Fixed Points of Functors

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Abstract

This is a survey on fixed points of endofunctors, including initial algebras and terminal coalgebras. We also consider the rational fixed point, a canonical domain of behavior for finitely presentable systems. In addition to the basic existence theorems for fixed points, several new results are presented. For example, the Smyth-Plotkin theorem that locally continuous endofunctors of DCPO have terminal coalgebras is derived from a new result stating that every locally monotone endofunctor with a fixed point has a terminal coalgebra. We introduce bounded endofunctors on abstract categories and prove that they have terminal coalgebras. We study well-founded coalgebras and prove that for set functors, the largest well-founded coalgebra of every fixed point is the initial algebra. Another new result concerns mixed fixed points: initial algebras and terminal coalgebras of a parametrized accessible functor always form accessible functors.

Keywords: fixed points of functors, initial algebra, terminal coalgebra, rational fixed point

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1. Introduction

Fixed points of endofunctors are applied in diverse fields of computer science. As *initial algebras*, they are important for modelling inductive data types and reasoning by structural recursion and induction on a general and conceptually clean level of abstraction. Probably the first application was Lawvere's definition of a natural numbers object as an initial algebra for the functor $X \mapsto 1 + X$. As *terminal coalgebras*, they model coinductive data types and provide abstract structural corecursion and coinduction; their importance for the coalgebraic theory of discrete systems was explained in Rutten's fundamental paper [58]. And the *rational fixed point* [16, 48], being the terminal finitely presentable coalgebra, plays, on the one hand, the role of a canonical domain of behaviour for finitely presented systems, and, on the other hand, it is the initial iterative algebra and therefore important in the theory of iterative monads.

But, first of all, what is a fixed point of an endofunctor F? It is neither simply an object X with $FX = X$ (this is too much to ask for!), nor an object X isomorphic to FX (this is too little, since from among the many isorphisms that might exist between X and FX only a specific one is useful in applications): a fixed point is an object X together with an isomorphism with FX. And as such, it can be viewed both as an algebra and as a coalgebra for F. Typically, applications of a given fixed point concern both of these views. We mentioned this for the rational fixed point already, while the initial algebra turns out to be the terminal well-founded coalgebra, and the terminal coalgebra is the initial completely iterative algebra.

Our paper is a survey of the main results on fixed points of functors. But new points of view have been taken, and some new results are proved. Let us briefly explain the contents of our paper.

After introducing some basic concepts in Section 2, we study the initial-algebra chain and the terminal-coalgebra (co)chain in Section 3. We recall the notion of a constructive class $\mathscr{M}$ of morphisms from [69] and the fact that if an endofunctor preserves $\mathscr{M}$-subobjects, then it has a initial algebra iff it has a fixed point. This is applied in Section 4, devoted to categories enriched over DCPO, to the class $\mathscr{M}$ of embeddings: we prove that every locally monotone endofunctor with a fixed point has a *canonical fixed point*; i.e., one that is an initial algebra and a terminal coalgebra at the same time. As a consequence we get the classical result of Smyth and Plotkin [64] that every locally continuous endofunctor has a canonical fixed point. Analogous results concerning categories enriched over complete metric spaces are briefly mentioned in Section 5.

The classical result of Kawahara and Mori [41] that bounded set functors have terminal coalgebras is generalized in Section 6: we introduce bounded endofunctors on abstract categories, prove that they have terminal coalgebras, and deduce that also all accessible functors do. (Recall from [19] that set functors are bounded iff they are accessible.)

Parametric terminal coalgebras are studied in Section 7, and we prove that for λ-accessible parametric functors the corresponding functors obtained by taking either initial algebras or terminal coalgebras are also λ-accessible, whenever λ is an uncountable regular cardinal. We also demonstrate that this result does not hold for finitary functors (i.e. $\aleph_0$-accessible ones).

Well-founded coalgebras are studied in Section 8. Besides recalling that initial algebras of set functors are characterized as well-founded fixed points [12], we prove that for every set functor F each algebra A has a largest well-founded coalgebra A_0. Moreover, if A is a fixed point of F, then A_0 is the initial algebra.

The last two sections are pure surveys: Section 10 is devoted to *completely iterative algebras* and presents results from [47, 15], and Section 11 treats *iterative algebras*, which were introduced in [16] and studied further in [48].

Further results. The authors are working on a monograph entitled 'Initial Algebras and Terminal Coalgebras'. So the theme of the monograph is the same as that of this paper. The present paper is a choice of some of the topics that emerged from this work. The monograph expands on almost every topic in this paper. In addition, there is a large number of important topics that we could not even touch in the present article and which are going to be treated in the monograph. We hope to finish it by the end of 2018.

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2. Initial and Terminal Fixed Points

Throughout our paper $\mathscr{C}$ denotes a category and F an endofunctor on it. As mentioned in the Introduction, a fixed point of an endofunctor F consists of an object X and an isomorphism between FX and X. Hence, it can be viewed as an algebra or a coalgebra for F. Fundamental examples are

$$F(\mu F) \xrightarrow{\iota} \mu F$$

the initial algebra for F, whose algebra structure ι is an isomorphism by Lambek's Lemma [43], and dually

$$\nu F \xrightarrow{\tau} F(\nu F)$$

the terminal coalgebra for F. Of course, both μF and νF are unique up to isomorphism, if they exist.

Example 2.1. One of the first applications of initial algebra was the concept of a natural number object, as introduced by Lawvere [45]. This is precisely the initial algebra for $FX = 1 + X$. That is, given a category with a terminal object 1 and with finite coproducts, we put

$$\mathbb{N} = \mu X.1 + X.$$

If this initial algebra exists, its structure provides a global element $z : 1 \to \mathbb{N}$ and an endomorphism $s : \mathbb{N} \to \mathbb{N}$, which are the components of $\iota = [z, s] : 1+\mathbb{N} \to \mathbb{N}$. In Set this initial algebra is carried by the set of natural numbers.

The terminal coalgebra of the same set functor is

$$\mathbb{N}^{\top} = \nu X.1 + X.$$

It is carried by the set $\mathbb{N} \cup \{\top\}$ where the coalgebra structure $\tau : \mathbb{N}^{\top} \to 1+\mathbb{N}^{\top}$ acts as follows: $\tau(0) = *$ is the unique element of 1, $\tau(n) = n - 1$ for all $n > 0$ and $\tau(\top) = \top$. This is a consequence of Theorem 2.5.

Example 2.2. Given sets Σ_0 and Σ_1, the functor $FX = \Sigma_0 + \Sigma_1 \times X$ has the initial algebra carried by $\mu F = \Sigma_1^* \times \Sigma_0$ and the terminal coalgebra carried by $\nu F = \Sigma_1^* \times \Sigma_0 + \Sigma_1^\omega$, where Σ_1^ω is the set of *streams* over Σ_1 (i.e. infinite sequences of elements of Σ_1) both with obvious structures. This also follows from Theorem 2.5 below.

Example 2.3. (1) The power-set functor $\mathscr{P} : \mathsf{Set} \to \mathsf{Set}$ does not have an initial algebra: Cantor's Theorem tells us that for all sets A, there is no map of A onto $\mathscr{P}A$.

(2) Let $\mathscr{P}_{\mathsf{f}}$, *the finite power-set functor*, be the subfunctor of $\mathscr{P}$ given by

$$\mathscr{P}_{\mathsf{f}} A = \{M \subseteq A : M \text{ is finite}\}.$$

The initial algebra $\mu\mathscr{P}_{\mathsf{f}}$ can be described as

$$V_\omega = \text{all hereditarily finite sets},$$

see Example 3.10 below. In concrete terms,

$$V_\omega \;=\; \emptyset \cup \mathscr{P}(\emptyset) \cup \mathscr{P}\mathscr{P}\emptyset \cup \cdots \cup \mathscr{P}^n\emptyset \cup \cdots .$$

This means that elements of V_ω are finite sets each element of which is a hereditarily finite set. Indeed, it is easy to see that $\mathscr{P}_{\mathsf{f}} V_\omega = V_\omega$. And V_ω with the identity map is the initial algebra for $\mathscr{P}_{\mathsf{f}}$.

The terminal coalgebra for $\mathscr{P}_{\mathsf{f}}$ was described by Worrell [74] as the coalgebra of all strongly extensional finitely branching trees. The coalgebra structure is the inverse of tree-tupling.

Among endofunctors on Set, the polynomial functors H_Σ play an important role. Let Σ be a finitary signature, i.e. $\Sigma = (\Sigma_n)_{n<\omega}$ is a sequence of sets where Σ_n is a set of operation symbols of *arity* n. The *polynomial functor* $H_\Sigma : \mathsf{Set} \to \mathsf{Set}$ given by

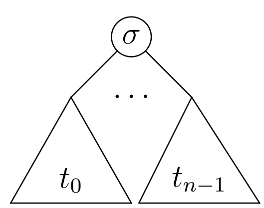
$$H_\Sigma X \;=\; \coprod_{n\in\mathbb{N}} \Sigma_n \times X^n. \tag{2.1}$$

This functor assigns to a function $f : X \to Y$ the function

$$H_\Sigma f \;=\; \coprod_{n\in\mathbb{N}} \mathsf{id}_{\Sigma_n} \times f \times \ldots \times f \;.$$

We describe the initial algebra and terminal coalgebra for H_Σ using Σ-*trees*, i.e. rooted and ordered trees whose nodes with n children are labelled in Σ_n. Σ-trees are always considered up to isomorphism (preserving the tree structure including the labelling). *Tree tupling* is the function that assigns to a tuple $(\sigma, t_0, \ldots, t_{n-1})$ that consists of an n-ary operation symbol $\sigma \in \Sigma_n$ and n Σ-trees $t_0, \ldots, t_{n-1}$ the Σ-tree obtained by joining the n given Σ-trees with a new root node labelled by σ:

$(\sigma, t_0, \ldots, t_{n-1}) \longmapsto$



Remark 2.4. Since we consider Σ-trees up to isomorphism, we can represent them as partial functions $t : \mathbb{N}^* \to \Sigma$ as usual: $t(\varepsilon)$ is the label of the root, $t(i)$ is the label of the ith child (if it exists, otherwise undefined), $t(ij)$ the label of the jth grandchild through i (if it exists), etc. The tree tupling of $(\sigma, t_0, \dots, t_{n-1})$ is then the partial function t with $t(\varepsilon) = \sigma$ and $t(iw) = t_i(w)$ for all $i < n$ (otherwise undefined).

For example, for a signature with $\Sigma_n \neq \emptyset$ the complete n-ary trees are the partial functions $t : \mathbb{N}^* \to \Sigma$ with the domain of definition $\{0, 1, \dots, n-1\}^*$.

Theorem 2.5. *For every finitary signature Σ we have*

$$\mu H_\Sigma = \textit{all finite } \Sigma\textit{-trees}$$

with the algebra structure $H_\Sigma(\mu H_\Sigma) \to \mu H_\Sigma$ given by tree tupling, and

$$\nu H_\Sigma = \textit{all (finite and infinite) } \Sigma\textit{-trees}$$

with the coalgebra structure $\nu H_\Sigma \to H_\Sigma(\nu H_\Sigma)$ given by the inverse of tree tupling.

Proof. (1) Algebras for H_Σ are the classical Σ-algebras of General Algebra. It is well known that the initial Σ-algebra is the algebra of all closed Σ-terms (i.e. Σ-terms without variables). There is an obvious isomorphism h between the sets of closed Σ-terms and finite Σ-trees defined inductively as follows: for a term $t \in \Sigma_0$, $h(t)$ is the root-only tree with label t; for $t = \sigma(t_0, \dots, t_{n-1})$ with $\sigma \in \Sigma_n$, $h(t)$ is the tree tupling applied to $(\sigma, h(t_0), \dots h(t_{n-1}))$.

(2) A coalgebra $a : A \to H_\Sigma A$ can be viewed as a set A together with two functions: one, denoted by $\mathsf{head} : A \to \coprod_n \Sigma_n$, indicates which of the operation symbols is assigned to the state $x \in A$. The other function, body, is defined on a subset of $A \times N$. For each $x \in A$, $body(x, i)$ is defined for $i \leq \mathsf{ar}(head(x))$. Put $\mathsf{body}(x, i) = y_i$ where $a(x) = (\mathsf{head}(x), y_0, \dots, y_{n-1})$. We now prove that a unique coalgebra morphism from A to T_Σ exists.

(2a) A coalgebra morphism $h : A \to T_\Sigma$ exists. We define partial functions $h(x) = t_x$ from $\mathbb{N}^*$ to A by structural recursion on words in $\mathbb{N}^*$ (for all $x \in A$ at once) as follows:

$$t_x(\varepsilon) = \mathsf{head}(x) \quad \text{and} \quad t_x(iw) = t_y(w) \text{ for } y = \mathsf{body}(x, i). \tag{2.2}$$

In the case where $\mathsf{body}(x, i)$ is undefined, so is $t_x(iw)$. It is easy to verify that all t_x are well-defined Σ-trees. To prove that h is a coalgebra morphism recall that τ^{-1} is the tree tupling. We verify $h = \tau^{-1} \cdot H_\Sigma h \cdot a$: given $x \in A$ with $a(x) = (\sigma, y_0, \ldots, y_{n-1})$, then the right-hand side is the tree tupling of $(\sigma, t_{y_0}, \ldots, t_{y_{n-1}})$ – and this is precisely $t_x = h(x)$.

(2b) Conversely, if a coalgebra homomorphism $h : A \to T_\Sigma$ is given, we prove for the trees t_x in (2.2) that $h(x) = t_x$ for every $x \in A$. Indeed, from $h = \tau^{-1} \cdot H_\Sigma h \cdot a$ we deduce that $h(x)$ is the tree-tupling of $H_\Sigma(a(x)) = (\sigma, h(y_0), \ldots, h(y_{n-1}))$ whenever $a(x) = (\sigma, y_0, \ldots, y_{n-1})$. Then (2.2) implies that this tree-tupling is precisely t_x. $\square$

Example 2.6. A signature Σ with arities 0 or 1 yields the polynomial functor $H_\Sigma X = \Sigma_0 + \Sigma_1 \times X$, cf. Example 2.2. In this case, finite Σ-trees are finite paths whose nodes are labelled in Σ_1 except for the last node which is labelled in Σ_0; an isomorphism with $\Sigma_1^* \times \Sigma_0$ is thus evident. Infinite Σ-trees are infinite paths with nodes labelled in Σ_1; here an isomorphism with Σ_1^ω is evident. Using Theorem 2.5, we obtain the formulas of Example 2.2. The special case with $\Sigma_0 = 1 = \Sigma_1$ yields Example 2.1.

