algebra monad, we prove that it is also finitary.

Recall that the *Borel subsets* of a metric space (X,d) form the smallest σ-algebra

$$\mathscr{B}(X,d)$$

containing all open sets. A *probability measure* on (X,d) is a σ-additive function

$$\mu\colon \mathscr{B}(X,d) \to [0,1].$$

with $\mu(X) = 1$. It is called *tight* if for every number $\varepsilon > 0$ there exists a compact set $C \subseteq X$ such that $\mu(X \setminus C) < \varepsilon$.

The Kantorovich functor $\mathscr{K}\colon \mathbf{CMS} \to \mathbf{CMS}$, see [12], is defined on a complete metric space (X, d) by

$$\mathscr{K}(X, d) = \text{all tight probability measures on } (X, d)$$

with the metric

$$d_{\mathscr{K}(X)}(\mu, \nu) = \sup_f \left(\int_X f d\mu - \int_X f d\nu \right)$$

where f ranges over all nonexpanding functions $f : X \to [0, 1]$. Given a nonexpanding map $f\colon (X, d) \to (\bar{X}, \bar{d})$, then $\mathscr{K} f$ assigns to a measure μ the measure

$$\mathscr{K} f(\mu)\colon B \mapsto \mu\big(f^{-1}(B)\big).$$

It was proved in [10] that $\mathscr{K}$ is a well-defined endofunctor of **CMS**.

Just like the Hausdorff functor, the Kantorovich functor has an algebraic characterization. By a *(metric) mean-value algebra* is meant a pair $(X, \oplus)$ of a space $X \in \mathbf{CMS}$ and a binary operation $\oplus : X \times X \to X$ that satisfies the following axioms for all $u, v, x, y \in X$:

(MV1) $u \oplus u = u$

(MV2) $u \oplus v = v \oplus u$

(MV3) $(u \oplus v) \oplus (x \oplus y) = (u \oplus x) \oplus (v \oplus y)$

(MV4) $d_X(x \oplus y, u \oplus v) \leq \frac{1}{2} \left(d_X(x, u) + d_X(y, v) \right)$

Observe that (MV4) implies that $\oplus$ is nonexpanding. Let $\mathscr{M}$ denote the category of all mean-value algebras and nonexpanding $\oplus$-preserving maps. For every $(X, d) \in \mathbf{CMS}$, the Kantorovich space $\mathscr{K}(X, d)$ is a mean-value algebra with respect to the operation

$$\mu_1 \oplus \mu_2 = \frac{\mu_1 + \mu_2}{2}.$$

Thus, we obtain the functor $\bar{\mathscr{K}} : \mathbf{CMS} \to \mathscr{M}$ that maps $(X, d) \in \mathbf{CMS}$ to $(\mathscr{K}(X, d), \oplus)$.

Theorem 4.1 (F. van Breugel et al. [8]). *$\bar{\mathscr{K}}$ is a left adjoint to the forgetful functor $U : \mathscr{M} \to$ **CMS**. Hence, $\mathscr{K}(X,d) = U\bar{\mathscr{K}}(X,d)$ is the free mean-value algebra in **CMS** on (X,d).*

The proof in [8] has a gap, so we include a full proof in Appendix A for the convenience of the reader. See also our discussion of related work in the Introduction.

Theorem 4.2. *The Kantorovich functor is finitary.*

Proof. By the previous theorem, $\mathscr{K}$ is the composite of the forgetful functor $U : \mathscr{M} \to$ **CMS** and its left adjoint $\bar{\mathscr{K}}$. Since left adjoints preserve colimits, all we need to prove is that U is finitary. To this end, consider the algebraic theory $\mathscr{T}_m$ given by the equations (MV1)-(MV3), i.e., its morphisms from n to 1 are the terms of the above equational theory in n variables.

Now U is the composite of the full embedding $\mathscr{M} \hookrightarrow \mathbf{Alg}_{\mathbf{CMS}}\, \mathscr{T}_m$ and the forgetful functor of $\mathbf{Alg}_{\mathbf{CMS}}\, \mathscr{T}_m$. The latter functor is finitary by Corollary 3.12. Thus, all we need to prove is that $\mathscr{M}$ is closed under filtered colimits in $\mathbf{Alg}_{\mathbf{CMS}}\, \mathscr{T}_m$.

(a) We start with an observation about filtered colimits $c_i\colon (X_i, d_i) \to (X, d)$, $i \in \mathsf{obj}\, I$, in **CMS**: for every pair $x, y \in X$ and every $\varepsilon > 0$ there exists an index i and a pair $\bar{x}, \bar{y} \in X_i$ such that

$$d(x, c_i(\bar{x})) < \varepsilon, \qquad d(y, c_i(\bar{y})) < \varepsilon \quad \text{and} \quad |d_i(x,y) - d_i(\bar{x},\bar{y})| < \varepsilon.$$

To see this, form first a colimit of the same diagram in **PMS**, say, $\hat{c}_i\colon (X_i, d_i) \to (\hat{X}, \hat{d})$. From Example 2.6(3) and Lemma 2.7 we know that $\hat{X} = \operatorname{colim} X_i$ is a filtered colimit in **Set** and $\hat{d}$ is the infimum metric. Thus, there exists a unique map $r\colon \hat{X} \to X$ with

$$c_i = r \cdot \hat{c}_i \qquad \text{for all } i.$$

Recall that **CMS** is a full reflective subcategory and observe that (due to Example 2.6(4)) the reflection maps are dense isometries. Since colimits in **CMS** are reflections of colimits in **PMS**, the above map r is a reflection of $(\hat{X}, \hat{d})$ in **CMS**. Thus, r is dense. Consequently, in order to prove the above property for all x, y and ε, it is clearly possible to restrict ourselves to pairs x, y in the dense set $r[\hat{X}]$: $x = r(\hat{x})$ and $y = r(\hat{y})$. Since $\hat{d}$ is the infimum metric on the colimit $\hat{X} = \operatorname{colim} X_i$ in **Set**, there exists i and elements $\bar{x}, \bar{y} \in X_i$ with $\hat{c}_i(\bar{x}) = \hat{x}, c_i(\bar{y}) = \hat{y}$ and $|\hat{d}(\hat{x},\hat{y}) - d_i(\bar{x},\bar{y})| < \varepsilon$. From $c_i = r \cdot \hat{c}_i$

we thus get the above three conditions: $x = c_i(\bar{x})$, $y = c_i(\bar{y})$ and, since $d(x,y) = \hat{d}(\hat{x},\hat{y})$ (recall that r is an isometry) also $|d(x,y) - d_i(\bar{x},\bar{y})| < \varepsilon$.

(b) We are now ready to prove that $\mathscr{M}$ is closed under filtered colimits in $\mathbf{Alg_{CMS}}\,\mathscr{T}_m$. Let $c_i\colon (X_i, d_i, \oplus) \to (X, d, \oplus)$, $i \in \mathsf{obj}\, I$, be a filtered colimit in $\mathbf{Alg_{CMS}}\,\mathscr{T}_m$ such that all domains are mean-value-algebras:

$$d_i(x \oplus y, u \oplus v) \leq \frac{1}{2}\big(d_i(x,u) + d_i(y,v)\big). \tag{4.3}$$

To prove that $(X, d, \oplus)$ is also a mean-value-algebra, we need to show that it satisfies (MV4). To this end it is sufficient to prove that given $x, y, u, v \in X$ and $\varepsilon > 0$ then

$$d(x \oplus y, u \oplus v) < \frac{1}{2}\big(d(x,u) + d(y,v)\big) + 4\varepsilon.$$

Using (a), we find $i \in I$ and elements $\bar{x}, \bar{y}, \bar{u}$ and $\bar{v}$ in X_i such that

$$d(x, c_i(\bar{x})) < \varepsilon \quad \text{and analogously for } y, u \text{ and } v \tag{4.4}$$

and

$$|d(x,y) - d_i(\bar{x},\bar{y})| < \varepsilon \quad \text{and} \quad |d(u,v) - d_i(\bar{u},\bar{v})| < \varepsilon. \tag{4.5}$$