Example 2.7. Let Σ be an input alphabet (i.e. a non-empty finite set). A *deterministic automaton* on a set S of states is given by a next-state function $\delta : S \times \Sigma \to S$ and a set $A \subseteq S$ of *accepting states*. We can present the next-state function in curried form as $\delta : S \to S^\Sigma$, and the set A via its characteristic function $\chi : S \to \{0, 1\}$. Thus, deterministic automata are precisely the coalgebras for

$$FX = \{0, 1\} \times X^\Sigma.$$

The terminal coalgebra for F is carried by the set of all formal languages over Σ:

$$\nu X.\{0, 1\} \times X^\Sigma = \mathscr{P}(\Sigma^*).$$

To obtain this as consequence of Theorem 2.5, let $n = |\Sigma|$ and consider the signature Γ of two n-ary operation symbols. The above endofunctor F is then naturally isomorphic to the polynomial functor H_Γ. A Γ-tree is precisely a complete n-ary tree (i.e., every node has precisely n children) with nodes labelled by 0 or 1. We can address the nodes of the complete n-ary tree by the words in Σ^*, see Remark 2.4. Labelling the nodes of this tree is thus equivalent to choosing a subset of Σ^* (of those nodes with label 1). Therefore, we have an isomorphism between the set of Γ-trees and the set $\mathscr{P}(\Sigma^*)$ of formal languages over Σ. Since $F \cong H_\Gamma$ and $\nu H_\Gamma \cong \mathscr{P}(\Sigma^*)$ we see that $\mathscr{P}(\Sigma^*)$ is a final F-coalgebra. Working out the meaning of tree tupling along the above isomorphism, we see that the structure of the terminal coalgebra $\tau = \langle o, t \rangle : \mathscr{P}(\Sigma^*) \to \{0,1\} \times \mathscr{P}\Sigma^*$ acts as follows: the accepting states are exactly those $L \in \mathscr{P}(\Sigma^*)$ containing ε, and the next-state functions takes (L, s) to $s^{-1}L = \{w \mid sw \in L\}$ the left language derivative of L by s.

Finally, note that the initial F-algebra is not interesting as it is empty.

We can define polynomial functors H_Σ also for infinitary signatures Σ: here Σ is a set of operation symbols so that to each $\sigma \in \Sigma$ a cardinal number $\mathsf{ar}(\sigma)$, its arity, is associated. Then we put

$$H_\Sigma = \coprod_{\sigma \in \Sigma} X^{\mathsf{ar}(\sigma)}. \tag{2.3}$$

The concept of a Σ-tree remains the same: a node with label $\sigma \in \Sigma$ has precisely $\mathsf{ar}(\sigma)$ children. To describe μH_Σ we need the following

Definition 2.8. A Σ-tree is *well-founded* if it has no infinite path.

For a finitary signature Σ, every well-founded Σ-tree is finite, due to König's Lemma.

Theorem 2.9. *For an every signature Σ we have*

$$\mu H_\Sigma = \textit{all well-founded } \Sigma\textit{-trees}, \qquad \textit{and} \qquad \nu H_\Sigma = \textit{all } \Sigma\textit{-trees.}$$

For νH_Σ the proof is precisely the same as in Theorem 2.5. For μH_Σ see Example 3.9 below.

An endofunctor that has an initial algebra necessarily has a fixed point. The converse is true for set functors:

Theorem 2.10 (Trnková et al. [69], Theorem II.4)**.** *A set functor has an initial algebra iff it has a fixed point.*

We prove this in Remark 3.15 below. However, a functor that has a fixed point need not have an initial algebra in general, as the next example, with base category Set^2, demonstrates.

Example 2.11. On the category of 2-sorted sets define the endofunctor F by

$$F(X,Y) = \begin{cases} (\emptyset, \mathscr{P}Y) & \text{if } X = \emptyset \\ (1,1) & \text{else} \end{cases}$$

and $F(\mathsf{id}_\emptyset, f) = (id_\emptyset, \mathscr{P}f)$ for all maps f. Although $(1,1)$ clearly is a fixed point of F, this functor does not have an initial algebra.

A characterization similar to Theorem 2.10 is not known for terminal coalgebras, and it is unlikely that one is possible. In fact, there exist two set functors that coincide on all sets (but not all maps) such that one has a terminal coalgebra and one does not, see [9].

However, the existence of two "neighbor" fixed points, i.e. whose cardinality is a power of two apart, suffices.

Definition 2.12. Given a set functor F, a *fixed point pair* is a pair of fixed points X and Y of F such that $|Y| = 2^{|X|}$.

Theorem 2.13 (Adámek and Koubek [8], Theorem 5)**.** *Assuming the Generalized Continuum Hypothesis, every set functor with a fixed-point pair has a terminal coalgebra.*

3. Initial-Algebra and Terminal-Coalgebra Chains

Notation 3.1. Throughout this section we denote by Ord the ordered class of all ordinal numbers considered as a category. We also consider each ordinal number as a category (consisting of all smaller ordinals).

Recall that a *chain* in a category $\mathscr{C}$ is any diagram $n \to \mathscr{C}$ where $n = \mathsf{Ord}$ or n is an ordinal number. We say that $\mathscr{C}$ has colimits of chains if it has colimits of chains $n \to \mathscr{C}$ for all $n \in \mathsf{Ord}$. In particular, $\mathscr{C}$ has an initial object, viz. the colimit of the empty chain $0 \to \mathscr{C}$. The dual notion should be called a *cochain*, but we abuse terminology and speak about chains $\mathsf{Ord}^{\mathsf{op}} \to \mathscr{C}$.

Definition 3.2 (Adámek [5], 2nd Proposition)**.** Let $\mathscr{C}$ be a category with colimits of chains. For every endofunctor F we define the *initial-algebra chain* $W : \mathsf{Ord} \to \mathscr{C}$. Its objects are denoted by W_i or $F^i 0$, its connecting morphisms by $w_{i,j} : W_i \to W_j$, $i \le j \in \mathsf{Ord}$:

$$\begin{aligned} W_0 &= 0, && \\ W_{j+1} &= FW_j && \text{for all ordinals } i, \\ W_j &= \operatorname*{colim}_{i<j} W_i && \text{for all limit ordinals } j, \end{aligned}$$

and

$$\begin{aligned} &w_{0,1}\colon 0 \to W_1 \text{ is unique,} \\ &w_{j+1,k+1} = Fw_{j,k}\colon FW_j \to FW_k, \\ &w_{i,j}\ (i < j) \text{ is the colimit cocone for limit ordinals } j. \end{aligned}$$

Remark 3.3. The above rules define an essentially unique chain $W : \mathsf{Ord} \to \mathscr{C}$. In fact, any two chains satisfying these rules are naturally isomorphic. We therefore speak of *the* initial-algebra chain of F.

Indeed, the equations for the connecting morphisms in Definition 3.2 imply all of the values not explicitly shown. For example, $w_{\omega,\omega+1} \colon W_\omega \to W_{\omega+1} = FW_\omega$ need not be specified: since W is a functor and thus preserves composition, we have $w_{\omega,\omega+1} \cdot w_{n+1,\omega} = w_{n+1,\omega+1} = Fw_{n,\omega}$ for all $n < \omega$, and this determines $w_{\omega,\omega+1}$ uniquely.

Definition 3.4. We say that the initial-algebra chain of a functor F *converges in λ steps* if $w_{\lambda,\lambda+1}$ is an isomorphism. We say that the chain converges in *exactly λ steps* if λ is the least such ordinal.

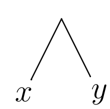
Theorem 3.5 (Adámek [5])**.** *Let $\mathscr{C}$ be a category with colimits of chains. If the initial-algebra chain W of a functor F converges in λ steps, then $F^\lambda 0$ is the initial algebra with the algebra structure $w^{-1}_{\lambda,\lambda+1} \colon FW_\lambda \to W_\lambda$.*

Proof. Given an algebra $a : FA \to A$, there exists a unique cocone $a_i : W_i \to A$ ($i \in \mathsf{Ord}$) of the initial-algebra chain such that $a_{i+1} = a \cdot Fa_i$ for all ordinals i; this is easy to show by transfinite induction. It follows that

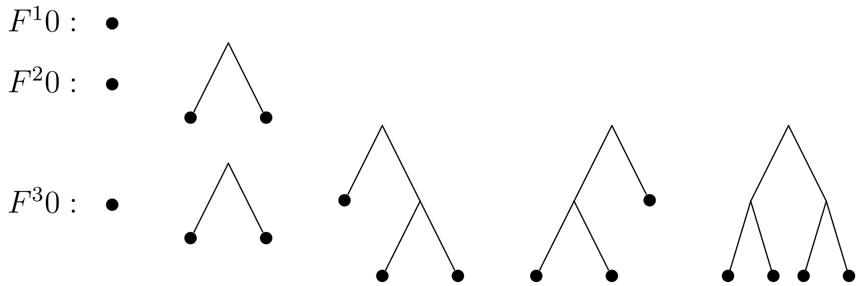
$$a_\lambda : (W_\lambda, w^{-1}_{\lambda,\lambda+1}) \to (A, a)$$

is an F-algebra morphism. To see uniqueness, let $h : (W_\lambda, w^{-1}_{\lambda,\lambda+1}) \to (A, a)$ be an F-algebra morphism. Then by an easy transfinite induction we see that $h \cdot w_{i,\lambda} = a_i$ for all ordinals $i \leq \lambda$. Hence, the case $i = \lambda$ yields $h = a_\lambda$ as desired. $\square$

Example 3.6. Let us consider the set functor $FX = 1 + X \times X$. When dealing with this functor and related ones, it is often useful to adopt a graphical notation. We shall draw $(x, y) \in X \times X \hookrightarrow FX$ as the ordered tree



and the unique element of 1 as $\bullet$. We start with $F^0 0 = \emptyset$. Now we picture the elements of $F^i 0$ for $i = 1$, 2, and 3:



In general, $F^n 0$ is the set of all binary trees of depth less than n and the connecting maps $F^n 0 \to F^{n+1} 0$ are the inclusions.

Then the carrier of the initial algebra μF may be taken to be the union $\bigcup_{n<\omega} F^n 0$, which are the finite binary trees as in Theorem 2.5.

Corollary 3.7. *If $\mathscr{C}$ has colimits of chains and F preserves colimits of λ-chains for some infinite ordinal λ, then F has an initial algebra $\mu F = W_\lambda$.*

Example 3.8 (Kleene's Fixed Point Theorem)**.** Let $\mathscr{C}$ be a poset with joins of chains. Then every monotone endofunction has a least fixed point. In fact, if $\lambda > |\mathscr{C}|$, then f preserves colimits of λ-chains: for cardinality reasons the join of a λ-chain must already be a member of the chain. Thus $\mu f = f^\lambda(\perp)$.

Example 3.9. (1) For finitary signatures Σ, the functor H_Σ preserves colimits of ω-chains. We then get a short proof of the description of μH_Σ in Theorem 2.5. In fact, we can identify $H^1_\Sigma 0 = H_\Sigma \emptyset = \Sigma_0$ with the set of all Σ-trees of depth 0, i.e., one-node trees labelled by an element of Σ_0. Moreover, we identify

$$H^2_\Sigma 0 = H_\Sigma \Sigma_0 = \Sigma_0 + \coprod_{k>0} \Sigma_k \times \Sigma_0^k$$

with the set of all Σ-trees of depth at most 1. More generally, $H^n_\Sigma 0$ can be identified with the set of Σ-trees of depth less than n. We obtain the chain of inclusion maps

$$H^0_\Sigma 0 \hookrightarrow H^1_\Sigma 0 \hookrightarrow H^2_\Sigma 0 \hookrightarrow \cdots$$

whose colimit (that is union in Set) is the set of all finite Σ-trees.

(2) For infinitary signatures Σ, the situation is completely analogous. Recall that the well-founded Σ-trees are Σ-trees without infinite paths. And for a well-founded Σ-tree t the *depth* is defined to be the following cardinal:

$$\mathsf{depth}(t) = 1 + \sup_{j \in J} \mathsf{depth}(t_j),$$

where $\{t_j \mid j \in J\}$ is the set of all maximal proper subtrees of t. The formula in item (1) remains valid for all ordinals i:

$$H_\Sigma^i \emptyset = \text{all } \Sigma\text{-trees of depth less than } i.$$

Now, it is easy to see that, for a regular cardinal λ, products of families of sets of cardinality less than λ commute with colimits of λ-chains. Therefore for $n < \lambda$, the Set-functors $X \mapsto X^n$ (taking the n-th power of a set X) preserve colimits of λ-chains. Consequently, if Σ is a λ-ary signature, i.e. all arities are smaller than λ, then the functor H_Σ preserves colimits of λ-chains. Hence, the initial-algebra chain converges in λ steps:

$$\mu H_\Sigma = H_\Sigma^\lambda \emptyset.$$

Moreover, since Σ is λ-ary, no tree has depth λ or more. Therefore, we obtain that

$$\mu H_\Sigma = \text{all well-founded } \Sigma\text{-trees}.$$

Example 3.10. (1) The power-set functor has no fixed point, thus, no initial algebra. Nonetheless, it has an interesting initial-algebra chain. It is given by the sets

$$W_0 = \emptyset, \qquad W_{i+1} = \mathscr{P} W_i, \qquad \text{and} \qquad W_i = \bigcup_{j<i} W_j \quad \text{for limit ordinals } i$$

and the inclusion functions as connecting maps $W_i \hookrightarrow W_j$ for $i \leq j$. In set theory these sets are usually denoted by V_i. They constitute a proper hierarchy: for all ordinals i, we have $i \in W_{i+1} \setminus W_i$. Due to the Foundation Axiom, every set belongs to W_{i+1} for some i. The least such i is called the *rank* of the set.

Of special interest is the set of all *hereditarily finite sets* of Example 2.3(2).

(2) Analogously, denote by

$$\mathscr{P}_{\mathsf{c}}$$

the subfunctor of $\mathscr{P}$ of all countable subsets. Since $\mathscr{P}_{\mathsf{c}}$ preserves colimits of ω_1-chains, where ω_1 is the first uncountable ordinal, the initial algebra for $\mathscr{P}_{\mathsf{c}}$ is the algebra W_{ω_1} of *hereditarily countable* sets.