We know that $\oplus$ is nonexpanding and $\bar{c}_i$ preserves $\oplus$ (since it is a homomorphism of algebras for $\mathscr{T}_m$). Thus, we have

$$d(x \oplus y, c_i(\bar{x} \oplus \bar{y})) \leq \max\{d(x, c_i(\bar{x})), d(y, c_i(\bar{y}))\} < \varepsilon \tag{4.6}$$

and analogously

$$d(u \oplus v, c_i(\bar{u} \oplus \bar{v})) < \varepsilon. \tag{4.7}$$

Consequently, we get the desired inequality as follows:

$$\begin{array}{rcll} d(x \oplus y, u \oplus v) & \leq & d(x \oplus y, c_i(\bar{x} \oplus \bar{y})) & \\ & & + d(c_i(\bar{x} \oplus \bar{y}), c_i(\bar{u} \oplus \bar{v})) & \\ & & + d(c_i(\bar{u} \oplus \bar{v}), u \oplus v) & \text{by triangle inequality} \\ & \leq & d(c_i(\bar{x} \oplus \bar{y}), c_i(\bar{u} \oplus \bar{v})) + 2\varepsilon & \text{by (4.6) and (4.7)} \\ & \leq & d_i(\bar{x} \oplus \bar{y}, \bar{u} \oplus \bar{v}) + 2\varepsilon & c_i \text{ nonexpanding} \\ & \leq & \frac{1}{2}\big(d_i(\bar{x},\bar{u}) + d_i(\bar{y},\bar{v})\big) + 2\varepsilon & \text{by (4.3)} \\ & < & \frac{1}{2}\big(d(x,u) + d(y,v)\big) + 4\varepsilon & \text{by (4.5).} \end{array}$$ □

Corollary 4.3. *All endofunctors of* **CMS** *defined from the functors* $\mathscr{H}$, $\mathscr{K}$, Id *and* Const_X *by composition, finite products or arbitrary coproducts, are finitary.*

In particular, each such endofunctor $F\colon \mathbf{CMS} \to \mathbf{CMS}$ has a final coalgebra obtained by some transfinite iteration of F on 1. This follows from the fact that F clearly preserves monomorphisms (since $\mathscr{H}$ and $\mathscr{K}$ do) and every finitary, mono-preserving endofunctor F of a locally presentable category has the final coalgebra of the form $F^i 1$ for some ordinal i, see [5].

5. Equational Presentation of Functors

Finitary set functors F can, as proved in [4], be presented by a signature Σ and a set of "flat" equations. Then F-algebras are precisely the Σ-algebras satisfying those equations. We recall this quickly and then generalize it to finitary endofunctors of all locally finitely presentable categories.

Example 5.1. The set functor

$$FX = \text{all unordered pairs in } X$$

is presented by a single binary operation, corresponding to the polynomial functor $HX = X \times X$, and the commutativity equation. The latter can be expressed by the parallel pair of morphisms

$$u, u'\colon 1 \to H2$$

(recall $2 = \{0, 1\}$ from Example 2.2(1)) representing the elements $(0, 1)$ and $(1, 0)$ of $H2$, respectively. In fact, the obvious natural transformation $\varepsilon\colon H \to F$ given by $(x, y) \mapsto \{x, y\}$ is universal w.r.t. the property that ε_2 merges u and u'.

Example 5.2. The functor $\mathscr{P}_f^+$ of all non-empty finite subsets can be presented by the signature Σ of one n-ary operation σ_n for every $n = 1, 2, \ldots$, corresponding to the polynomial functor

$$H_\Sigma X = X + X \times X + \cdots = X^+$$

via all the equations

$$\sigma_l(x_0, \ldots, x_{l-1}) = \sigma_k(y_0, \ldots, y_{k-1})$$

where for any $l \leq k$ in $\mathbb{N}$ we have $\{x_0, \ldots, x_{l-1}\} = \{y_0, \ldots, y_{k-1}\}$. Again, each such an equation corresponds to a parallel pair

$$u, u'\colon 1 \to H_\Sigma k,$$

and the obvious natural transformation $\varepsilon\colon H_\Sigma \to \mathscr{P}_f^+$ is universal w.r.t.

$$\varepsilon_k \cdot u = \varepsilon_k \cdot u' \qquad \text{for each } u, u' \text{ above.}$$

Remark 5.3. (1) Recall that for every signature $\Sigma = (\Sigma_k)_{k \in \mathbb{N}}$ the classical Σ-algebras are precisely the algebras for the *polynomial endofunctor* on **Set** given by

$$H_\Sigma X = \Sigma_0 + \Sigma_1 \times X + \Sigma_2 \times X^2 + \cdots$$

(2) An *equation* $u = u'$ is just a notation for a pair of terms. We call it *flat* if both of the terms have the form $\sigma(x_1, \dots, x_n)$ for some $\sigma \in \Sigma_n$ and some n-tuple of variables. This is precisely a parallel pair

$$u, u'\colon 1 \to H_\Sigma X, \qquad \text{where } X = \{x_1, \dots, x_n\}.$$

Definition 5.4 (See [4])**.** A set functor F is **presented** by a signature Σ and a set of flat equations $u_i, u_i'\colon 1 \to H_\Sigma X_i$ $(i \in I)$ provided that there exists a natural transformation $\varepsilon\colon H_\Sigma \to F$ universal w.r.t. the commutativity of the squares

$$\begin{array}{ccc} 1 & \xrightarrow{\;u_i\;} & H_\Sigma X_i \\ {\scriptstyle u_i'}\downarrow & & \downarrow{\scriptstyle \varepsilon_{X_i}} \\ H_\Sigma X_i & \xrightarrow[\varepsilon_{X_i}]{} & F X_i \end{array} \tag{5.8}$$

Remark 5.5. We will see later that every presentation defines a finitary set functor, and every finitary set functor has a presentation.

Remark 5.6. As we pointed out in Remark 2.3(1), the category of finitary functors on a locally finitely presentable category $\mathscr{A}$ is equivalent to the presheaf category $\mathscr{A}^{\mathscr{F}}$. Hence, from now on we will not distinguish between a finitary endofunctor on $\mathscr{A}$ and the corresponding presheaf.

Observation 5.7. The signature Σ can be considered as a functor from $\mathbb{F}^0$, the discrete category of natural numbers, into **Set**. We thus obtain the category

$$\mathbf{Sgn} = \mathbf{Set}^{\mathbb{F}^0}$$

of signatures as a functor category. Its morphisms are functions $f\colon \Sigma \to \Sigma'$ preserving the arity (or, equivalently, collections of functions $f_n\colon \Sigma_n \to \Sigma_n'$ indexed by natural numbers).

Denote by

$$I\colon \mathbb{F}^0 \to \mathbb{F}$$

the non-full embedding. Then the polynomial functor H_Σ, considered as an object of $\mathbf{Set}^{\mathbb{F}}$, can be characterized, up to natural isomorphism, as the left Kan extension of $\Sigma\colon \mathbb{F}^0 \to \mathbf{Set}$ along I:

$$H_\Sigma = \mathrm{Lan}_I\, \Sigma.$$

That is, given a finitary functor considered as $G \in \mathbf{Set}^{\mathbb{F}}$, then natural transformations $\alpha\colon H_\Sigma \to G$ correspond bijectively to natural transformations $\bar{\alpha}\colon \Sigma \to G{\cdot}I$ via precomposition with I (see Lemma 5.17).

Definition 5.8 (M. Kelly and J. Power [13])**.** Let $\mathscr{A}$ be a locally finitely presentable category. By a **signature** is meant a collection $\Sigma = (\Sigma_n)_{n\in\mathscr{F}}$ of objects of $\mathscr{A}$ indexed by representatives of finitely presentable objects.

A **Σ-algebra** is an object A of $\mathscr{A}$ together with a function assigning to morphisms in $\mathscr{A}(n, A)$ morphisms in $\mathscr{A}(\Sigma_n, A)$:

$$\frac{n \xrightarrow{f} A}{\Sigma_n \xrightarrow{\hat{f}} A} \qquad \text{for every } n \in \mathscr{F}.$$

Given another Σ-algebra B, a **Σ-homomorphism** is a morphism $h\colon A \to B$ of $\mathscr{A}$ satisfying

$$h{\cdot}\hat{f} = \widehat{h{\cdot}f} \qquad \text{for all } n \in \mathscr{F} \text{ and } f\colon n \to A.$$

We now provide examples for various categories $\mathscr{A}$.

Example 5.9. $\mathscr{A} = \mathbf{Set}$. For $\mathbb{F}$ of Example 2.2, the notion of a signature has the usual meaning. And the same holds for Σ-algebras: given a set A with n-ary operations $\sigma^A\colon A^n \to A$ for all $\sigma \in \Sigma_n$, we obtain a map assigning to every n-tuple in A, $f\colon n \to A$, the function

$$\hat{f}\colon \Sigma_n \to A, \qquad \sigma \mapsto \sigma^A(f).$$

Conversely, given a Σ-algebra A as in Definition 5.8, define for every $\sigma \in \Sigma_n$ the n-ary operation $\sigma^A\colon f \mapsto \hat{f}(\sigma)$.

Under this bijective translation, homomorphisms in the sense of Definition 5.8 are the usual homomorphisms of Σ-algebras.

Example 5.10. $\mathscr{A} = \mathbf{Pos}$. Here Σ is indexed by (representatives of) finite posets. We denote for every $n \in \mathbb{N}$ by $c(n)$ the chain of length n and by $d(n)$ the discretely ordered set of n elements. We also denote by 0 the initial (empty) poset and by $1 = d(1)$ the terminal one.

(1) The signature Σ with

$$\Sigma_{d(2)} = c(2) \qquad \text{and} \qquad \Sigma_n = 0 \quad \text{for all } n \neq d(2)$$

corresponds to algebras on posets $(A, \leq)$ with two binary operations σ, τ satisfying $\sigma(x,y) \leq \tau(x,y)$ for all pairs (x,y). In fact, this is the same as giving a function

$$\frac{d(2) \xrightarrow{f} A}{c(2) \xrightarrow{\hat{f}} A}$$

An example of a Σ-algebra is $\mathbb{N} \setminus \{0,1\}$ where σ is addition and τ is multiplication.

(2) The signature Σ' with

$$\Sigma'_{c(2)} = 1 \qquad \text{and} \qquad \Sigma_n = 0 \quad \text{for all } n \neq c(2)$$

corresponds to algebras given by a binary operation σ defined iff the pair (x,y) satisfies $x \leq y$. This is the same as giving a function

$$\frac{c(2) \xrightarrow{f} A}{1 \xrightarrow{\hat{f}} A}$$

An example of a Σ-algebra is $\mathbb{N} \setminus \{0\}$ ordered by divisibility, where the operation σ is division.

Example 5.11. $\mathscr{A} = K\text{-}\mathbf{Vec}$. Here $\mathscr{F} = \{K^n; n \in \mathbb{N}\}$ and signatures thus have the same form as in **Set**. However, due to the coincidence of binary products and coproducts, formally different signatures can yield equal categories of algebras. For example, let us consider the signature that in **Set** corresponds to one binary and one unary operation:

$$\Sigma_{K^2} = K, \qquad \Sigma_K = K \qquad \text{and} \qquad \Sigma_n = 0 \quad \text{else.}$$

Then a Σ-algebra is given by a vector space A and two linear functions $A \times A \to A$ and $A \to A$. This is equivalent to giving three linear functions $A \to A$, thus, the signature

$$\Sigma'_K = K^3 \qquad \text{and} \qquad \Sigma'_n = 0 \quad \text{else}$$

yields the same algebras.

Example 5.12. $\mathscr{A} = \mathbf{MS}$. Let δ be the metric space of two elements of distance $\frac{1}{2}$. The signature

$$\Sigma_\delta = 1 \qquad \text{and} \qquad \Sigma_n = 0 \quad \text{else}$$

corresponds to algebras on a metric space (A, d) with one binary operation defined on precisely the pairs (x, y) with $d(x, y) \leq \frac{1}{2}$.

Definition 5.13. The **polynomial functor**

$$H_\Sigma \colon \mathscr{A} \to \mathscr{A}$$

of a given signature Σ is defined on objects X by

$$H_\Sigma X = \coprod_{n \in \mathscr{F}} \mathscr{A}(n, X) \bullet \Sigma_n$$

where $M \bullet \Sigma_n$ denotes a copower of M copies of the object Σ_n.

Example 5.14. For **Set** this is the formula of Remark 5.3, since $\mathbf{Set}(n, X) \bullet \Sigma_n = \Sigma_n \times X^n$.

The polynomial functors of Example 5.10 are

$$H_\Sigma(X, \leq) = (X \times X) \bullet c(2) \qquad \text{and} \qquad H_{\Sigma'}(X, \leq) = \mathbf{Pos}\big(c(2), (X, \leq)\big).$$

Lemma 5.15. *The category* Σ-**Alg** *of* Σ*-algebras and homomorphisms is isomorphic to* $\mathbf{Alg}\, H_\Sigma$.

Proof. Every Σ-algebra A defines for every $n \in \mathscr{F}$ a morphism from $\mathscr{A}(n, A) \bullet \Sigma_n$ to A whose component at $f \colon n \to A$ is $\hat{f} \colon \Sigma_n \to A$. We thus obtain a H_Σ-algebra where

$$\alpha \colon \coprod_{n \in \mathscr{F}} \mathscr{A}(n, A) \bullet \Sigma_n \to A$$

has the above components. Conversely, given an H_Σ-algebra $\alpha \colon H_\Sigma A \to A$ the function

$$\frac{n \xrightarrow{f} A}{\Sigma_n \xrightarrow{\hat{f}} A}$$

is defined by having $\hat{f}$ equal to the component of α corresponding to $f \in \mathscr{A}(n, A)$.

It is easy to see that the above functions extend to functors Σ-**Alg** $\to$ **Alg** H_Σ and **Alg** $H_\Sigma \to \Sigma$-**Alg** which form an isomorphism of categories. In fact, a homomorphism h of H_Σ-algebras as in the following diagram

$$\begin{array}{ccc} \coprod \mathscr{A}(n,A) \bullet \Sigma_n & \xrightarrow{\ \alpha\ } & A \\ {\scriptstyle \coprod \mathscr{A}(n,h)\bullet \mathrm{id}} \big\downarrow & & \big\downarrow {\scriptstyle h} \\ \coprod \mathscr{A}(n,B) \bullet \Sigma_n & \xrightarrow[\ \beta\ ]{} & B \end{array}$$

is precisely a morphism $h\colon A \to B$ in $\mathscr{A}$ such that $h\cdot \hat{f} = \widehat{h\cdot f}$ for every $n \in \mathscr{F}$ and $f \in \mathscr{A}(n, A)$. $\square$

Notation 5.16. Generalizing Observation 5.7, $\mathscr{F}^0$ denotes the discrete category on objects from $\mathscr{F}$, and

$$I\colon \mathscr{F}^0 \to \mathscr{F}$$

is the non-full embedding.

A signature is nothing else than a functor from $\mathscr{F}^0$ to $\mathscr{A}$, thus we call $\mathscr{A}^{\mathscr{F}^0}$ the *category of signatures.*

Lemma 5.17. *For every signature Σ the polynomial endofunctor H_Σ can, as an object of $\mathscr{A}^{\mathscr{F}}$, be characterized as the left Kan extension of Σ:*

$$H_\Sigma = \mathrm{Lan}_I\, \Sigma.$$

Proof. It is our task to show that for every finitary endofunctor considered as $G \in \mathscr{A}^{\mathscr{F}}$ the natural transformations α from Σ to $G\cdot I$ (i.e., collections of morphisms $\alpha_n\colon \Sigma_n \to G(n)$ indexed by $n \in \mathscr{F}$) correspond bijectively to natural transformations from H_Σ to G. Indeed, to give a natural transformation

$$\coprod_{n\in\mathscr{F}} \mathscr{A}(n,-) \bullet \Sigma_n \to G$$

means to give, for every $n \in \mathscr{F}$, a natural transformation $\beta\colon \mathscr{A}(n,-) \bullet \Sigma_n \to G$. By the Yoneda Lemma, β is determined by the id_n-component $\alpha_n\colon \Sigma_n \to G(n)$ of β_n. $\square$

Remark 5.18. In particular, given a signature morphism $u\colon \bar{\Sigma} \to \Sigma = H_\Sigma \cdot I$ the corresponding natural transformation $\bar{u}\colon H_{\bar{\Sigma}} \to H_\Sigma$ has components

$$\bar{u}_A = [H_\Sigma f \cdot u_n]\colon \coprod_{n\in\mathscr{F}} \coprod_{f\colon n\to A} \bar{\Sigma}_n \to H_\Sigma A$$

for all A.