We already discussed in the previous section that, by Lambek's Lemma, the existence of a fixed point is necessary for the existence of an initial algebra, but that the converse need not hold in general (see Example 2.11).

However, for endofunctors on "reasonable" categories (such as sets, posets, graphs and presheaves) preserving monomorphisms, the converse does hold.

Definition 3.11 (Trnková [68], Definition II.3)**.** (1) A class $\mathscr{M}$ of monomorphisms in a category $\mathscr{C}$ is called *constructive* provided that it is closed under composition, contains all isomorphisms, and for every chain of monomorphisms in $\mathscr{M}$, (i) a colimit exists and is formed by monomorphisms in $\mathscr{M}$, and (ii) the unique morphism from the colimit induced by any cocone of monomorphisms in $\mathscr{M}$ is again a monomorphism in $\mathscr{M}$.

(2) We say that an endofunctor *preserves* $\mathscr{M}$ if $m \in \mathscr{M}$ implies $Fm \in \mathscr{M}$.

Note that an initial object 0 exists and all morphisms $0 \to X$ lie in $\mathscr{M}$ if $\mathscr{C}$ has a constructive class of monomorphisms.

Example 3.12. (i) In the categories of sets, graphs, posets, and semigroups the class of all monomorphisms is constructive.

(ii) In the category of sets and partial functions, the split monomorphisms form a constructive class. (These are precisely the monomorphisms of Set.)

(iii) In contrast, the collection of all monomorphisms is not constructive in the category of rings. There are non-injective homomorphisms whose domain is the initial object, viz. the ring $\mathbb{Z}$ of integers.

Theorem 3.13 (Trnková et al. [69], Theorem II.4)**.** *Let $\mathscr{C}$ be a category with colimits of chains, and let $\mathscr{M}$ be a constructive class of monomorphisms such that $\mathscr{C}$ is $\mathscr{M}$-wellpowered. Then for every $\mathscr{M}$-preserving endofunctor F on $\mathscr{C}$ the following conditions are equivalent:*

(1) F has an initial algebra,

(2) F has a fixed point,

(3) the initial-algebra chain converges.

Proof. (2) $\Rightarrow$ (3). Let $a : FA \xrightarrow{\cong} A$ be a fixed point of F. $\mathscr{M}$-well-poweredness implies that we have a poset of all subobjects of A represented by morphisms in $\mathscr{M}$, and we know that this poset has joins (= colimits) of chains, and in particular, the least element $\perp$. The following endomap f on $\mathscr{M}$-subobjects of A defined by

$$f(m) = a \cdot Fm \qquad \text{for every } m : B \to A \text{ in } \mathscr{M}$$

is clearly monotone. Thus, by Example 3.7 we conclude that $\mu f = f^{\lambda}(\perp)$ for some ordinal λ.

Now recall the canonical cocone $a_i : W_i \to A$ from Theorem 3.5: it is easy to verify by transfinite induction that

$$a_i = f^i(\perp) \qquad \text{for all } i \in \mathsf{Ord}.$$

Consequently, the initial chain converges in λ steps.

For the implications (3) $\Rightarrow$ (1) $\Rightarrow$ (2) use Theorem 3.5 and Lambek's Lemma. □

Theorem 3.14 (Trnková [68], Propositions III.5 and II.4)**.** *For every set functor F there exists an essentially unique set functor $\bar{F}$ which coincides with F on nonempty sets and functions and preserves finite intersections (whence monomorphisms).*

Remark 3.15. (1) We call $\bar{F}$ the *Trnková hull* of F. From the proof in *op. cit.* it follows that

(1) if $F\emptyset \neq \emptyset$ then $\bar{F}\emptyset \neq \emptyset$,

(2) given the empty map $p_A : \emptyset \to A$ with $A \neq \emptyset$ we have $Fp_A[F\emptyset] \subseteq \bar{F}p_A[\bar{F}\emptyset]$; and

(3) if F is finitary (i.e. it preserves filtered colimits), so if $\bar{F}$.

(2) Note that one can obtain Theorem 2.10 as a consequence of Theorem 3.5 based on Theorem 3.14.

Corollary 3.16 (Trnková et al. [69]). *A set functor has an initial algebra iff it has a fixed point; the initial-algebra chain then converges.*

Proof. The initial-algebra chains of a set functor F coincides with that of its Trnková hull F' from step ω onwards (see [22, Proposition 5.14]). If F has a fixed point $X \neq \emptyset$, then X is a fixed point of F', thus by Theorem 3.13 an initial algebra for F' exists and this is also an initial algebra for F. If F has the fixed point $\emptyset$, then $\mu F = \emptyset$. □

Example 3.17. How long does one need to construct the initial algebra for a set functor?

(a) For every infinite regular cardinal λ take the signature Σ of one n-ary operation symbol for every cardinal $n < \lambda$. The initial-algebra chain for H_Σ converges in exactly λ steps; this follows from Example 3.9.

(b) The initial-algebras chain takes precisely

- 0 steps for $FX = X$,
- 1 steps for F constant to 1,
- 2 steps for F constant to 1 except for $F\emptyset = \{0, 1\}$,
- 3 steps for F with $F\emptyset = \{0, 1, 2\}$ and $FX = \{0\} \cup \{A \subseteq X \mid |A| = 3\}$ for $X \neq \emptyset$. On morphisms $f : X \to Y$ with $X \neq \emptyset$ put $Ff(A) = 0$ if $|f[A]| < 3$ else $Ff(A) = f[A]$, and if $X = \emptyset$ then $Ff = \mathsf{const}_0$.

These are all possible values of the exact convergence:

Theorem 3.18 (Adámek and Trnková [22], Theorem 6.1). *If the initial chain of a set functor converges in exactly λ steps, then $\lambda \leq 3$ or λ is an infinite regular cardinal.*

Remark 3.19. In contrast, for every ordinal number λ there is an endofunctor of the category of many-sorted sets whose initial-algebra chain converges in exactly λ steps, see [22].

The following definition is nothing else than the dual of the initial-algebra chain of Definition 3.2. It was formulated explicitly by Barr [26].

Definition 3.20. Let $\mathscr{C}$ be a category with limits of chains (in particular, $\mathscr{C}$ has then a terminal object 1). For every endofunctor F the *terminal-coalgebra chain* is the transfinite chain in $\mathscr{C}$ indexed by Ord^{op}, having the following objects $V_j = F^j 1$ and connecting morphism $v_{j,i}$, $j \geq i$:

$$\begin{aligned} V_0 &= 1, \\ V_{j+1} &= FV_j && \text{for all ordinals } j, \\ V_j &= \lim_{i<j} V_i && \text{for all limit ordinals } j, \end{aligned}$$

and

$$\begin{aligned} &v_{1,0} : F1 \to 1 \text{ is unique,} \\ &v_{k+1,j+1} = Fv_{k,j} : FV_k \to FV_j, \\ &v_{j,i}\ (j > i) \text{ is the limit cone for limit ordinals } j. \end{aligned}$$

We say that the terminal-coalgebra chain *converges in λ steps* if $v_{\lambda+1,\lambda}$ is an isomorphism. It converges in *exactly λ steps* if λ is the least such ordinal.

Theorem 3.21 (Dual of Theorem 3.5)**.** *Let $\mathscr{C}$ be a category with limits of chains. If the terminal-coalgebra chain of a functor F converges in λ steps, then $F^\lambda 1$ is the terminal coalgebra with the coalgebra structure $v^{-1}_{\lambda,\lambda+1} : V_\lambda \to FV_\lambda$.*

Corollary 3.22 (Dual of Corollary 3.7)**.** *Let $\mathscr{C}$ be a category with limits of chains. If an endofunctor F preserves limits of λ^{op}-chains for some infinite ordinal λ, then the terminal-coalgebra chain converges in λ steps, hence,*

$$\nu F = F^\lambda 1.$$

Example 3.23. (1) For polynomial functors H_Σ (even the non-finitary ones) of (2.3) we conclude that the terminal coalgebra is constructed by the finitary terminal-coalgebra chain:

$$\nu H_\Sigma = H^\omega_\Sigma 1.$$

Indeed, the functors $X \mapsto X^k$ preserve limits of ω^{op}-chains for all cardinals k, hence, so do all polynomial functors.

(2) For the finite power-set functor $\mathscr{P}_{\mathsf{f}}$, the terminal coalgebra chain $\mathscr{P}^k_{\mathsf{f}} 1$ needs more than ω steps. Worrell proved that it converges in exactly

$\omega+\omega$ steps (see also Theorem 6.13 below) and he described the terminal coalgebra

$$\nu\mathscr{P}_{\mathsf{f}} = \lim_{k<\omega+\omega} \mathscr{P}_{\mathsf{f}}^k 1$$

as the coalgebra of all finitely branching, strongly extensional trees [74]. Whereas the limit at ω, which Abramsky [1] denotes by

$$\mathbb{F} = \lim_{k<\omega} \mathscr{P}_{\mathsf{f}}^k 1$$

has a number of interesting descriptions which that paper presents. Moreover, $\mathbb{F}$ can also be described as the coalgebra of all compactly branching, strongly extensional trees [74].

(3) Let λ be an infinite regular cardinal. For the functor $\mathscr{P}_\lambda$ of all subsets of cardinality less than λ, the terminal-coalgebra chain converges in $\lambda+\omega$ steps as proved by Worrell [74]. Indeed, it converges in exactly $\lambda+\omega$ steps [10]. Worrell described $\nu\mathscr{P}_\omega$ as the coalgebra of all strongly extensional finitely branching trees.

(4) The terminal-coalgebra chain converges in exactly λ steps for the functor $\widetilde{\mathscr{P}_\lambda}$ that coincides with $\mathscr{P}_\lambda$ on objects and is defined on a morphism $f : X \to Y$ by $\widetilde{\mathscr{P}_\lambda} f(M) = f[M]$ if $f|_M$ is monomorphic, and otherwise $\widetilde{\mathscr{P}_\lambda} f(M) = \emptyset$ (see [9]).

Remark 3.24. A full characterization similar to Theorem 3.18 of the index of exact convergence of the terminal-coalgebra chain is not known. In [9], for every pair $\mu < \lambda$ of regular cardinals a set functor is constructed whose terminal-coalgebra chain converges in exactly $\lambda + \mu$ steps.

4. DCPO-Enriched Categories

This section studies an order-enriched setting and endofunctors on it satisfying an appropriate continuity condition.

Notation 4.1. We denote by

$$\mathsf{DCPO}$$

the category of *(directed) complete partial orders* (dcpo's, for short), i.e. posets with joins of directed subsets. The morphisms of this category are maps that are the *continuous* maps, i.e. those ones that preserve directed joins.

A category $\mathscr{C}$ is DCPO-enriched if each hom-set $\mathscr{C}(X,Y)$ carries the structure of a dcpo such that composition is continous (in both variables). And $\mathscr{C}$ is called *strict* DCPO-enriched if, in addition, each $\mathscr{C}(X,Y)$ has a least element $\bot_{XY}$ such that composition is *left-* and *right-strict*: for every $f : X \to X'$ and every object Z:

$$f \cdot \bot_{ZX} = \bot_{ZY} \qquad \text{and} \qquad \bot_{YZ} \cdot f = \bot_{XZ}.$$

Example 4.2. The category $\mathsf{DCPO}_\bot$ of directed complete partial orders with least element and continuous functions that are *strict*, i. e., preserve that least element, is a strict DCPO-enriched category. Also, Pfn and Rel, the categories of sets and partial functions and sets and relations are strict DCPO-enriched w.r.t. the ordering of morphisms by inclusion.

By a classical result of Smyth and Plotkin [64], every locally continuous endofunctor F on a strict DCPO-enriched category has a canonical fixed point $\mu F = \nu F$. In Theorem 4.13 we prove a new result (using the same technique): every locally monotone endofunctor with a fixed point has a canonical one.

Definition 4.3. A *canonical fixed point* of an endofunctor is an initial algebra that coincides with the terminal coalgebra. More precisely, it is an initial algebra $\iota : F(\mu F) \to \mu F$ such that the inverse $\iota^{-1} : \mu F \to F(\mu F)$ is a terminal coalgebra for F.

Before we proceed let us note the following lemma. We include a proof as this is the place where we need left- and right-strictness of composition. Recall that a *zero object* of a category is an initial object that is also a terminal object.

Lemma 4.4 (Barr [25], Proposition 4.7)**.** *Every strict* DCPO*-enriched category with* ω*-colimits has a zero object* $0 = 1$*.*

Proof. Choose any object X and form the ω-chain with the connecting morphisms $\bot_{XX}$. The colimit $c_n : X \to C$ then fulfils $c_n = c_{n+1} \cdot \bot_{XX} = \bot_{XC}$ for all n. Thus C is initial: for every object Y the unique morphism from C is $\bot_{CY}$. Indeed, given $f : C \to Y$ then $f \cdot c_n = f \cdot \bot_{CX} = \bot_{XY} = \bot_{CY} \cdot c_n$ for all n, thus, $f = \bot_{CY}$ since the colimit injections c_n are jointly epimorphic. And C is terminal: the unique morphism into C is $\bot_{YC}$ because clearly $\bot_{CC} = \mathsf{id}_C$, hence, given $f : Y \to C$ we have $f = \mathsf{id}_C \cdot f = \bot_{CC} \cdot f = \bot_{YC}$. $\square$

Definition 4.5 (Scott [61], Definition 3.6)**.** In a DCPO-enriched category, a morphism $e\colon X \to Y$ is called an *embedding* if there exists a morphism $\widehat{e}\colon Y \to X$ with

$$\widehat{e} \cdot e = \mathsf{id}_X \quad \text{and} \quad e \cdot \widehat{e} \sqsubseteq \mathsf{id}_Y. \tag{4.1}$$

Note that $\widehat{e}$ is uniquely determined by (4.1). It is called the *projection* for e. The pair $(e, \widehat{e})$ is called an *embedding-projection pair.*

Remark 4.6. Objects of a DCPO-enriched category $\mathscr{C}$ and embeddings form a category $\mathscr{C}^E$. The point is that a composite $e \cdot f$ of embeddings is itself an embedding with

$$\widehat{e \cdot f} \quad = \quad \widehat{f} \cdot \widehat{e}. \tag{4.2}$$

$\mathscr{C}^E$ is a DCPO-enriched category, since given a directed set $\{e_i \mid i \in I\}$ of embeddings in $\mathscr{C}(X, Y)$, the join $\bigsqcup_i e_i$ is an embedding with projection $\bigsqcup_i \widehat{e}_i$.

The following lemma stems from [64]. We present a full proof because we actually need a more general result (see our discussion after Corollary 4.9) whose proof is, however, completely analogous.

Fundamental Lemma 4.7 (Smyth and Plotkin [64], Theorem 3)**.** *Let $\mathscr{C}$ be a DCPO-enriched category. Consider an ω-chain of embeddings in $\mathscr{C}$*

$$E_0 \xrightarrow{e_0} E_1 \xrightarrow{e_1} E_2 \xrightarrow{e_2} \cdots \tag{4.3}$$

Let $c_i : E_i \to C$ be a cocone. Then the following are equivalent:

(1) (c_i) is a colimit cocone.

(2) Each c_i is an embedding, the composites $c_i \cdot \widehat{c}_i$ form an ω-chain in $\mathscr{C}(C, C)$, and

$$\bigsqcup_{i<\omega} c_i \cdot \widehat{c}_i = \mathsf{id}_C. \tag{4.4}$$

Proof. (1) $\Rightarrow$ (2): For every i we have the following cocone of the shortened chain (all E_j, $j < i$, deleted) with codomain E_i:

$$\begin{tikzcd}
E_i \arrow[r, "e_i"] \arrow[dd, equals] & E_{i+1} \arrow[r, "e_{i+1}"] \arrow[d, equals] \arrow[ddl, "\widehat{e}_i"'] & E_{i+2} \arrow[r, "e_{i+2}"] \arrow[dl, "\widehat{e}_{i+1}"] & \cdots \\
 & E_{i+1} \arrow[dl, "\widehat{e}_i"] & & \\
E_i & & &
\end{tikzcd} \tag{4.5}$$

Since the shortened chain has the same colimit as the original one, namely C, there exists a unique morphism $\widehat{c}_i : C \to E_i$ such that for all $k \geq i$ the triangles on the left below commute:

$$\begin{tikzcd} E_k \arrow[d, "c_k"'] \arrow[dr, "\widehat{e}_i \cdot \widehat{e}_{i+1} \cdots \widehat{e}_{k-1}"] & \\ C \arrow[r, "\widehat{c}_i"'] & E_i \end{tikzcd} \quad \Bigg| \quad \begin{tikzcd} E_i \arrow[d, "c_i"'] \arrow[dr, "e_k \cdot e_{k-1} \cdots \cdot e_i"] & \\ C \arrow[r, "\widehat{c}_k"'] & E_k \end{tikzcd} \tag{4.6}$$

In particular

$$\widehat{c}_i \cdot c_i \quad = \quad \mathsf{id}_{E_i}. \tag{4.7}$$

We verify the triangles on the right in (4.6) commute by induction on $n = k - i$. (4.7) is the base case. Assuming the commutativity when the difference is $n-1$, we get it when the difference is n:

$$(e_k \cdot e_{k-1} \cdot \cdots \cdot e_{i+1}) \cdot e_i \quad = \quad \widehat{c}_k \cdot c_{i+1} \cdot e_i \quad = \quad \widehat{c}_k \cdot c_i.$$

Comparing the triangles in (4.6) for $\widehat{c}_i$ and $\widehat{c}_{i+1}$, we see that given $k \geq i+1$ we have

$$\widehat{c}_i \cdot c_k \quad = \quad \widehat{e}_i \cdot \widehat{c}_{i+1} \cdot c_k$$

Now $(c_k)_{k \geq i+1}$ is a colimit cocone and therefore a jointly epic family. Thus, we conclude

$$\widehat{c}_i \quad = \quad \widehat{e}_i \cdot \widehat{c}_{i+1}. \tag{4.8}$$

Hence, the morphisms $c_i \cdot \widehat{c}_i$ form a chain in $\mathscr{C}(C, C)$:

$$c_i \cdot \widehat{c}_i \quad = \quad (c_{i+1} \cdot e_i) \cdot (\widehat{e}_i \cdot \widehat{c}_{i+1}) \quad \sqsubseteq \quad c_{i+1} \cdot \widehat{c}_{i+1}$$

due to $e_i \cdot \widehat{e}_i \sqsubseteq \mathsf{id}$. We next prove that the join of this chain is id_C, thus also proving that $c_i \cdot \widehat{c}_i \sqsubseteq \mathsf{id}$; this inequation together with (4.7) establishes that c_i is an embedding. To see that id_C is the desired join, it is sufficient to prove that for every k

$$\Big(\bigsqcup_i c_i \cdot \widehat{c}_i\Big) \cdot c_k \quad = \quad c_k.$$

And for this, we need only show that for $i \geq k$, $c_i \cdot \widehat{c}_i \cdot c_k = c_k$. Using the triangle on the right in (4.6) and interchanging k and i, we see that

$$c_i \cdot \widehat{c}_i \cdot c_k \quad = \quad c_i \cdot e_i \cdot e_{i-1} \cdot \cdots \cdot e_k \quad = \quad c_k.$$

We have proved (4.4). This verifies all parts of (2) in our lemma.

(2) $\Rightarrow$ (1): Suppose we are given a cocone $b_i : E_i \to B$ of the given chain:

$$\begin{tikzcd}
 & C \arrow[dl, "\widehat{c}_0"'] \arrow[d, "\widehat{c}_1"'] \arrow[dr, "\widehat{c}_2"] & & \\
E_0 \arrow[r, "e_0"] \arrow[dr, "b_0"'] & E_1 \arrow[r, "e_1"] \arrow[d, "b_1"'] & E_2 \arrow[r] \arrow[dl, "b_2"] & \cdots \\
 & B & &
\end{tikzcd}$$

We first observe that the morphisms $b_i \cdot \widehat{c}_i$ form a chain in $\mathscr{C}(C, B)$. Since the c_i's are embeddings and $c_i = c_{i+1} \cdot e_i$, from 4.2 we get (4.8). Thus

$$b_i \cdot \widehat{c}_i \;=\; (b_{i+1} \cdot e_i) \cdot (\widehat{e}_i \cdot \widehat{c}_{i+1}) \;=\; b_{i+1} \cdot (e_i \cdot \widehat{e}_i) \cdot \widehat{c}_{i+1} \;\sqsubseteq\; b_{i+1} \cdot \widehat{c}_{i+1}.$$

Define

$$b \;=\; \bigsqcup_i (b_i \cdot \widehat{c}_i) : C \to B. \tag{4.9}$$

We show that this is the desired factorization: we fix k and show that $b \cdot c_k = b_k$. For this, we may restrict the above join to $i \geq k$. So we shall show that

$$b \cdot c_k \;=\; \bigsqcup_{i \geq k} (b_i \cdot \widehat{c}_i \cdot c_k) \;=\; b_k \tag{4.10}$$

Recall that (c_i) is a cocone. Thus for $i \geq k$, the top triangle below commutes:

$$\begin{tikzcd}
E_k \arrow[r, "e_k"] \arrow[d, "c_k"'] & E_{k+1} \arrow[r, "e_{k+1}"] & \cdots \arrow[r, "e_{i-1}"] & E_i \arrow[llld, "c_i"] \arrow[llldd, "\mathsf{id}"] \arrow[lllddd, "b_i"] \\
C \arrow[d, "\widehat{c}_i"'] & & & \\
E_i \arrow[d, "b_i"'] & & & \\
B & & &
\end{tikzcd}$$

The other triangles clearly commute, and so the outside does, too. And as (b_i) is a cocone, we have

$$b_i \cdot \widehat{c}_i \cdot c_k \;=\; b_i \cdot (e_{i-1} \cdot \cdots \cdot e_k) \;=\; b_k.$$

This for all $i \geq k$ shows (4.10). We have shown that b is a factorization morphism of our original cocone (b_i).

The factorization is unique: Given $b' : C \to B$ with $b' \cdot c_i = b_i$ for all $i < \omega$, we have, due to (4.4)

$$b' \;=\; b' \cdot \bigsqcup_i c_i \cdot \widehat{c_i} \;=\; \bigsqcup_i b_i \cdot \widehat{c_i} \;=\; b,$$

which completes the proof. □

Remark 4.8. The morphism b of (4.9) is an embedding. Indeed, we have $c_i = c_{i+1} \cdot e_i$ and $\widehat{b}_i = \widehat{e}_i \cdot \widehat{b}_{i+1}$. Thus

$$c_i \cdot \widehat{b}_i \;=\; c_{i+1} \cdot (e_i \cdot \widehat{e}_i) \cdot \widehat{b}_{i+1} \;\sqsubseteq\; c_{i+1} \cdot \widehat{b}_{i+1}.$$

We define $\widehat{b}$ to be $\bigsqcup_i c_i \cdot \widehat{b}_i$. This is a projection for b:

$$b \cdot \widehat{b} \;=\; (\textstyle\bigsqcup_i b_i \cdot \widehat{c}_i) \cdot (\bigsqcup_i c_i \cdot \widehat{b}_i) \;=\; \bigsqcup_i b_i \cdot \widehat{c}_i \cdot c_i \cdot \widehat{b}_i \;=\; \bigsqcup_i b_i \cdot \widehat{b}_i \;\sqsubseteq\; \mathsf{id}_B.$$

and also $\widehat{b} \cdot b = \mathsf{id}_C$, due to (4.4):

$$\widehat{b} \cdot b \;=\; \bigsqcup_i c_i \cdot \widehat{b}_i \cdot b_i \cdot \widehat{c}_i \;=\; \bigsqcup_i c_i \cdot \widehat{c}_i \;=\; \mathsf{id}_C.$$

Corollary 4.9. *In every strict* DCPO*-enriched category with colimits of* ω*-chains, the class of embedding is constructive (see Definition 3.11).*

Indeed, both the Fundamental Lemma and Remark 4.8, although formulated for ω-chains, hold for arbitrary chains. Observe that the case of the empty chain follows using Lemma 4.4: the unique $0 \to X$ is an embedding with projection the unique $X \to 1 = 0$.

Definition 4.10. An endofunctor F on a DCPO-enriched category $\mathscr{C}$ is called *locally monotone* (*locally continuous*, resp.) if all hom-mappings

$$\mathscr{C}(X, Y) \xrightarrow{F} \mathscr{C}(HX, HY), \qquad f \mapsto Ff$$

are monotone (continuous, resp.).

Example 4.11. (1) An example of a locally monotone endofunctor on $\mathsf{DCPO}_\perp$ that is not locally continuous is the ideal completion Idl: this functor assigns to every dcpo the poset of all ideals; i.e., down-closed directed subsets (ordered by inclusion), and it works on morphisms by taking down-closures of images of sets.

(2) Here are some examples of locally continuous endofunctors on DCPO-enriched categories:

   (a) Id is locally continous, and so is every constant functor.

   (b) A composite, product or coproduct of locally continous functors is locally continuous.

   (c) The endofunctor $X \mapsto X_\perp$ on DCPO given by adding a new bottom element is locally continous.

Remark 4.12. The following theorem entails that every locally monotone functor with a fixed point has a canonical one. A related result was obtained in the setting of ω-CPO-enriched Kleisli categories for monads on Set by Hasuo, Jacobs and Sokolova [40, Theorem 3.3]. Recall that an ω-cpo is a partially ordered set having joins of all ω-chains. In their setting one only assumes left-strictness of composition, and they consider a locally monotone endofunctor $\bar{F}$ on a Kleisli category that is lifted from an endofunctor F on Set. They show that the initial algebra for F lifts to the Kleisli category yielding an initial algebra for $\bar{F}$ and a canonical fixed point of the lifting. While this setting is not comparable to our present one, it seems that the following result essentially generalizes the result in *loc. cit.*

Theorem 4.13. *For every locally monotone endofunctor F on a strict* DCPO*-enriched category with colimits of ω-chains the following are equivalent:*

(1) F has a fixed point,

(2) F has an initial algebra,

(3) F has a terminal coalgebra,

(4) F has a canonical fixed point.

Proof. Since F is locally monotone, it preserves embeddings: we have $F\widehat{e} \cdot Fe = F(\widehat{e} \cdot e) = F\mathsf{id} = \mathsf{id}$ as well as

$$Fe \cdot F\widehat{e} = F(e \cdot \widehat{e}) \sqsubseteq F(\mathsf{id}) = \mathsf{id}.$$

Thus, using Corollary 4.9, we can apply Theorem 3.13 with $\mathscr{M}$ the class of embeddings, which gives the equivalence of (1) and (2). Equivalently, the initial-algebra chain converges, and using Lemma 4.4 we see this chain coincides with the terminal-coalgebra chain. Hence, we obtain the equivalence with (3) and (4) as desired. ∎

Corollary 4.14 (Smyth and Plotkin [64]). *Every locally continuous endofunctor F on a strict* DCPO*-enriched category has a canonical fixed point.*

Indeed, F has an initial algebra because the first ω steps of the initial-algebra chains consist of embeddings. Therefore, by the Fundamental Lemma 4.7, F preserves the colimit of this chain; to see this use condition (2): from (4.4) we get

$$\bigsqcup_i Fc_i \cdot F\widehat{c_i} = \bigsqcup_i F(c_i \cdot \widehat{c_i}) = F(\bigsqcup_i c_i \cdot \widehat{c_i}) = \mathsf{id}$$

by local continuity.

5. CMS-enriched categories

Results analogous to those for categories enriched over DCPO can be obtained when enriching over the category CMS of complete metric spaces with distances bounded by 1 and non-expansive maps. A category is CMS-*enriched* if each hom-set carries the structure of a complete metric space such that composition is non-expansive (in both variables).

Example 5.1. (a) CMS itself is CMS-enriched with respect to the supremum metric

$$d_{X,Y}(f, g) = \sup_{x \in X} d_Y(f(x), g(x)).$$

(b) $\mathsf{CMS}_\perp$, the category of pointed complete metric spaces and non-expanding maps preserving $\perp$, is also CMS-enriched using the supremum metric.

(c) If $\mathscr{C}$ is CMS-enriched then so is $\mathscr{C}^{op}$, where the metric of $\mathscr{C}^{op}(A, B)$ is that of $\mathscr{C}(B, A)$.