Definition 5.19. By a **flat equation** in a signature Σ is meant a parallel pair

$$u, u'\colon n \to H_\Sigma k \qquad \text{for } n, k \in \mathscr{F}.$$

A finitary functor considered as G in $\mathscr{A}^{\mathscr{F}}$ is said to be **presented** by a signature Σ and flat equations $u_i, u_i'\colon n_i \to H_\Sigma k_i$ $(i \in I)$ provided that there exists a natural transformation $\varepsilon\colon H_\Sigma \to G$ universal w.r.t. the commutativity of the squares

$$\begin{array}{ccc} n_i & \xrightarrow{\ u_i\ } & H_\Sigma k_i \\ {\scriptstyle u_i'}\downarrow & & \downarrow{\scriptstyle \varepsilon_{k_i}} \\ H_\Sigma k_i & \xrightarrow[\ \varepsilon_{k_i}\ ]{} & G k_i \end{array} \qquad (i \in I). \tag{5.9}$$

Example 5.20. The endofunctor G of **Pos** defined by

$$G(X, \leq) = \big\{(x,y) \in X^2; x < y\big\} \cup \{*\}$$

on objects and on morphisms $f\colon (X,\leq) \to (Y,\preceq)$ by

$$Gf(x,y) = \big(f(x), f(y)\big) \qquad \text{if} \quad f(x) \preceq f(y)$$

whereas Gf has else the value $*$, has the presentation by the signature Σ' of Example 5.10 and the flat equation

$$\sigma(x,x) = \sigma(y,y).$$

Construction 5.21. A finitary endofunctor presented by a signature Σ and a given set $u_i, u_i'\colon n_i \to H_\Sigma k_i$ $(i \in I)$ of flat equations.

Define a signature $\bar{\Sigma}$ by

$$\bar{\Sigma}_k = \coprod_{i\in I, k_i = k} n_i \qquad \text{for all } k \in \mathscr{F}.$$

Then the morphisms u_i define a natural transformation from $\bar{\Sigma}$ to $H_\Sigma \cdot I$: its component at $k \in \mathscr{F}$ is simply

$$[u_i] \colon \coprod_{i \in I, k_i = k} n_i \to H_\Sigma k.$$

From Lemma 5.17 we obtain the corresponding natural transformation

$$\bar{u} \colon H_{\bar{\Sigma}} \to H_\Sigma.$$

Analogously for $\bar{u'} \colon H_{\bar{\Sigma}} \to H_\Sigma$. In the (cocomplete) category $\mathscr{A}^{\mathscr{F}}$ of all finitary endofunctors, form the coequalizer ε:

$$H_{\bar{\Sigma}} \underset{\bar{u'}}{\overset{\bar{u}}{\rightrightarrows}} H_\Sigma \xrightarrow{\ \varepsilon\ } F.$$

Then F is presented by the given flat equations.

Indeed, for every k the equation $\varepsilon_k \cdot u_k = \varepsilon_k \cdot u'_k$ guarantees that ε satisfies the equations u_i, u'_i for all $i \in I$. Conversely, let $\varepsilon' \colon H_\Sigma \to F'$ be a natural transformation with $\varepsilon'_{k_i} \cdot u_i = \varepsilon'_{k_i} \cdot u'_i$ for all $i \in I$. Then obviously $\varepsilon' \cdot \bar{u} = \varepsilon' \cdot \bar{u'}$, thus, ε' factorizes uniquely through ε.

Proposition 5.22. *Every finitary endofunctor of a locally finitely presentable category has a presentation by a signature and a set of flat equations.*

Proof. Precomposition with $I \colon \mathscr{F}^0 \to \mathscr{F}$ defines a functor

$$-\cdot I \colon \mathscr{A}^{\mathscr{F}} \to \mathscr{A}^{\mathscr{F}^0}$$

which is monadic. Indeed, this functor has both a left and a right adjoint, and it reflects isomorphisms: given a morphism $\alpha \colon F \to G$ in $\mathscr{A}^{\mathscr{F}}$ which is invertible in $\mathscr{A}^{\mathscr{F}^0}$ (i.e., has invertible components), then α is a natural isomorphism, i. e., it is invertible in $\mathscr{A}^{\mathscr{F}}$. Thus, monadicity follows from Beck's Theorem, see e.g. [7], Theorem 4.4.4. Consequently, finitary endofunctors of $\mathscr{A}$ are precisely the monadic algebras of the corresponding monad $\mathbb{T}$ on the category $\mathscr{A}^{\mathscr{F}^0}$ of signatures. It follows from Lemma 5.17 that this monad assigns to every signature Σ the signature $T(\Sigma) = H_\Sigma \cdot I$. The free $\mathbb{T}$-algebra on Σ is then H_Σ.

Every finitary endofunctor F, i.e., every Eilenberg-Moore algebra for $\mathbb{T}$, is a coequalizer of a parallel pair of homomorphisms between free $\mathbb{T}$-algebras:

$$H_{\bar{\Sigma}} \underset{u'}{\overset{u}{\rightrightarrows}} H_\Sigma \xrightarrow{\ \varepsilon\ } F.$$

Consider, for every $k \in \mathscr{F}$, the object $H_{\bar{\Sigma}}k$ as a filtered colimit of objects n_i $(i \in I_k)$ in $\mathscr{F}$ with the colimit cocone $v_i^{(k)} \colon n_i \to H_{\bar{\Sigma}}k$. Then the flat equations

$$u_k \cdot v_i^{(k)}, u'_k \cdot v_i^{(k)} \colon n_i \to H_\Sigma k$$

(where k ranges through $\mathscr{F}$ and i through I_k) form an equational presentation of F.

Indeed, from $\varepsilon \cdot u = \varepsilon \cdot u'$ it follows that each of the above parallel pairs is merged by ε_k. Let $\hat{\varepsilon} \colon H_\Sigma \to \hat{F}$ another morphism of $\mathscr{A}^{\mathscr{F}}$ such that each of the above parallel pairs is merged by $\hat{\varepsilon}_k$. To prove that $\hat{\varepsilon}$ uniquely factorizes through ε we need to verify that $\hat{\varepsilon} \cdot u = \hat{\varepsilon} \cdot u'$. Equivalently, $\hat{\varepsilon}_k \cdot u_k = \hat{\varepsilon}_k \cdot u'_k$ for every $k \in \mathscr{F}$. This follows from $\hat{\varepsilon}_k \cdot u_k \cdot v_i^{(k)} = \hat{\varepsilon}_k \cdot u'_k \cdot v_i^{(k)}$ since the cocone $v_i^{(k)}$, $i \in I_k$, is collectively epic (being a colimit cocone). □

Remark 5.23. (1) The above proof shows that we always have a canonical presentation of a finitary functor F: take the signature Σ defined by

$$\Sigma_n = F(n)$$

for all $n \in \mathscr{F}$. Obtain a canonical natural transformation $\varepsilon \colon H_\Sigma \to F$ whose component

$$\varepsilon_k \colon \coprod_{n \in \mathscr{F}} \mathscr{A}(n,k) \bullet Fn \to Fk$$

is given by $[Ff] \colon \coprod_{f \in \mathscr{A}(n,k)} Fn \to Fk$. Then consider all the flat equations formed by all parallel pairs

$$u, u' \colon n \to H_\Sigma k \qquad (n, k \in \mathscr{F}) \qquad \text{with } \varepsilon_k \cdot u = \varepsilon_k \cdot u'.$$

(2) We can, as we have seen e.g. in Example 5.1, often obtain a much simpler equational presentation. Here is another example:

Example 5.24. Let F be the set-functor obtained from $X \mapsto X \times X$ by merging the diagonal to a single element $*$:

$$FX = \big\{(x,y); x, y \in X, x \neq y\big\} \cup \{*\}.$$

F has a presentation using a single binary operation σ and the equation

$$\sigma(x,x) = \sigma(y,y).$$

Definition 5.25. A Σ-algebra A is said to **satisfy** a flat equation $u, u' \colon n \to H_\Sigma k$ provided that its algebra structure $\alpha \colon H_\Sigma A \to A$ merges $H_\Sigma f \cdot u$ and $H_\Sigma f \cdot u'$ for all $f \colon k \to A$.