(d) A product $\mathscr{C}' \times \mathscr{C}''$ of CMS-enriched categories (with metrics $d'_{A',B'}$ and $d''_{A'',B''}$ on hom-sets, respectively) is CMS-enriched using the maximum metric: the distance of (f', f'') and (g', g'') is $\max\{d'(f', g'), d''(f'', g'')\}$.

At the root of our study is the classical *Banach Fixed Point Theorem* stating that a contracting endofunction on a non-empty complete metric space has a unique fixed point. Recall that a function $f : X \to Y$ is *contracting* for some $\varepsilon < 1$, if $d(fx, fy) < \varepsilon \cdot d(x, y)$ for all $x, y \in X$.

Definition 5.2 (America and Rutten [24])**.** An endofunctor F of a CMS-enriched category $\mathscr{C}$ is *locally contracting* if there exists $\varepsilon < 1$ such that for all objects X and Y and parallel pairs $f, g : X \to Y$ we have

$$d(Ff, Fg) \leq \varepsilon \cdot d(f, g).$$

Example 5.3. (1) For every $\varepsilon < 1$, the functor which takes a space (X, d) and shrinks the metric by ε, giving $(X, \varepsilon \cdot d)$ is locally contracting.

(2) Every composite, coproduct, and finite product of locally contracting endofunctors is locally contracting.

(3) As a consequence of (1) and (2), any polynomial functor H_Σ on Set has a *contracting lifting* H'_Σ to CMS. This means that the following square

$$\begin{tikzcd} \mathsf{CMS} \arrow[r, "H'_\Sigma"] \arrow[d, "U"'] & \mathsf{CMS} \arrow[d, "U"] \\ \mathsf{Set} \arrow[r, "H_\Sigma"'] & \mathsf{Set} \end{tikzcd}$$

commutes, where U is the forgetful functor taking a metric space to its set of points.

The following theory is based on ideas of de Bakker and Zucker [31], and their method was further developed by America and Rutten in their seminal paper [24] on this topic. Their results were extended in [20], and we sharpen them a bit. First a surprising result about uniqueness of fixed points: In the following theorem a category is called *connected* if its hom-sets are all non-empty.

Theorem 5.4 (America and Rutten [24], Theorem 4.2)**.** *Let $\mathscr{C}$ be a* CMS-*enriched connected category. Every fixed point of a locally contracting endofunctor of $\mathscr{C}$ is canonical.*

Proof. (1) Let $\alpha : FA \to A$ be a fixed point of a locally contracting endofunctor F with contraction factor ε. We prove that this is the initial algebra of F.

Given an algebra $\beta : FB \to B$, define an endofunction k of $\mathscr{C}(A, B)$ as follows:

$$k : (A \xrightarrow{f} B) \longmapsto (A \xrightarrow{\alpha^{-1}} FA \xrightarrow{Ff} FB \xrightarrow{\beta} B).$$

Then k is ε-contracting: for every parallel pair $f, g : A \to B$ we have

$$\begin{aligned} d(k(f), k(g)) &= d(\beta \cdot Ff \cdot \alpha^{-1}, \beta \cdot Fg \cdot \alpha^{-1}) \\ &\le d(Ff, Fg) \\ &\le \varepsilon \cdot d(f, g) \end{aligned}$$

since composition is non-expanding and F is ε-contracting. Thus, by the Banach Fixed Point Theorem, k has a fixed point:

$$h : A \to B \text{ with } h = \beta \cdot Fh \cdot \alpha^{-1}.$$

This implies that h is an algebra homomorphism. Conversely, every algebra homomorphism ($\alpha \cdot h = \beta \cdot Fh$) is a fixed point of k. By Banach's Theorem, $h = k$, thus (A, α) is an initial algebra.

(2) The proof that $\alpha^{-1} : A \to FA$ is a terminal coalgebra is by duality: if $\mathscr{C}$ is a CMS-enriched category, then so is $\mathscr{C}^{op}$ (using the same metric on hom-sets). And F^{op} is a contracting endofunctor. By applying (1) to it we see that (A, α^{-1}) is an initial algebra for F^{op}, i.e. a terminal coalgebra for F. □

Notation 5.5. Let $\mathscr{C}$ be a CMS-enriched category. We denote by $\mathscr{C}^E$ the category of all objects of $\mathscr{C}$ where the morphisms from A to B are all pairs $(e, \widehat{e})$ of morphisms in $\mathscr{C}$, with $e : A \to B$, and $\widehat{e} : B \to A$, and such that $\widehat{e} \cdot e = \mathsf{id}_A$:

$$\begin{tikzcd} A \arrow[loop left, "\mathsf{id}"] \arrow[r, shift left, "e"] & B \arrow[l, shift left, "\widehat{e}"] \end{tikzcd}$$

This is the category with which America and Rutten work in [24].

Definition 5.6. An ω-chain in the category $\mathscr{C}^E$

$$\begin{tikzcd} E_0 \arrow[r, shift left, "e_0"] & E_1 \arrow[l, shift left, "\widehat{e}_0"] \arrow[r, shift left, "e_1"] & E_2 \arrow[l, shift left, "\widehat{e}_1"] \arrow[r, shift left, "e_2"] & \cdots \arrow[l, shift left, "\widehat{e}_2"] \end{tikzcd} \tag{5.1}$$

is called *contracting* if there exists $\varepsilon < 1$ such that for all i we have

$$d(e_{i+1} \cdot \widehat{e}_{i+1}, \mathsf{id}_{E_{i+1}}) \le \varepsilon \cdot d(e_i \cdot \widehat{e}_i, \mathsf{id}_{E_i}).$$

Example 5.7. Let F be a locally contracting endofunctor on a CMS-enriched category $\mathscr{C}$ with a terminal object 1 and such that $\mathscr{C}(1, F1)$ is non-empty. Given $e : 1 \to F1$, the chain

$$1 \underset{!}{\overset{e}{\rightleftarrows}} F1 \underset{F!}{\overset{Fe}{\rightleftarrows}} F^2 1 \underset{F^2!}{\overset{F^2 e}{\rightleftarrows}} \cdots \tag{5.2}$$

is contracting. Indeed, $d(F^{i+1}e \cdot F^{i+1}!, \mathsf{id}) \leq \varepsilon \cdot d(F^i e \cdot F^i !, \mathsf{id})$.

Fundamental Lemma 5.8 (America and Rutten [24], Lemma 3.8)**.** *Let $\mathscr{C}$ be a CMS-enriched category. Consider a contracting chain*

$$E_0 \underset{\widehat{e}_0}{\overset{e_0}{\rightleftarrows}} E_1 \underset{\widehat{e}_1}{\overset{e_1}{\rightleftarrows}} E_2 \underset{\widehat{e}_2}{\overset{e_2}{\rightleftarrows}} \cdots \tag{5.3}$$

in $\mathscr{C}^E$. Let $c_i : E_i \to C$ be a cocone of $(e_i)_{i \in \mathbb{N}}$ in $\mathscr{C}$. Then the following are equivalent:

(1) (c_i) is a colimit cocone in $\mathscr{C}$.

(2) There is a cone $(\widehat{c}_i)$ of $(\widehat{e}_i)$ in $\mathscr{C}$ such that each $\widehat{c}_i$ is a splitting of c_i, and $\lim_i (c_i \cdot \widehat{c}_i) = \mathsf{id}_C$.

The following is a variant of Theorem 4.4 of America and Rutten [24]:

Theorem 5.9 (Adámek and Reiterman [20], Theorem 2)**.** *Every locally contracting endofunctor on a complete and connected CMS-enriched category has a canonical fixed point.*

Proofsketch. Let $F : \mathscr{C} \to \mathscr{C}$ be a locally contracting endofunctor on the complete and connected CMS-enriched category $\mathscr{C}$. Let $e : 1 \to F1$, and consider the chain in $\mathscr{C}^E$ in (5.2). This chain is contracting, as shown in Example 5.7. By completeness of $\mathscr{C}$, the chain determined by the morphisms $F^n e$ has a limit. We denote its limit by $F^\omega 1$ and the limit projections by $\widehat{c}_i : F^\omega 1 \to F^i 1$.

For every $i < \omega$ we obtain a cone $b_j : F^i 1 \to F^j 1$, $j < \omega$ as follows:

$$b_j = \begin{cases} F^j ! \cdot \ldots \cdot F^{i-1}! & \text{for } j < i \\ \mathsf{id}_{F^i 1} & \text{for } j = i \\ F^{j-1} e \cdot \ldots \cdot F^i e & \text{for } j > i. \end{cases}$$

Hence, the universal property of the limit induces a morphism $c_i : F^i 1 \to F^\omega 1$. It is not difficult to see that c_i is a split monomorphism with splitting $\widehat{c}_i$. Furthermore the c_i form a cocone, and one readily shows that $\lim_i (c_i \cdot \widehat{c}_i) = \mathsf{id}_{F^\omega 1}$. Consider the image of our chain under F; this is the same as the original chain, but without its first term. We also have a cocone $(Fc_i) : F^{n+1}1 \to F^{\omega+1}1$. The morphisms $F\widehat{c}_i$ constitute a cone, and each $F\widehat{c}_i$ splits the corresponding Fc_i. Moreover, the fact that F is contracting implies that $\lim_i (Fc_i \cdot F\widehat{c}_i) = F(\lim_i(c_i \cdot \widehat{c}_i)) = \mathsf{id}$. We now use (2) $\Rightarrow$ (1) in Lemma 5.8 to see that (Fc_i) is a colimit cocone of the second chain, hence also of the first. Hence the connecting morphism $v_{\omega+1,\omega} : F^{\omega+1}1 \to F^\omega 1$ from Definition 3.20 is an isomorphism. In particular, the terminal coalgebra-chain converges, and F has a fixed point. Since $\mathscr{C}$ is connected, we can apply Theorem 5.4 to conclude the proof. □

Corollary 5.10 (Adámek and Reiterman [20], Proposition 1)**.** *Every locally contracting endofunctor on* CMS *has a canonical fixed point.*

6. Bounded and Accessible Endofunctors

In Corollary 3.7 we saw that an endofunctor preserving colimits of λ-chains on a cocomplete category has an initial algebra. The preservation requirement is fairly mild, whereas the requirement for terminal coalgebras in the dual Corollary 3.22 that a functor preserve limits of λ-chains is satisfied much more rarely.

Here we present a proof that, under suitable conditions on the base category, every λ-accessible functor (i.e., one preserving λ-filtered colimits) has a terminal coalgebra. Here and below, λ is an arbitrary infinite regular cardinal. This result follows from the Limit Theorem of Makkai and Paré [46] from which it follows that the category $\mathsf{Coalg}\, F$ is locally presentable, whence complete. Consequently, it has a terminal object. This is explained more in detail e.g. by van Breugel et al. [72, Theorem 8] and also mentioned by Barr [26], but it seems that an explicit self-contained proof has never been published.

We also prove that every *bounded* endofunctor (a concept generalizing the the bounded set functors of Kawahara and Mori [41]) has a terminal coalgebra.

Remark 6.1. Recall the following form of Freyd's Adjoint Functor Theorem (see e.g. [44]): if $\mathscr{C}$ is a cocomplete and cowellpowered category with a *weakly* *terminal set* $\mathscr{S}$ of objects (meaning that every object of $\mathscr{C}$ has a morphism into some object of $\mathscr{S}$), then $\mathscr{C}$ has a terminal object. Applying this to $\mathsf{Coalg}\, F$, whose colimits are computed on the level of $\mathscr{C}$, we get:

Theorem 6.2. *Let $\mathscr{C}$ be a cocomplete and cowellpowered category, and let F have a weakly terminal set of coalgebras. Then F has a terminal coalgebra.*

Definition 6.3. A functor $F\colon \mathscr{C} \to \mathscr{C}$ is called *bounded* by a set $\mathscr{G}$ of objects of $\mathscr{C}$ if for every coalgebra $a\colon A \to FA$ and every morphism $g\colon G \to A$ with $G \in \mathscr{G}$, there exists a coalgebra $a' : A' \to FA'$ with $A' \in \mathscr{G}$ and a coalgebra homomorphism $k\colon (A', a') \to (A, a)$ through which g factorizes:

$$\begin{tikzcd} & A' \arrow[r, "a'"] \arrow[d, "k"] & FA' \arrow[d, "Fk"] \\ G \arrow[ur, dashed, "\exists"] \arrow[r, "g"'] & A \arrow[r, "a"'] & FA \end{tikzcd}$$

In [41] this was defined for $\mathscr{C} = \mathsf{Set}$ and $\mathscr{G}$ the (essentially small) collection of all sets of cardinality at most λ. We then say that F is *bounded by* λ. Observe that this means precisely that for every element $x \in A$ there exists a subcoalgebra $A' \hookrightarrow A$ containing x and with $|A'| \leq \lambda$.

Example 6.4. (1) A set functor is bounded by λ iff it is λ^+-accessible, see [19].

(2) The polynomial functor H_Σ is bounded by λ whenever the arities of all symbols in Σ are less than λ. Moreover, bounded set functors are precisely the quotients of polynomial functors, as proved in [19].

(3) The functor $\mathscr{P}_\mathsf{f}$ is bounded (by $\lambda = \omega$), but $\mathscr{P}$ is not.

Recall the concept of a *generator* of a category. This is a set $\mathscr{G}$ of objects such that whenever parallel morphisms $p, q : X \to Y$ are distinct, there is an object $G \in \mathscr{G}$ and a morphism $g : G \to X$ such that $p \cdot g \neq q \cdot g$.

Remark 6.5. For every endofunctor F on a cocomplete category $\mathscr{C}$ the category $\mathsf{Coalg}\, F$ is also cocomplete. Indeed, the forgetful functor $U : \mathsf{Coalg}\, F \to \mathscr{C}$ *creates* colimits. That means that for any given diagram D of coalgebras $a_i : A_i \to FA_i$ $(i \in I)$ with UD having a colimit cocone $c_i : A_i \to C$ $(i \in I)$, there exists a unique coalgebra structure $c : C \to FC$ such that each c_i is a coalgebra morphism. Moreover, (C, c) is then the colimit of D.

Theorem 6.6. *Let $\mathscr{C}$ be a cocomplete and cowellpowered category with a terminal object. Every endofunctor bounded by a generator of $\mathscr{C}$ has a terminal coalgebra.*

Proof. This uses Freyd's Special Adjoint Functor Theorem (see Remark 6.1): we prove that the canonical forgetful functor $U\colon \mathsf{Coalg}\,F \to \mathscr{C}$ has a right adjoint $U \dashv R$ (then $R1$ is terminal in $\mathsf{Coalg}\,F$). Since $\mathsf{Coalg}\,F$ is cocomplete and cowellpowered and U preserves colimits, we need only exhibit a generating set in $\mathsf{Coalg}\,F$. In fact, let $\mathscr{G}$ be a generator in $\mathscr{C}$ such that F is bounded by it. Then the set of all coalgebras $a\colon A \to FA$ with $A \in \mathscr{G}$ is generating. Indeed, given homomorphisms $p, q\colon (B, b) \to (\bar{B}, \bar{b})$ if $p \neq q$ there exists $g\colon G \to B$ with $G \in \mathscr{G}$ and $p \cdot g \neq q \cdot g$. Let $m\colon (B', b') \to (B, b)$ be a subcoalgebra with $B' \in \mathscr{G}$ such that g factorizes through m, then clearly $p \cdot m \neq q \cdot m$. $\square$

Definition 6.7 (Gabriel, Ulmer [36], Definition 7.1)**.** Let λ be an infinite regular cardinal.

(1) A category $\mathscr{D}$ is called λ-*filtered* if every subcategory with less than λ morphisms has a cocone in $\mathscr{D}$. A λ-filtered diagram is a diagram $\mathscr{D} \to \mathscr{C}$ with $\mathscr{D}$ λ-filtered. For example, every α-chain is α-filtered.

(2) A functor is called λ-*accessible* if it preserves λ-filtered colimits.

(3) An object X of $\mathscr{C}$ is λ-*presentable* if its hom-functor $\mathscr{C}(X, -)$ is λ-accessible.

(4) A category $\mathscr{C}$ is called *locally* λ-*presentable* if it is cocomplete and has a set $\mathscr{C}_\lambda$ of λ-presentable objects whose closure under λ-filtered colimits is all of $\mathscr{C}$. And $\mathscr{C}$ is called *locally presentable* if it is locally λ-presentable for some λ.

Note that an ω-presentable object is called a *finitely presentable*, and a locally ω-presentable category $\mathscr{C}$ is called *locally finitely presentable.*

Examples 6.8. (1) A set is λ-presentable in Set iff its cardinality is less than λ. Thus Set is locally finitely presentable.

Recall from [19] that a set functor is λ-accessible iff for every set X and every element $x \in FX$ there exists a subset $u : U \hookrightarrow X$ with $|U| < \lambda$ such that x lies in the image of Fu.

(2) Similarly, the categories of posets and graphs are locally finitely presentable with finite posets and finite graphs being the finitely presentable objects.

(3) In a variety $\mathscr{C}$ of (finitary) algebras an algebra is λ-presentable iff it has a presentation by less than λ-generators and less than λ relations [21, Corollary 3.13]. Every variety is locally finitely presentable. Concrete instances are the categories of monoids, groups, vector spaces over a fixed field etc.

(4) The categories of cpos (i.e. posets with joins of ω-chains and continuous maps), cpos with $\perp$ (i.e. cpos with a least element and continuous maps preserving $\perp$), and CMS are locally ω_1-presentable. However, the categories DCPO and $\mathsf{DCPO}_\perp$ are not locally λ-presentable for any λ.

Remark 6.9. Recall from [21] the following properties of a locally presentable category $\mathscr{C}$:

(1) $\mathscr{C}$ is complete, cocomplete and cowellpowered,

(2) the set $\mathscr{C}_\lambda$ in Definition 6.7 is a generator,

(3) $\mathscr{C}$ is locally μ-presentable for every regular cardinal $\mu \geq \lambda$.

(4) A λ-*small* diagram is one whose diagram scheme has less than λ objects and morphisms. A colimit of a λ-small diagram of λ-presentable objects is λ-presentable, too.

(5) Every morphism $f : A \to \operatorname{colim} D$, where D is a λ-filtered diagram and A a λ-presentable object, factorizes through some colimit injection.

Furthermore, note that every λ-accessible functor is μ-accessible for every $\mu \geq \lambda$, but the converse may fail. For example the countable power-set functor is ω_1-accessible but not ω-accessible.

Theorem 6.10. *Let $\mathscr{C}$ be a locally presentable category. Every accessible endofunctor on $\mathscr{C}$ has an initial algebra and a terminal coalgebra.*

Proof. (1) The existence of an initial algebra follows from Theorem 3.5: given a λ-accessible functor, the initial-algebra chain converges in at most λ steps because the first λ steps of the initial-algebra chain form a λ-filtered diagram.

(2) To prove that a terminal coalgebra exists, choose (using Remark 6.9(3) above) a regular cardinal λ such that $\mathscr{C}$ is locally λ-presentable, and the given endofunctor is λ-accessible. Put $\mathscr{G} = \mathscr{C}_\lambda$ in Definition 6.7(4), form the closure $\overline{\mathscr{G}}$ of $\mathscr{G}$ under colimits of μ-chains for all limit ordinals $\mu < \lambda$ and verify the assumption of Theorem 6.6 for $\overline{\mathscr{G}}$; note that $\mathscr{G}$ and $\overline{\mathscr{G}}$ are both small. We only need to prove that F is bounded by (the set of objects of) $\overline{\mathscr{G}}$. Let $a\colon A \to FA$ be a coalgebra and $g\colon G \to A$ a morphism, where G is in $\overline{\mathscr{G}}$. We define a λ-chain B_i, $i < \lambda$, of λ-presentable objects together with a cocone $k_i\colon B_i \to A$ $(i < \lambda)$ and a natural transformation $b_i : B_i \to FB_{i+1}$ $(i < \lambda)$ for which the square below commutes:

$$\begin{tikzcd} B_i \arrow[r, "b_i"] \arrow[d, "k_i"'] & FB_{i+1} \arrow[d, "Fk_{i+1}"] \\ A \arrow[r, "a"'] & FA \end{tikzcd} \tag{6.1}$$

We first express A as a λ-filtered colimit of λ-presentable objects $G_t \in \mathscr{G}$, $t \in T$, and denote by $g_t : G_t \to A$ the colimit cocone. Since F is λ-accessible, we conclude $FA = \operatorname{colim}_{t\in T} FG_t$. We now define a chain $(B_i)_{i<\lambda}$ of λ-presentable objects with connecting morphisms $b_{ij}, i \leq j$ by transfinite recursion:

First step: put $B_0 = G$ and $k_0 = g$. To define B_1, k_1 and b_{01}, use the λ-filtered colimits $A = \operatorname{colim} G_t$ and $FA = \operatorname{colim} FG_t$: since G is λ-presentable the morphisms

$$G \xrightarrow{g} A \quad \text{and} \quad G \xrightarrow{a\cdot g} FA$$

factorize through colimit injections. Using filteredness, we may assume that common $t \in T$ is given such that g factorizes through g_t and $a \cdot g$ through Fg_t:

$$\begin{tikzcd} B_0 = G \arrow[rr, dashed, "b_{01}"] \arrow[dr, "k_0=g"'] & & G_t \arrow[dl, "g_t"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, dashed, "b_1"] \arrow[d, "g"'] & FG_t \arrow[d, "Fg_t"] \\ A \arrow[r, "a"'] & FA \end{tikzcd}$$

Now put $B_1 = G_t$, $k_1 = g_t$ and b_1, b_{01} as above.

Isolated step: factorize the morphisms $k_i : B_i \to A$ and $a \cdot k_i : B_i \to FA$ through colimit injections. By filteredness, we can assume that we have a $t \in T$ with factorizations $k_i = g_t \cdot u$ and $a \cdot k_i = Fg_t \cdot v$. For every ordinal

$j < i$ consider the diagram below:

$$\begin{tikzcd}
B_j \arrow[rrr, "b_j"] \arrow[dr, "k_j"] \arrow[dd, "b_{ji}"'] & & & FB_{j+1} \arrow[dl, "Fk_{j+1}"'] \arrow[d, "Fb_{j+1,i}"] \\
& A \arrow[r, "a"] & FA & FB_i \arrow[l, "Fk_i"'] \arrow[d, "Fu"] \\
B_i \arrow[ur, "k_i"'] \arrow[rrr, "v"'] & & & FG_t \arrow[ul, "Fg_t"]
\end{tikzcd}$$

All inner parts of this diagram commute: its upper part is (6.1) for j, the left-hand and upper right-hand triangles commute by induction, and the remaining two parts commute by the definition of u and v, respectively. It follows that the outside of the diagram postcomposed by Fg_t commutes. Since B_j is λ-presentable, it follows that there exists an $s \in T$ and a connecting morphism g_{ts} in the diagram defining A such that Fg_{ts} merges the two morphisms that form the outside of the diagram:

$$Fg_{ts} \cdot v \cdot b_{ji} = Fg_{ts} \cdot Fu \cdot Fb_{j+1,i} \cdot b_j. \tag{6.2}$$

We may choose s independent of j because our diagram is λ-filtered and the number of all ordinals $j < i$ is less than λ. (Note that this holds since λ is assumed to be regular!) Put

$$B_{i+1} = G_s, \quad k_{i+1} = g_s \quad \text{and} \quad b_i = (B_i \xrightarrow{v} FG_t \xrightarrow{Fg_{ts}} FG_s).$$

The connecting morphism is defined by

$$b_{i,i+1} = (B_i \xrightarrow{u} G_t \xrightarrow{g_{ts}} G_s),$$

and this yields $k_{i+1} \cdot b_{i,i+1} = k_i$:

$$\begin{tikzcd}
B_i \arrow[drr, bend left, "k_i"] \arrow[dr, "u"] \arrow[dd, "b_{i,i+1}"'] & & \\
& G_t \arrow[r, "g_t"] \arrow[dl, "g_{ts}"'] & A \\
B_{i+1} = G_s \arrow[urr, bend right, "g_s = k_{i+1}"'] & &
\end{tikzcd}$$

The square (6.1) commutes since

$$Fk_{i+1} \cdot b_i = Fg_s \cdot b_i = Fg_s \cdot (Fg_{ts} \cdot v) = Fg_t \cdot v = a \cdot k_i,$$

and the naturality square for b_i follows from (6.2):

$$\begin{tikzcd}
B_j \arrow[rrr, "b_{ji}"] \arrow[dd, "b_j"'] & & & B_i \arrow[dl, "v"'] \arrow[dd, "b_i"] \\
 & FB_i \arrow[r, "Fu"] \arrow[drr, "Fb_{i,i+1}"] & FG_t \arrow[dr, "Fg_{ts}"] & \\
FB_{j+1} \arrow[ur, "Fb_{j+1,i}"'] \arrow[rrr, "Fb_{j+1,i+1}"'] & & & FG_s = FB_{i+1}
\end{tikzcd}$$

Limit step: for limit ordinals j put $B_j = \operatorname{colim}_{i<j} B_i$ with colimit cocone b_{ij} $(i < j)$. Since the $k_i : B_i \to A$, $i < j$, form a compatible cocone we obtain the unique induced morphism $k_j \colon B_j \to A$ such that $k_j \cdot b_{ij} = k_i$ for all $i < j$. Furthermore, naturality of the $b_i : B_i \to FB_{i+1}$ $(i < j)$ implies that $Fb_{i+1,j+1} \cdot b_i : B_i \to FB_j$ form a compatible cocone; thus we obtain a unique $b_j : B_j \to FB_{j+1}$ such that the desired naturality square commutes:

$$\begin{tikzcd}
B_i \arrow[r, "b_{ij}"] \arrow[d, "b_i"'] & B_j \arrow[d, "b_j"] \\
FB_{i+1} \arrow[r, "Fb_{i+1,j+1}"'] & FB_{j+1}
\end{tikzcd}$$

It remains to verify square (6.1) for j; to this end consider the diagram below:

$$\begin{tikzcd}
B_i \arrow[rrr, "b_i"] \arrow[dr, "b_{ij}"] \arrow[dd, "k_i"'] & & & FB_{i+1} \arrow[dl, "Fb_{i+1,j+1}"'] \arrow[dd, "Fk_{i+1}"] \\
 & B_j \arrow[r, "b_j"] \arrow[dl, "k_j"'] & FB_{j+1} \arrow[dr, "Fk_{j+1}"] & \\
A \arrow[rrr, "a"'] & & & FA
\end{tikzcd}$$

The outside of this diagram commutes, being (6.1) for i, both triangles commute since the k_i form a cocone, and the upper part commutes by naturality.

Therefore, the lower part commutes when precomposed by b_{ij} for every $i < j$. Thus, using that the colimit injections b_{ij} $(i < j)$ form an epimorphic family we see that this part commutes as desired.

Finally, for $j = \lambda$ we form a colimit

$$B = \operatorname*{colim}_{i<\lambda} B_i \in \overline{\mathscr{G}}$$

which is λ-filtered, thus, preserved by F. The morphisms b_i above yield a coalgebra structure

$$\operatorname*{colim}_{i<\lambda} b_i \colon B \to FB$$

such that

$$k = \operatorname{colim} k_i \colon B \to A$$

is a coalgebra homomorphism. We have a factorization of g through k: recall that $g = k_0$. $\square$

Remark 6.11. A slightly stronger result follows from Kelly [42, Theorem 1.8.1]. He showed that it is not necessary to assume that F preserve λ-filtered colimits: it is sufficient that it preserve λ-directed unions. Moreover, the base category only needs to be *locally ranked*; we do not recall that notion here but note that this includes all locally presentable categories as well as the category of topological spaces.

What about the terminal-coalgebra chain for λ-accessible functors? Worrell proved in [73] that for $\mathscr{C} = \mathsf{Set}$, that chain converges in at most $\lambda + \lambda$ steps. We present below a simpler proof from [9]. But first, let us mention a result for general coalgebras.

Theorem 6.12 (Adámek and Trnková [23], Theorem 2.6)**.** *Let F be an accessible endofunctor of a locally presentable category preserving monomorphisms. Then the terminal chain converges:*

$$\nu F = F^j 1 \qquad \textit{for some ordinal number } j.$$

Theorem 6.13 (Worrell [74], Theorem 11)**.** *Let λ be an infinite regular cardinal, and let F be a λ-accessible set functor.*

(1) The terminal-coalgebra chain has monomorphic connecting morphisms from λ onwards.

(2) It converges after at most $\lambda + \lambda$ steps.