Example 5.26. In **Set** this is the usual concept of fulfilling an equation with k variables: given any interpretation $f \colon k \to A$ of the variables, the elements of A computed from the two sides of the equation are equal.

Lemma 5.27. *If a functor F is presented by a signature Σ and flat equations u_i, u_i' $(i \in I)$, then the category $\mathbf{Alg}\,F$ of F-algebras is equivalent to the category of all Σ-algebras satisfying those equations.*

Proof. Recall the coequalizer ε of $\bar{u}$ and $\bar{u'}$ from Construction 5.21. Its components are (regular) epimorphisms. We can identify an F-algebra $\alpha \colon FA \to A$ with an H_Σ-algebra whose structure, $\bar{\alpha}$, factorizes through ε_A:

$$\begin{array}{ccc} H_\Sigma A & \xrightarrow{\ \bar{\alpha}\ } & A \\ {\scriptstyle \varepsilon_A}\downarrow & \nearrow{\scriptstyle \alpha} & \\ FA & & \end{array}$$

In that sense $\mathbf{Alg}\,F$ is a full subcategory of the category $\mathbf{Alg}\,H_\Sigma$ of Σ-algebras, see Lemma 5.15.

(1) If $\bar{\alpha}$ factorizes through ε_A, then $(A, \bar{\alpha})$ satisfies the given equations: for every $f \colon k_i \to A$ we have $\bar{\alpha} \cdot H_\Sigma f = \alpha \cdot Ff \cdot \varepsilon_{k_i}$, thus, from $\varepsilon_{k_i} \cdot u_i = \varepsilon_{k_i} \cdot u_i'$ we conclude $\bar{\alpha} \cdot H_\Sigma f \cdot u_i = \bar{\alpha} \cdot H_\Sigma f \cdot u_i'$.

(2) Conversely, given an H_Σ-algebra $\alpha \colon H_\Sigma A \to A$ satisfying all the given flat equation morphisms, we prove that α factorizes through ε_A. From Construction 5.21 we know that ε is the coequalizer of the natural transformations $\bar{u}, \bar{u'} \colon H_{\bar{\Sigma}} \to H_\Sigma$ obtained from the given flat equations. Since coequalizers in $\mathscr{A}^{\mathscr{F}}$ are formed object-wise, all we need to prove is that α merges $\bar{u}_A$ and $\bar{u'}_A$. This follows from the fact that for every $f \colon k_i \to A$ we have $\alpha \cdot H_\Sigma f \cdot u_i = \alpha \cdot H_\Sigma f \cdot u_i'$: recall from Remark 5.18 that for the natural transformation $[u_i]$ of Construction 5.21 we have $\bar{u}_A = [H_{\bar{\Sigma}} f \cdot u_n]$ and analogously for $\bar{u'}_A$ and compute as follows.

$$\begin{array}{rcll} \alpha \cdot \bar{u}_A & = & \alpha \cdot [H_{\bar{\Sigma}} F \cdot u_n] & \text{by Remark 5.18} \\ & = & [\alpha \cdot H_\Sigma f \cdot u_n] & \\ & = & [\alpha \cdot H_\Sigma f \cdot u_n'] & \\ & = & \alpha \cdot [H_\Sigma f \cdot u_n'] & \\ & = & \alpha \cdot \bar{u'}_A & \text{by Remark 5.18.} \end{array}$$

□

Remark 5.28. There is an alternative definition of what it means for a Σ-algebra A to satisfy a flat equation—and fortunately, the result is the same as above. This is based on the following idea of M. Kelly and J. Power [13]: given objects A and B of $\mathscr{A}$, let $\langle A, B\rangle$ be the endofunctor of $\mathscr{A}$ assigning to X the power of B to the set $\mathscr{A}(X, A)$:

$$\langle A, B\rangle X = B^{\mathscr{A}(X,A)}.$$

In other words, $\langle A, B\rangle$ is the following composite

$$\mathscr{A} \xrightarrow{\mathscr{A}(-,A)} \mathbf{Set}^{\mathrm{op}} \xrightarrow{B^{(-)}} \mathscr{A}.$$

Then natural transformations from F to $\langle A, A\rangle$ are, for every endofunctor F, in a canonical bijective correspondence to F-algebra structures on A. In fact, to every algebra $\alpha\colon FA \to A$ assign $\alpha^*\colon F \to \langle A, A\rangle$ where the components $\alpha^*_X\colon FX \to A^{\mathscr{A}(X,A)}$ are given by

$$\alpha^*_X = \langle \alpha{\cdot}Ff\rangle_{f\colon X\to A}\colon FX \to \langle A, A\rangle X = A^{\mathscr{A}(X,A)}.$$

It is now natural to say that a Σ-algebra $\alpha\colon H_\Sigma A \to A$ satisfies a flat equation $u, u'\colon n \to H_\Sigma k$ iff

$$\alpha^*_k{\cdot}u = \alpha^*_k{\cdot}u'\colon n \to \langle A, A\rangle k.$$

But this tells us precisely that $\alpha{\cdot}H_\Sigma f{\cdot}u = \alpha{\cdot}H_\Sigma f{\cdot}u'$ for all $f\colon X \to A$.

Remark 5.29. Everything above generalizes without any problem from finitary functors to accessible ones. Let $\mathscr{A}$ be a locally λ-presentable category (see Definition 2.5).

By a *λ-ary signature* is meant a collection $\Sigma = (\Sigma_n)_{n\in\mathscr{A}_\lambda}$ of objects of $\mathscr{A}$. The corresponding polynomial endofunctor H_Σ is given by $H_\Sigma X = \coprod_{n\in\mathscr{A}_\lambda} \mathscr{A}(n, X) \bullet \Sigma_n$. A *flat λ-ary equation* is a parallel pair of morphisms $u, u'\colon n \to H_\Sigma k$ with $n, k \in \mathscr{A}_\lambda$. A λ-accessible endofunctor F is said to have a *λ-ary presentation* if there exists a λ-ary signature Σ and a collection $u_i, u_i'\colon n_i \to H_\Sigma k_i$ of λ-ary flat equations such that there is a universal natural transformation $\varepsilon\colon H_\Sigma \to F$ w.r.t. $\varepsilon_{k_i}{\cdot}u_i = \varepsilon_{k_i}{\cdot}u_i'$ for every i.

Proposition 5.30. *Every λ-accessible endofunctor of a locally λ-presentable category has a λ-ary presentation, and every λ-ary presentation defines a λ-accessible endofunctor.*

The proof is completely analogous to Construction 5.21 and Proposition 5.22.

Example 5.31. The functor $\mathscr{P}_c^+$ of all non-empty countable subsets can be represented by the signature Σ of one $\aleph_0$-ary operation σ corresponding to the polynomial functor

$$H_\Sigma X = X^\omega + 1$$

via all the equations

$$\sigma(x_0, x_1, x_2 \dots) = \sigma(y_0, y_1, y_2 \dots) \qquad \text{whenever} \quad \{x_i\}_{i\in\mathbb{N}} = \{y_i\}_{i\in\mathbb{N}}.$$

Again, each such an equation corresponds to a parallel pair

$$u, u' \colon 1 \to H_\Sigma k,$$

and the obvious natural transformation $\varepsilon \colon H_\Sigma \to \mathscr{P}_c^+$ is universal w.r.t.

$$\varepsilon_k \cdot u = \varepsilon_k \cdot u' \qquad \text{for each } u, u' \text{ above.}$$

6. An Equational Presentation of the Hausdorff Functor

There is another "natural" approach to generalizing finitary signatures and the corresponding algebras to a category $\mathscr{A}$. Suppose $\mathscr{A}$ has finite products and Σ is a finitary signature (in **Set**). Then we denote by K_Σ the endofunctor

$$K_\Sigma X = \coprod_{n<\omega} \Sigma_n \bullet X^n.$$

An algebra is now an object A together with a morphism $f^A \colon A^n \to A$ for every n-ary symbol f in Σ. Denote by U_Σ the forgetful functor from the category of K_Σ-algebras into $\mathscr{A}$.