Proof. If $F1 = \emptyset$, then F is constant with value $\emptyset$, and the statements hold trivially. So we may assume $F1 \neq \emptyset$. Then there is a nonempty coalgebra, and therefore the empty coalgebra cannot be terminal. Hence all members of the terminal-coalgebra chain V are nonempty: if we had $V_i = \emptyset$, then we should also have $V_{i+1} = \emptyset$ by presence of the map $v_{i+1,i}$, which would be invertible, whence $\emptyset$ would be a terminal coalgebra.

As to (1): It is sufficient to prove that $v_{\lambda+1,\lambda}$ is monic. Indeed then $v_{\lambda+1,\lambda}$ is a split monomorphism, hence $v_{\lambda+2,\lambda+1}$ is monic, *etc.* We obtain by transfinite induction that all $v_{i,\lambda}$ with $i \geq \lambda$ are monic.

We prove that two distinct elements x and y of $V_{\lambda+1} = FV_\lambda$ remain distinct under $v_{\lambda+1,\lambda}$. By Example 6.8(1) there exists a nonempty subset $u : U \hookrightarrow V_\lambda$ such that $|U| < \lambda$ and x and y lie in the image of Fu. Since $(v_{\lambda,i})_{i<\lambda}$ is a limit cone, thus collectively monic, every pair of distinct elements of U remains distinct under $v_{\lambda,i}$ for some $i < \lambda$. From $|U \times U| < \lambda$ we conclude that one $i_0 < \lambda$ can be chosen for *all* distinct pairs in U. (This is precisely where we use the regularity of λ.) In other words, $v_{\lambda,i_0} \cdot u$ is a monomorphism. It splits, thus $Fv_{\lambda,i_0} \cdot Fu$ is monic, which implies that Fv_{λ,i_0} keeps x and y distinct. Thus, $v_{\lambda+1,\lambda}$ too keeps them distinct, because

$$Fv_{\lambda,i_0} = v_{\lambda+1,i_0+1} = v_{\lambda,i_0+1} \cdot v_{\lambda+1,\lambda}.$$

As to (2): To prove that $v_{\lambda+\lambda+1,\lambda+\lambda}$ is invertible is to prove that F preserves the limit $V_{\lambda+\lambda}$ of $(V_i)_{i<\lambda+\lambda}$. We can disregard the first λ members of that chain:

$$V_{\lambda+\lambda} = \lim_{i<\lambda} V_{\lambda+i}.$$

The connecting morphisms

$$v_i = v_{\lambda+i,\lambda} : V_{\lambda+i} \to V_\lambda$$

of that last λ-chain are monic by (1). So $V_{\lambda+\lambda} = \bigcap_{i<\lambda} V_{\lambda+i}$, and we are to prove that F preserves this intersection.

Given an element $x \in FV_\lambda$ lying in the image of Fv_i, for all $i < \lambda$, we have the task to prove that x lies in the image of $Fv_{\lambda+\lambda,\lambda}$. Using Example 6.8(1), choose a subset $u : U \hookrightarrow V_\lambda$, $|U| < \lambda$, such that x lies in the image of Fu. By Remark 3.15 we may assume that F preserves finite intersections. Then for $u_i = u \cap v_i$ we know that $x \in \mathrm{im}(Fu_i)$ for all $i < \lambda$. But the u_i's form a decreasing λ-chain of subsets of U. Since $|U| < \lambda$, this chain converges at some $i_0 < \lambda$. It follows that $u_{i_0} = u_\lambda \subseteq v_{\lambda+\lambda,\lambda}$, whence indeed x lies in the image of $Fv_{\lambda+\lambda,\lambda}$. $\square$

Definition 6.14. A functor F is *finitary* if it preserves filtered colimits. That is, F is λ-accessible (see Definition 6.7) for $\lambda = \aleph_0$.

For example, a polynomial set functor associated to a finitary signature is finitary. Among the accessible functors, the finitary ones are especially important.

Corollary 6.15. *For a finitary set functor the initial-algebra chain converges in ω steps and the terminal-coalgebra chain in $\omega + \omega$ steps.*

Example 6.16. The full $\lambda + \lambda$ steps are needed for some λ-accessible set functors, see [9]. For example, this holds for the functor assigning to every set X the set of all λ-small filters on X, i.e. filters having a member of cardinality less than λ.

Remark 6.17. In Section 11 we study, for a finitary endofunctor F, the rational fixed point, which is the colimit of all coalgebras carried by finitely presentable objects. In contrast, the analogous fixed point for λ-accessible functors with λ uncountable is just νF:

Theorem 6.18 (Adámek, Milius and Velebil [16], Proposition 5.16)**.** *For every λ-accessible endofunctor on a locally λ-presentable category, where λ is an uncountable regular cardinal, the terminal coalgebra is the colimit of all coalgebras carried by λ-presentable objects.*

7. Parametric Fixed Points

In many applications one is interested in mixed inductive/coinductive definitions, i.e. a mix of taking initial algebras and terminal coalgebras. For example, given a functor $F : \mathscr{C}^3 \to \mathscr{C}$ one would like to form

$$\nu X_1.\mu X_2.\nu X_3.F(X_1, X_2, X_3). \tag{7.1}$$

More generally, one would like to iteratively take, for an endofunctor with several arguments, the initial algebra in some arguments and the terminal coalgebra in others.

In this section we shall prove that such mixed fixed points exist provided that $\mathscr{C}$ is a locally presentable category and F an accessible functor. More precisely, fixing all but one argument and taking an initial algebra or terminal coalgebra gives a functor which is accessible; see Theorem 7.4. We then iterate this procedure.

Actually, we consider a slightly more general setting. For the rest of this section we assume that $\mathscr{C}$ and $\mathscr{D}$ are locally λ-presentable categories and that

$$F : \mathscr{D} \times \mathscr{C} \to \mathscr{C}$$

is a λ-accessible functor.

Our aim is to prove that by taking parametrized initial algebras or terminal-coalgebras in the second argument of F one obtains another λ-accessible functor $T : \mathscr{D} \to \mathscr{C}$.

Notation 7.1. We denote by $T : \mathscr{D} \to \mathscr{C}$ the functor defined on objects by

$$TA = \mu F(A, -),$$

i.e. TA is the initial algebra for the endofunctor $F(A, -) : \mathscr{C} \to \mathscr{C}$. These initial algebras exist by Theorem 6.10, and we denote their structure by

$$\iota_A : F(A, TA) \to TA.$$

Furthermore, T assigns to every $f : A \to B$ the unique $F(A, -)$-algebra morphism displayed below:

$$\begin{tikzcd}
F(A,TA) \arrow[rr, "\iota_A"] \arrow[d, "FTf"'] & & TA \arrow[d, "Tf"] \\
F(A,TB) \arrow[r, "F(f,TB)"'] & F(B,TB) \arrow[r, "\iota_B"'] & TB
\end{tikzcd}$$

That this assignment is functorial is then easy to verify using the universal property of initial algebras.

Theorem 7.2. *The functor $T : \mathscr{D} \to \mathscr{C}$ is λ-accessible.*

Proof. For every object A let us denote the components of the initial-algebra chain by $T_i A = F(A, -)^i(0)$. Recall from Corollary 3.7 that this chain converges in λ steps, whence $TA = T_\lambda A$. By an easy transfinite induction one shows that every T_i is a functor: for any given $f : A \to B$ define $T_i f$ by transfinite induction as follows

$$T_0 f = \mathsf{id}_0, \qquad T_{i+1} f = (T_{i+1}A = F(A, T_iA) \xrightarrow{F(f, T_i f)} F(B, T_iB) = T_{i+1}B),$$

and on limit ordinals j, $T_j f$ is uniquely determined such that all connecting morphisms $w_{i,j} : T_i \to T_j$ are natural transformations by the universal property of the colimits T_iA. We thus obtain a chain in the category $[\mathscr{D}, \mathscr{C}]$ of all functors from $\mathscr{D} \to \mathscr{C}$.

Another easy transfinite induction now shows that every T_i is λ-accessible; here one uses that F is λ-accessible and that the full subcategory of $[\mathscr{D}, \mathscr{C}]$ given by all λ-accessible functors is cocomplete (with colimits formed objectwise). Thus $T = T_\lambda$ is λ-accessible. □

The desired result for terminal coalgebras is more subtle. The key to this is the following construction of the terminal coalgebras for $F(A, -)$.

Construction 7.3. For any given object A of $\mathscr{C}$ denote by

$$\mathsf{Coalg}_\lambda\, F(A, -)$$

the full subcategory of the category of all coalgebras for $F(A, -)$ given by all coalgebras $X \to F(A, X)$ with a λ-presentable carrier X.

Since, by Remark 6.5, $\mathsf{Coalg}_\lambda\, F(A, -)$ has colimits of all diagrams with less than λ objects and morphisms, we see that it is a λ-filtered category, whence the inclusion functor

$$I_A : \mathsf{Coalg}_\lambda\, F(A, -) \hookrightarrow \mathsf{Coalg}\, F(A, -)$$

is a λ-filtered diagram. We define the coalgebra $\tau_A : SA \to F(A, SA)$ as its colimit:

$$SA = \operatorname{colim} I_A \quad \text{with colimit injections } c^\sharp : (X, c) \to (SA, \tau_A).$$

Note that by Remark 6.5, the underlying object of the coalgebra SA is the colimit $\operatorname{colim} UI_A$ of the underlying objects of all coalgebras $c : X \to F(A, X)$ in $\mathsf{Coalg}_\lambda\, F(A, -)$ equipped with the unique coalgebra structure $\tau_A : SA \to F(A, SA)$ such that all colimit injections $c^\sharp$ are coalgebra morphisms.

The following result is a consequence of Theorem 6.18:

Theorem 7.4. *Suppose that λ is an uncountable regular cardinal. Then SA is the final coalgebra for $F(A,-)$.*

Proof. Since λ is uncountable, for every λ-accessible endofunctor $H : \mathscr{C} \to \mathscr{C}$ the category $\mathsf{Coalg}\, H$ is locally λ-presentable, and a coalgebra is λ-presentable iff its carrier is λ-presentable in $\mathscr{C}$ (see [19, Corollary 4.2]).

Applying this to the functor $H = F(A,-)$, which is λ-accessible by assumption, we see that the above inclusion I_A is a dense subcategory of a locally λ-presentable category, i.e. every coalgebra for H is a λ-filtered colimit of coalgebras from $\mathsf{Coalg}_\lambda\, H$.

Consequently, we only need to verify that for every coalgebra (X, c) in $\mathsf{Coalg}_\lambda\, H$ there exists a unique coalgebra morphism into (SA, τ_A). The colimit injection $c^\sharp : X \to SA$ is a coalgebra morphism, and the proof that it is unique is completely analogous to the proof of [16, Proposition 3.2] $\square$

Notation 7.5. Let λ be uncountable. We denote by $S : \mathscr{D} \to \mathscr{C}$ the functor defined on objects by $SA = \nu F(A,-)$ and on morphisms dually to that in Notation 7.1.

Theorem 7.6. *If λ is an uncountable regular cardinal, then $S : \mathscr{D} \to \mathscr{C}$ is a λ-accessible functor.*

Proof. To see that S is a functor use the argument dual to that in the proof of Theorem 7.2. We proceed to prove that S is λ-accessible. Suppose that K is a λ-filtered diagram and $a_k : A_k \to A$, $k \in K$, form its colimit. Let us consider the category $\mathscr{E}$ whose objects are morphisms $c : X \to F(A_k, X)$ with X λ-presentable and whose morphisms from c to $d : Y \to F(A_\ell, Y)$ are pairs (u, h) where $h : X \to Y$ is a morphism of $\mathscr{C}$ and $u : A_k \to A_\ell$ is a connecting morphism in the diagram K such that the following square commutes:

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "h"'] & F(A_k, X) \arrow[d, "F(u,h)"] \\ Y \arrow[r, "d"'] & F(A_\ell, Y) \end{tikzcd}$$

We have the forgetful functor $E : \mathscr{E} \to \mathscr{C}$ mapping objects $c : X \to F(A_k, X)$ to X. Observe that the colimit

$$\operatorname*{colim}_{k \in K} SA_k = \operatorname*{colim}_{k \in K} \operatorname{colim} I_{A_k}$$

is precisely the colimit of E. Further, we have a functor $J : \mathscr{E} \to \mathsf{Coalg}_\lambda\, F(A,-)$ assigning to $c : X \to F(A_k, X)$ in $\mathscr{E}$ the following coalgebra

$$X \xrightarrow{c} F(A_k, X) \xrightarrow{F(a_k,X)} F(A, X)$$

and with $J(u,h) = h$ on morphism. The desired proof of $SA = \mathrm{colim}_{k\in K}\ SA_k$ will be complete once we prove that J is cofinal. Since I_A in Construction 7.3 is a λ-filtered diagram, this requires that we verify the following two properties:

(1) for every object $c : X \to F(A,X)$ of $\mathsf{Coalg}_\lambda\, F(A,-)$ there is some $k \in K$ and $d : Y \to F(A_k, Y)$ such that (X,c) has a morphism to $J(Y,d)$ in $\mathsf{Coalg}_\lambda\, F(A,-)$.

(2) given $c : X \to F(A,X)$ and two morphisms in $\mathsf{Coalg}_\lambda\, F(A,-)$

$$J(Y,d) \xleftarrow{f} (X,c) \xrightarrow{f'} J(Y',d')$$

for some $k,\ell \in K$, $d : Y \to F(A_k, Y)$ and $d' : Y' \to F(A_\ell, Y')$ in $\mathscr{E}$, there exists an $m \in K$, an object $d'' : Y'' \to F(A_m, Y'')$ and morphisms

$$(Y,d) \xrightarrow{(u,g)} (Y'',d'') \xleftarrow{(u',g')} (Y',d')$$

such that the square below commutes:

$$\begin{tikzcd}
(X,c) \arrow[r, "f"] \arrow[d, "f'"'] & J(Y,d) \arrow[d, "J(u,g)"] \\
J(Y',d') \arrow[r, "J(u',g')"'] & J(Y'',d'')
\end{tikzcd}$$

Ad (1). Given c we know that $F(A,X) = \mathrm{colim}_{k\in K}\, F(A_k, X)$ because $F(-,X)$ is λ-accessible. Since X is λ-presentable we thus obtain a factorization of c through some colimit injection $F(a_k, X)$, i.e. there exists $k \in K$ and $d : X \to F(A_k, X)$ such that $c = F(a_k, X) \cdot d$, i.e. (a_k, X) is a morphism in $\mathscr{E}$ from $J(X,d)$ to (X,c), and we are done.