Remark 6.1. (1) Every term t in k variables (in the classical sense) defines a natural transformation $\hat{t}$ from U_Σ^k to U_Σ in the expected sense:

- if t is the i-th variable, then $\hat{t}$ is the i-th projection;
- if $t = f(t_1, \dots, t_n)$, then the component $\hat{t}_A$ in an algebra A is the composite

$$\hat{t}_A = (\, A^k \xrightarrow{\langle (\widehat{t_1})_A, \dots, (\widehat{t_n})_A \rangle} A^n \xrightarrow{\ f\ } A).$$

(2) We thus have a "natural" interpretation of classical equations $t = u$ in the category of K_Σ-algebras: an algebra A satisfies that equation iff $\hat{t}_A = \hat{u}_A$.

Definition 6.2. (1) We say that a natural transformation $\varepsilon : K_\Sigma \to F$ *respects* the equation $t = u$ provided that the functor $\varepsilon^* : \mathbf{Alg}\, F \to \mathbf{Alg}\, K_\Sigma$ given by precomposition with ε fulfils $\hat{t}\varepsilon^* = \hat{u}\varepsilon^*$.

(2) A *classical presentation* of an endofunctor F of $\mathscr{A}$ consists of a (classical) signature Σ and equations E (in **Set**) such that there exists a natural transformation $\varepsilon : K_\Sigma \to F$ universal w.r.t. respecting all the given equations.

The Hausdorff functor has a classical presentation identical with the presentation of the non-empty finite power-set functor in Example 5.2. More precisely, consider the parallel pairs $u, u' : 1 \to k^+$ presenting $\mathscr{P}_f$ in Example 5.2. We interpret $X^+ = \coprod_{n>0} X^n$, using the coproduct of finite powers in **CMS**; this is the disjoint union of the spaces of finite tuples with the maximum metric (see Remark 6.5 for a discussion of the appropriate signature).

We now use the same family of pairs $u, u' : 1 \to k^+$ as in Example 5.2. We claim that the joint coequalizer in **CMS** of this family is the natural transformation $\varepsilon : (-)^+ \to \mathscr{H}$ given by

$$\varepsilon_X : X^+ \to \mathscr{H}X \qquad (x_1, \dots, x_n) \mapsto \{x_1, \dots, x_n\}.$$

Indeed, the set $\mathscr{P}_f(X)$ of all non-empty finite subsets of X is dense in $\mathscr{H}X$ for any space X. To see this, let C be a non-empty compact subset of X. Fix $\delta > 0$. The collection of open balls of radius δ which meet C covers C. By compactness, there is a finite subcollection which covers C. The set of centers gives a finite, hence compact, $F \subseteq X$, and its distance to C in the Hausdorff metric is at most δ.

Since $\varepsilon{\cdot}u = \varepsilon{\cdot}u'$ for all the above pairs $u, u' \colon 1 \to k^+$, our claim that ε is their joint equalizer follows from the following fact.

Lemma 6.3. *For every pair M, N of non-empty finite subsets of a complete metric space X, there are words $m, n \in X^+$ so that $\varepsilon_X(m) = M$, $\varepsilon_X(n) = N$, and*

$$d_{X^+}(m, n) = d_{\mathscr{H}X}(M, N).$$

Proof. Put $M = \{x_0, \dots, x_{p-1}\}$ and $N = \{y_0, \dots, y_{q-1}\}$. We may assume that x_0 and y_0 are such that $d_{\mathscr{H}X}(M, N) = d_X(x_0, y_0)$. Define $(p+q)$-tuples in X by

$$\begin{array}{rcl} m & = & (x_0, \ldots, x_{p-1}, x[y_0], \ldots, x[y_{q-1}]) \\ n & = & (y[x_0], \ldots, y[x_{p-1}], y_0, \ldots, y_{q-1}) \end{array}$$

where we choose $y[x_i] \in N$ such that $d(x_i, y[x_i]) \leq d(x_0, y_0)$ for all i, and analogously we choose $x[y_j] \in M$ with $d(x[y_j], y_j) \leq d(x_0, y_0)$ for all j. Then $d_{X^+}(m, n) = d(x_0, y_0)$. Moreover, $\varepsilon_X(m) = M$ and $\varepsilon_X(n) = N$. □

Proposition 6.4. *The Hausdorff functor has a classical presentation by all the parallel pairs $u, u' : 1 \to H_\Sigma k$ of Example 5.2, with the discrete metric on $k = \{0, ..., k-1\}$.*

Proof. It is our task to prove that for the above endofunctor $(-)^+$ of **CMS**, the natural transformation $\varepsilon : (-)^+ \to \mathscr{H}$ is a joint coequalizer of all pairs u_i, u_i'. Clearly ε is non-expanding. Let X and Y be complete metric spaces, and let $f : X^+ \to Y$ satisfy $f \cdot u_i = f \cdot u_i'$ for all $i \in I$. There is a unique $g_0 : \mathscr{P}_f(X) \to Y$ such that $g_0 \cdot \varepsilon_{X^+} = f$. By Lemma 6.3 and since f is non-expanding, g_0 is non-expanding, too. Since $\mathscr{P}_f(X)$ is dense in $\mathscr{H}(X)$, g_0 extends to a unique $g : \mathscr{H}(X) \to Y$. □

Remark 6.5. (1) We find it surprising that $\mathscr{H}$ has the same presentation in **CMS** that $\mathscr{P}_f$ has in **Set**. Let us observe that, nonetheless, this presentation is not a finitary presentation in the sense of Definition 5.4 for two reasons: (a) $(-)^+$ is not a polynomial functor and (b) no non-empty space is finitely presentable (see Remark 2.10).

(2) J. Velebil und A. Kurz define the notion of presentation in an enriched setting in [19]. Then $(-)^+$ is indeed a polynomial functor so that the presentation of the Hausdorff functor we showed in this section is then an ω-ary presentation in the sense of Remark 5.29 (even though only operations of finite arity are used).

(3) Velebil and Kurz also provide in [19, Proposition 5.4], a presentation (in the enriched setting) of a related functor mapping a complete metric space X to the space of its closed and separable subsets with the Hausdorff metric. Their presentation is countable, using besides n-ary operations as above also an ω-ary operation.

7. Conclusions

We have shown that finitary endofunctors of locally finitely presentable categories have an equational presentation using finitary signatures in the sense of M. Kelly and J. Power [13]. There are important categories which are not locally finitely presentable, but are locally countably presentable, e.g. ω **CPO** and **CMS** (complete metric spaces). There every countably accessible endofunctor has an equational presentation using signatures of countable arity. The main results of our paper is that the Hausdorff and the Kantorovich functor on **CMS**, which were proved to be accessible by F. van Breugel et al. [8], are in fact finitary. Moreover, the Hausdorff functor has a presentation which is completely analogous to the presentation of the finite power set functor on **Set**.

References

[1] Abramsky, S., A. Jung, A.: Domain Theory. In Handbook of Logic in Computer Science, Clarendon Press, Oxford 1994

[2] Adámek, J.: Free algebras and automata realizations in the language of categories. Comment. Math. Univ. Carolinae **15** (1974) 589–609

[3] Adámek, J., Rosický, J.: Locally Presentable and Accessible Categories. Cambridge University Press 1993

[4] Adámek, J., Trnková, V.: Automata and Algebras in Categories. Kluwer Academic Publ., Dordrecht 1990

[5] Adámek, J., Trnková, V.: Relatively terminal coalgebras. J. Pure Appl. Algebra 216 (2012), 1887–1895

[6] Barr, M.: Terminal coalgebras in well-founded set theory. Theoret. Comput. Sci. **114** (1993), 299–315

[7] Borceux, F.: Handbook of Categorical Algebra, Vol. 2. Cambridge University Press 1994

[8] van Breugel, F., Hermida, C., Makkai, M., Worrell, J.: Recursively defined metric spaces without contraction. Theoret. Comput. Sci. **380** (2007), 143–163

[9] van Breugel, F., Hermida, C., Makkai, M., Worrell, J.: Addendum to Recursively defined metric spaces without contraction. To appear in Theoret. Comput. Sci.

[10] van Breugel, F., J. Worrell, J.: A behavioral pseudometric for probabilistic transition systems. Theoret. Comput. Sci. **331** (2005) 115–142

[11] Heckmann, R.: Probabilistic Domains. CAAP 1994 142-156

[12] Kantorovich, L. V.: On the translocation of masses. Dokl. Akad. Nauk SSSR **37** (1942) 227–229

[13] Kelly, G. M, Power, J.: Adjunctions whose counits are coequalizers and presentations of finitary enriched monads. J. Pure Appl. Algebra **89** (1993) 163–179

[14] Korte, B., Vygen, J.: Combinatorial Optimization. Springer 2010

[15] Lawvere, F. W.: Functional semantics of algebraic theories. Dissertation, Columbia University 1963

[16] Makkai, M., Paré, R.: Accessible Categories. Amer. Math. Soc., Providence, Rhode Island 1989

[17] Parthasarathy, K.R.: Probability Measures on Metric Spaces. AP 1989

[18] Urbat, H.: Kantorovich Functor (Master's thesis), Technische Universität Braunschweig, October 2012. Available under *https://www.tu-braunschweig.de/Medien-DB/iti/dpl_kantorovich.pdf*

[19] Velebil, J., Kurz, A. Equational presentations of functors and monads. Math. Structures Comput. Sci. **21** (2011), 363–381

[20] Whitney, H.: Analytic extensions of functions defined in closed sets. Transactions of the American Mathematical Society **36** (1) 63-89 Amer. Math. Soc., Providence, Rhode Island 1989

Appendix A. Full proof of Theorem 4.1

We present a detailed proof of the characterization of $\mathscr{K}$ as the free mean-value algebra monad in **CMS**, based on ideas from [8]. The key to our proof is an observation about Kantorovich distances of uniform discrete probability measures. In the sequel, let $\eta_X(x)$ denote the Dirac measure on a metric space X at $x \in X$, defined by

$$\eta_X(x)(A) = \begin{cases} 1, & x \in A; \\ 0, & x \notin A \end{cases}$$

for Borel sets $A \subseteq X$. Note that

$$\int_X f d\eta_X(x) = fx$$

for all measurable functions $f : X \to \mathbb{R}$.

Lemma A.1. *Let $X \in$ **MS** and $x_1, \ldots, x_{2n} \in X$. Then the probability measures*

$$\mu := \frac{1}{n} \sum_{i=1}^{n} \eta_X(x_i), \quad \nu := \frac{1}{n} \sum_{j=n+1}^{2n} \eta_X(x_j)$$

have Kantorovich distance

$$d_{\mathscr{K}(X)}(\mu, \nu) = \frac{1}{n} \min_{\pi} \sum_{i=1}^{n} d_X(x_i, x_{\pi(i)}),$$

where the minimum ranges over all bijections $\pi : \{1, \ldots, n\} \to \{n+1, \ldots, 2n\}$.

In other words, computing $d_{\mathscr{K}(X)}(\mu, \nu)$ amounts to finding a perfect matching between the points $x_1, \ldots, x_n$ and $x_{n+1}, \ldots, x_{2n}$ that minimizes the average distance.

Proof. For all bijections $\pi : \{1, \ldots, n\} \to \{n+1, \ldots, 2n\}$, we have

$$\begin{aligned} d_{\mathscr{K}(X)}(\mu, \nu) &= \frac{1}{n} \sup_{f \text{ nonexp.}} \left(\sum_{i=1}^{n} fx_i - \sum_{j=n+1}^{2n} fx_j \right) && (\text{def. } d_{\mathscr{K}(X)}) \\ &= \frac{1}{n} \sup_{f \text{ nonexp.}} \left(\sum_{i=1}^{n} (fx_i - fx_{\pi(i)}) \right) && (\pi \text{ bijective}) \\ &\leq \frac{1}{n} \sum_{i=1}^{n} d_X(x_i, x_{\pi(i)}) && (f \text{ nonexpanding}) \end{aligned}$$

and therefore

$$d_{\mathscr{K}(X)}(\mu,\nu) \leq \frac{1}{n}\min_{\pi}\sum_{i=1}^{n} d_X(x_i, x_{\pi(i)}).$$

To prove that this holds with equality, we show that there exists a nonexpanding function $f : X \to [0,1]$ with

$$\sum_{i=1}^{n} fx_i - \sum_{j=n+1}^{2n} fx_j = \min_{\pi}\sum_{i=1}^{n} d_X(x_i, x_{\pi(i)})$$

For this purpose, it suffices to prove that the linear program

$$\begin{array}{l} \max \sum_{i=1}^{n} y_i - \sum_{j=n+1}^{2n} y_j \\ y_i - y_j \leq d_X(x_i, x_j) \quad (1 \leq i,j \leq 2n) \end{array} \tag{A.1}$$

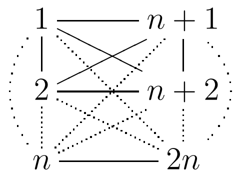
has an optimal solution of value $\min_\pi \sum_{i=1}^{n} d_X(x_i, x_{\pi(i)})$. (In fact, given such a solution $(y_1, \dots, y_{2n})$, where w.l.o.g. $y_i \in [0,1]$ for all i since X is 1-bounded, we obtain a nonexpanding function $f' : \{x_1, \dots, x_{2n}\} \to [0,1]$ by setting $f'(x_i) := y_i$. Note that f' is well-defined since $x_i = x_j$ implies $y_i - y_j \leq d_X(x_i, x_j) = 0$. Then we can extend f' to the nonexpanding function

$$f : X \to [0,1], \quad f(x) := \min\{1, \min_{i=1,\dots,2n}\{f'(x_i) + d_X(x_i, x)\}\}$$

which has the desired property.) Now consider the dual of (A.1):

$$\begin{array}{rcll} \multicolumn{4}{l}{\min \sum_{i=1}^{2n}\sum_{j=1}^{2n} z_{ij} d_X(x_i, x_j)} \\ \sum_{j=1}^{2n} z_{ij} - \sum_{j=1}^{2n} z_{ji} & = & 1 & (i = 1, \dots, n) \\ \sum_{j=1}^{2n} z_{ij} - \sum_{j=1}^{2n} z_{ji} & = & -1 & (i = n+1, \dots, 2n) \\ z_{ij} & \geq & 0 & (1 \leq i,j \leq 2n) \end{array} \tag{A.2}$$

By the duality theorem of linear programming [14, Theorem 3.20], both (A.1) and (A.2) have an optimal solution, and the optimal values agree. Observe that (A.2) is equivalent to the minimum-cost flow problem (cf. [14, Chapter 9]) on the complete graph with $2n$ nodes,



where the nodes $1, \dots, n$ are sources with supply 1, the nodes $n+1, \dots, 2n$ are sinks with demand 1, and the cost per unit of the edge ij is $d_X(x_i, x_j)$ $(1 \le i, j \le 2n)$. Every bijection $\pi : \{1, \dots, n\} \to \{n+1, \dots, 2n\}$ induces a feasible flow z^π of value $\sum_{i=1}^n d_X(x_i, x_{\pi(i)})$, namely, z^π sends one unit of flow from i to $\pi(i)$ for $i = 1, \dots, n$. Thus we are done once we establish the following

Claim. For every feasible flow z, there exists a bijection $\pi : \{1, \dots, n\} \to \{n+1, \dots, 2n\}$ with $\sum_{i=1}^n d_X(x_i, x_{\pi(i)}) \le \sum_{i=1}^{2n} \sum_{j=1}^{2n} z_{ij} d_X(x_i, x_j)$

Proof. Since the supplies and demands of the above flow network are integers, (A.2) has an integral optimal solution ([14, Corollary 8.7]), so we may assume that z is an integral flow. By [14, Theorem 8.8], there exists a set $\mathcal{P} = \{P_1, \dots, P_n\}$ of directed paths with the following properties:

1. P_i starts in the node i $(i = 1, \dots, n)$.
2. For each sink $j \in \{n+1, \dots, 2n\}$, there is a path in $\mathcal{P}$ that ends in j.
3. For each edge ij, the number of paths in $\mathcal{P}$ containing ij is at most z_{ij}.

Thus we obtain a bijection $\pi : \{1, \dots, n\} \to \{n+1, \dots, 2n\}$ by mapping $i \in \{1, \dots, n\}$ to the last node of P_i. It follows that

$$\begin{aligned} \sum_{i=1}^n d_X(x_i, x_{\pi(i)}) &\le \sum_{i=1}^n \sum_{kl \in P_i} d_X(x_k, x_l) && \text{(triangle inequality)} \\ &\le \sum_{i=1}^{2n} \sum_{j=1}^{2n} z_{ij} d_X(x_i, x_j) && \text{(property 3 of } \mathcal{P}\text{)} \end{aligned}$$ □

Let $(X, \oplus)$ be a metric mean-value algebra. Following [11], one can extend the binary operation $\oplus : X^2 \to X$ to the 2^n-ary operations $\oplus_n : X^{2^n} \to X$ $(n \ge 0)$ which are inductively defined by

- $\oplus_0(x) = x$
- $\oplus_{n+1}((x_i)_{i=1,\ldots,2^{n+1}}) = (\oplus_n((x_i)_{i=1,\ldots,2^n})) \oplus (\oplus_n((x_i)_{i=2^n+1,\ldots,2^{n+1}}))$

The following lemma summarizes some elementary properties of $\oplus_n$. In the statement, S_{2^n} denotes the set of bijections of $\{1, \ldots, 2^n\}$.

Lemma A.2. *Let $(X, \oplus)$ be a metric mean-value algebra.*

(a) For all $\pi \in S_{2^n}$ and $(x_i) \in X^{2^n}$,

$$\oplus_n((x_i)_{i=1,\ldots,2^n}) = \oplus_n((x_{\pi(i)})_{i=1,\ldots,2^n}).$$

(b) For all $(x_i), (x'_i) \in X^{2^n}$,

$$d_X(\oplus_n((x_i)_{i=1,\ldots,2^n}), \oplus_n((x'_i)_{i=1,\ldots,2^n})) \leq 2^{-n} \sum_{i=1}^{2^n} d_X(x_i, x'_i).$$

(c) Morphisms of mean-value algebras preserve $\oplus_n$.

Proof. See [11] for (a); the statements (b) and (c) are easily verified by induction on n. □

By part (a) of the previous lemma, we can view $\oplus_n$ as an operation $\oplus_n : \mathcal{M}_{2^n}(X) \to X$, where $\mathcal{M}_{2^n}(X)$ is the set of all 2^n-element multisets over X. In particular, for the multiset

$$M = \{n_i \cdot \eta_X(x_i) : i \in I\} \in \mathcal{M}_{2^n}(\mathscr{K}(X)),$$

we obtain

$$\oplus_n(M) = \sum_i \frac{n_i}{2^n} \eta_X(x_i).$$

Hence the probability measures of the form $\oplus_n(M)$ are precisely the measures with finite support and dyadic probabilities. By [17], Theorem 6.3, these form a dense subalgebra of $(\mathscr{K}(X), \oplus)$ that we denote $\mathcal{D}(X)$. Now we are prepared to prove

Theorem 4.1. $\bar{\mathscr{K}}$ is a left adjoint to the forgetful functor $U : \mathscr{M} \to \mathbf{CMS}$.

Proof. Let $X \in \mathbf{CMS}$, and let $\eta_X : X \to \mathscr{K}(X) = U\bar{\mathscr{K}}(X)$ be the function that maps $x \in X$ to the Dirac measure $\eta_X(x)$. Note that η_X is nonexpanding since, for all $x, y \in X$,

$$\begin{aligned} d_{\mathscr{K}(X)}(\eta_X(x), \eta_X(y)) &= \sup_{f \text{ nonexp}} \left(\int_X f d\eta_X(x) - \int_X f d\eta_X(y) \right) && (\text{def. } d_{\mathscr{K}(X)}) \\ &= \sup_{f \text{ nonexp}} (fx - fy) \\ &\leq \sup_{f \text{ nonexp}} d_X(x, y) && (f \text{ nonexpanding}) \\ &= d_X(x, y) \end{aligned}$$

We show that for each metric mean-value algebra $(Y, \oplus)$ and nonexpanding map $f : X \to Y$, there exists a unique mean-value algebra morphism $\bar{f} : \mathscr{K}(X) \to Y$ with $\bar{f}\eta_X = f$. Since $\mathcal{D}(X)$ is a dense subalgebra of $\mathscr{K}(X)$, and since $\oplus$ is continuous, it is sufficient to define $\bar{f}$ on $\mathcal{D}(X)$. The only possible definition of $\bar{f}$ is

$$\bar{f}(\oplus_n((\eta_X x_i)_{i=1,\dots,2^n})) := \oplus_n((f x_i)_{i=1,\dots,2^n})$$

We need to verify that $\bar{f}$ is well-defined, nonexpanding and $\oplus$-preserving.

1. $\bar{f}$ is well-defined: Suppose that $\oplus_n((\eta_X x_i)_{i=1,\dots,2^n}) = \oplus_m((\eta_X x'_j)_{j=1,\dots,2^m})$ (i.e., $2^{-n} \sum_{i=1}^{2^n} \eta_X(x_i) = 2^{-m} \sum_{i=1}^{2^m} \eta_X(x'_j)$) for some $m, n \geq 0$ and $x_i, x'_j \in X$. By the idempotence of $\oplus$, we can assume $m = n$. Now two discrete measures are equal if and only if they have the same support and the same probability at each point of their support. In this case, this means that the tuples (x_i) and (x'_j) contain the same elements with the same multiplicity, i.e., they agree as multisets. Then the same holds for the sequences $(f x_i)$ and $(f x'_j)$, so $\oplus_n((f x_i)_{i=1,\dots,2^n}) = \oplus_n((f x'_j)_{j=1,\dots,2^n})$ by Lemma A.2(a).

2. $\bar{f}$ is nonexpanding: For $\oplus_n((\eta_X x_i)_{i=1,\ldots,2^n}), \oplus_n((\eta_X x'_j)_{j=1,\ldots,2^n}) \in \mathcal{D}(X)$, we compute

$$\begin{aligned}
& d_Y(\bar{f}(\oplus_n((\eta_X x_i)_i)), \bar{f}(\oplus_n((\eta_X x'_j)_j))) \\
& = d_Y(\oplus_n((f x_i)_i), \oplus_n((f x'_j)_j)) && (\text{def. } \bar{f}) \\
& \leq 2^{-n} \min_{\pi \in S_{2^n}} \{\sum_{i=1}^{2^n} d_Y(f x_i, f x'_{\pi(i)})\} && (\text{Lemma } A.2) \\
& = d_{\mathscr{K}(Y)}(\sum_{i=1}^{2^n} 2^{-n} \eta_X f x_i, \sum_{j=1}^{2^n} 2^{-n} \eta_X f x'_j) && (\text{Lemma } A.1) \\
& = d_{\mathscr{K}(Y)}(\mathscr{K}(f)(\sum_{i=1}^{2^n} 2^{-n} \eta_X x_i), \mathscr{K}(f)(\sum_{j=1}^{2^n} 2^{-n} \eta_X x'_j)) && (\text{def. } \mathscr{K}(f)) \\
& \leq d_{\mathscr{K}(X)}(\sum_{i=1}^{2^n} 2^{-n} \eta_X x_i, \sum_{j=1}^{2^n} 2^{-n} \eta_X x'_j) && (\mathscr{K}(f) \text{ nonexpanding}) \\
& = d_{\mathscr{K}(X)}(\oplus_n((\eta_X x_i)_i), \oplus_n((\eta_X x'_j)_j)) && (\text{def. } \oplus_n)
\end{aligned}$$

3. $\bar{f}$ preserves $\oplus$: Let $\oplus_n((\eta_X x_i)_{i=1,\ldots,2^n}), \oplus_n((\eta_X x'_j)_{j=1,\ldots,2^n}) \in \mathcal{D}(X)$. Then using ; for concatenation of tuples we get

$$\begin{aligned}
& \bar{f}((\oplus_n((\eta_X x_i)_i)) \oplus (\oplus_n((\eta_X x'_j)_j))) \\
& = \bar{f}(\oplus_{n+1}((\eta_X x_i)_i; (\eta_X x'_j)_j)) && (\text{def. } \oplus_{n+1}) \\
& = \oplus_{n+1}((f x_i)_i; (f x'_j)_j) && (\text{def. } \bar{f}) \\
& = (\oplus_n((f x_i)_i)) \oplus (\oplus_n((f x'_j)_j)) && (\text{def. } \oplus_{n+1}) \\
& = \bar{f}(\oplus_n((\eta_X x_i)_i)) \oplus \bar{f}(\oplus_n((\eta_X x'_j)_j)) && (\text{def. } \bar{f})
\end{aligned}$$

□