Ad (2). Given c, f and f', since $\mathsf{Coalg}_\lambda\, F(A,-)$ is λ-filtered there exists $e : Z \to F(A,Z)$ with Z λ-presentable and $F(A,-)$-coalgebra morphisms $g : J(Y,d) \to (Z,e)$ and $g' : J(Y',d') \to (Z,e)$ such that $g \cdot f = g' \cdot f'$. Since Z is λ-presentable we can choose some $m \in K$ and $e' : Z \to F(A_m, Z)$ such that $e = F(a_m, Z)\cdot e'$. Further, since the A_k form a λ-filtered diagram we may assume without loss of generality that we have two connecting morphisms $u : A_k \to A_m$ and $v : A_\ell \to A_m$ of the diagram K.

Now in the following diagram

$$\begin{tikzcd}
Y \arrow[r, "d"] \arrow[d, "g"'] & F(A_k, Y) \arrow[r, "F(a_k,Y)"] \arrow[d, "F(u,g)"] & F(A, Y) \arrow[d, "F(A,g)"] \\
Z \arrow[r, "e'"'] & F(A_m, Z) \arrow[r, "F(a_m,Z)"'] \arrow[r, bend right, "e"'] & F(A, Z)
\end{tikzcd}$$

the outside and right-hand squares commute, hence the left-hand square commutes when extended by $F(a_m, Z)$. We have a analogous diagram for (Y', d'), g' and v. Thus by λ-filteredness we can choose a connecting morphism $w : A_m \to A_n$ of K such that $F(w, Z)$ merges the left-hand square above as well as that for (Y', d'), g' and v. This means that we have a commutative diagram

$$\begin{tikzcd}
Y \arrow[rr, "d"] \arrow[d, "g"'] & & F(A_k, Y) \arrow[dl, "F(u,g)"'] \arrow[d, "F(w\cdot u,g)"] \\
Z \arrow[r, "e'"'] & F(A_m, Z) \arrow[r, "F(w,Z)"'] & F(A_n, Z)
\end{tikzcd}$$

Thus, $(w \cdot u, g)$ is a morphism from (Y, d) to $(Z, F(w, Z)\cdot e')$ in $\mathscr{E}$. Similarly $(w \cdot v, g')$ is a morphism from (Y', d') to $(Z, F(w, Z)\cdot e')$ in $\mathscr{E}$. Consequently, we have the object $(Z, F(w, Z) \cdot e')$ of $\mathscr{E}$ and the two desired morphisms $(w \cdot u, g)$ and $(v \cdot u, g')$ with

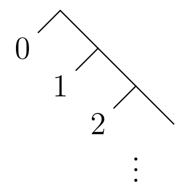
$$J(w \cdot u, g) \cdot f = g \cdot f = g' \cdot f' = J(v \cdot u, g') \cdot f'. \qquad \square$$

Example 7.7. Coming back to our motivating example in (7.1), let us consider an accessible functor $F : \mathscr{C}^n \to \mathscr{C}$. Then any mixed fixed point

$$\chi_1 X_1.\chi_2 X_2.\cdots\chi X_n.F(X_1, \ldots, X_n)$$

where $\chi_i \in \{\mu, \nu\}$ for $i = 1, \ldots, n$, exists. Indeed, take $\mathscr{D} = \mathscr{C}^{n-1}$ and apply Theorem 7.2 to obtain the accessible functor $T = \mu X_n.F(-, X_n) : \mathscr{C}^{n-1} \to \mathscr{C}$ if $\chi_n = \mu$ or Theorem 7.6 to obtain the accessible functor $S = \nu X_n.F(-, X_n) : \mathscr{C}^{n-1} \to \mathscr{C}$ if $\chi_n = \nu$, respectively. And so on for another $n - 1$ steps.

Example 7.8. Theorem 7.6 does not hold for $\lambda = \omega$. For a counterexample consider the finitary functor $F : \mathsf{Set} \times \mathsf{Set} \to \mathsf{Set}$ with $F(A, X) = X \times X + A$. Then SA is the set of all binary trees with leaves labelled in A (cf. Theorem 2.5). The functor S is not finitary. To see this, take $A = \mathbb{N}$ and write it as the directed union of the chain of finite ordinals n. Then the tree



lies in $T\mathbb{N}$ but not in any Tn for n a finite ordinal.

However, we have the following result.

Corollary 7.9. *If $\mathscr{C}$ and $\mathscr{D}$ are locally $\aleph_1$-presentable and $F : \mathscr{D} \times \mathscr{C} \to \mathscr{C}$ is finitary, then T is $\aleph_1$-accessible.*

Indeed, F is also $\aleph_1$-accessible, and so one can apply Theorem 7.6 with $\lambda = \aleph_1$.

Example 7.10. Let $\mathscr{C} = \mathscr{D} = \mathsf{Set}$. By λ-ary polynomial functors in $[\mathsf{Set}^2, \mathsf{Set}]$ we mean the smallest class of functors $F : \mathsf{Set}^2 \to \mathsf{Set}$ containing all constant functors and the projections $P_i(X_1, X_2) = X_i$, $i = 1, 2$, and which is closed under products of size less than λ and arbitrary coproducts. For every polynomial functor F the functor S is a polynomial functor on Set (associated to a λ-ary signature if λ is uncountable and to an $\aleph_1$-ary signature if $\lambda = \omega$). This follows from Theorem 2.5.

For example, for $F(A, X) = A \times X$, which is finitary, $SA = A^\omega$ is the polynomial functor associated to a signature with one ω-ary operation symbol.

Examples 7.11. (1) Mixed fixed points were considered by several authors. For example, Ghani et al. [37] consider the functor $F(X, Y) = B \times X + Y^A$ on Set and form $\nu X.\mu Y.F(X, Y)$. By Theorem 2.5, we know that $\mu Y.F(X, Y)$ consists of all A-branching well-founded trees with leaves labelled in $B \times X$. According to Example 7.10, this is the polynomial functor for the signature whose n-ary operations symbols are all A-branching trees with n leaves labelled in B. It follows from Theorem 2.5 that the desired mixed fixed point $\nu X.\mu Y.F(X, Y)$ consists of all partial labellings of the nodes of the complete A-branching tree by elements of B such that on every path from the root one finds an infinite sequence of labels in B.

(2) Note that the order of taking fixed points in mixed inductive/coinductive definitions matters. To see this, let us consider the functor $F(X,Y) = \{0,1\} \times X + \{2\} \times Y$. Then $\mu X.\nu Y.F(X,Y) = \{0,1\}^*$ is countable while $\nu Y.\mu X.F(X,Y) = (\{0,1\}^*)^\omega$ is uncountable.

8. Well-Founded Coalgebras

Throughout this section $\mathscr{C}$ denotes a complete and wellpowered category with constructive monomorphisms (see Definition 3.11).

We now discuss well-founded coalgebras, and we shall show that the initial algebra is precisely a terminal well-founded coalgebra. We have seen in Example 3.9(2) that for polynomial set functors the initial algebra is formed by all well-founded Σ-trees. This is no coincidence: in this section we describe elements of initial algebras as certain well-founded coalgebras.

The concept of *well-foundedness* is well known for graphs: it means that the graph has no infinite directed paths. Similarly for relations; for example, the elementhood relation $\in$ of set theory is well-founded; this is precisely the Foundation Axiom. Osius [56] observed that well-foundedness has an elegant categorical formulation based on the concept of *subcoalgebra*; this means a coalgebra homomorphism into the given coalgebra carried by a monomorphism in the base category.

For general coalgebra the concept of well-foundedness was introduced by Taylor [66]. We use now a somewhat simpler formulation:

Definition 8.1. Let $a : A \to FA$ be a coalgebra.

(a) A *cartesian subcoalgebra* is a subcoalgebra $m : (B,b) \to (A,a)$ for which the square below is a pullback.

$$\begin{tikzcd} B \arrow[r, "b"] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & FB \arrow[d, "Fm"] \\ A \arrow[r, "a"'] & FA \end{tikzcd} \tag{8.1}$$

We have indicated this with the "corner" symbol.

(b) *Well-founded coalgebras* are coalgebras without any proper cartesian subcoalgebras. That is, if $m : B \to A$ is a cartesian subcoalgebra, then m is an isomorphism.

Remark 8.2. A closely related concept are *recursive coalgebras* studied in [66] and [28]: these are coalgebras $a : A \to FA$ such that for every algebra $b : FB \to B$ a unique coalgebra-to-algebra morphism exists, i.e. a unique morphism $h : A \to B$ such that $h = b \cdot Fh \cdot a$. As proved in [12], if F preserves monomorphisms, then

$$\text{well-founded} \implies \text{recursive}.$$

And if F preserves finite intersections, then the converse holds, too.

Examples 8.3. (1) A graph is a well-founded coalgebra for the power-set functor $\mathscr{P}$ iff it is well-founded as a graph, i.e. it has no infinite paths.

(2) If an initial algebra μF exists, then (considered as a coalgebra) it is well-founded. Indeed, in every pullback (8.1), since a is invertible, so is b. The unique algebra homomorphism from μF to the algebra $b^{-1} : FB \to B$ is clearly inverse to m.

(3) A deterministic automaton, as a coalgebra for $F = \{0,1\} \times (-)^S$, is well-founded iff it is empty. Indeed, since $F\emptyset = \emptyset$, the empty subcoalgebra of any coalgebra is clearly cartesian.

(4) Non-deterministic automata can also be considered as coalgebras for $F = \{0,1\} \times (\mathscr{P}-)^S$ on Set. Such a coalgebra is well-founded iff the state transition graph is well-founded.

Theorem 8.4 (Adámek et al. [12], Theorems 2.48 and 2.46)**.** *The initial algebra can also be characterized as the terminal well-founded coalgebra. This holds for all set functors, and for all endofunctors of complete categories preserving finite intersections.*

Definition 8.5. Let F be a set functor preserving (wide) intersections.

(1) For each coalgebra $a : A \to FA$ and each $x \in FA$, there is a least subcoalgebra $B_x \subseteq A$ such that $x \in FB$. Indeed,

$$B_x = \bigcap\{A_0 \subseteq A : A_0 \text{ is a subcoalgebra, and } x \in FA_0\}.$$

We call B_x the *subcoalgebra generated by* x.

(2) For every coalgebra $a\colon A \to FA$ one defines the *canonical graph* on A: the neighbors of $x \in A$ are precisely those elements of A which lie in the least subset $m\colon M \hookrightarrow A$ with $a(x) \in Fm[FM] \subseteq FA$.

Gumm [39] observed that we obtain a "subnatural" transformation τ from F to the power-set functor $\mathscr{P}$ by defining the maps $\tau_X : FX \to \mathscr{P}X$ with

$$\tau_X(x) = \text{the least subset } m : M \hookrightarrow X \text{ with } x \in Fm[FM].$$

These maps do not form a natural transformation in general, but for every monomorphism $m : X \to Y$ we have a commutative square

$$\begin{tikzcd} FX \arrow[r, "\tau_X"] \arrow[d, "Fm"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{P}X \arrow[d, "\mathscr{P}m"] \\ FY \arrow[r, "\tau_Y"'] & \mathscr{P}Y \end{tikzcd}$$

which even is a pullback. The canonical graph of a coalgebra $a : A \to FA$ is $\tau_A \cdot a : A \to \mathscr{P}A$.

Taylor [66] proved the following result for functors preserving intersections and inverse images; the proof here is a small modification.

Lemma 8.6. *A coalgebra for a set functor preserving intersections is well-founded iff its canonical graph is well-founded.*

Proof. Suppose first that the coalgebra (A, a) is well-founded, and let us prove that $(A, \tau_A \cdot a)$ is a well-founded $\mathscr{P}$-coalgebra. To this end we consider a cartesian subcoalgebra $m : (B, b) \to (A, \tau_A \cdot a)$:

$$\begin{tikzcd} B \arrow[r, dotted, "\bar{b}"] \arrow[d, "m"'] \arrow[rr, bend left, "b"] \arrow[dr, phantom, "\lrcorner", very near start] & FB \arrow[r, "\tau_B"] \arrow[d, "Fm"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{P}B \arrow[d, "\mathscr{P}m"] \\ A \arrow[r, "a"'] & FA \arrow[r, "\tau_A"'] & \mathscr{P}A \end{tikzcd}$$

We know that the right-hand square is a pullback. The universal property of pullbacks implies that we have a morphism $\bar{b} : B \to FB$ making the above diagram commute. Since the outside and the right-hand part are pullbacks, so is the left-hand one. Thus, since (A, a) is a well-founded coalgebra, m is an isomorphism. This proves that the canonical graph $(A, a \cdot \tau_A)$ is well-founded.

Conversely, suppose the $\mathscr{P}$-coalgebra $(A, \tau_A \cdot a)$ is well-founded, and let (B, b) be a cartesian subcoalgebra of (A, a) via m. Then the left-hand and right-hand squares above are pullbacks, thus so is the outside. Hence, we obtain from the well-foundedness of the canonical graph that m is an isomorphism. $\square$

Remark 8.7. Recall that König's Lemma states that if a finitely branching graph has no infinite path, then it is finite. We generalize this as follows:

Generalized König's Lemma 8.8. *Every well-founded coalgebra of a finitary set functor F is finite.*

Proof. Let (A, a) be a well-founded F-coalgebra. Suppose first that F preserves all intersections. Then we construct the canonical graph $G = (A, \tau_A \cdot a)$. Since F is also finitary, G is finitely-branching graph (use Example 6.8(1) for $\lambda = \omega$). Moreover, G is well-founded by Lemma 8.6. And so by the ordinary König Lemma, G has only finitely many points. So the set A is finite.

In general, the finitary functor F need not preserve intersections. In this case, consider the Trnková hull $\bar{F}$ from Remark 3.15. Since $\bar{F}$ preserves finite intersections and filtered colimits, it preserves intersections. Indeed, for every element $x \in \bar{F}X$, there exists a finite set $m : Y \hookrightarrow X$ with x contained in Fm, see Example 6.8(1). Preservation of finite intersections implies that we can chose Y to be the least one. Preservation of intersections follows immediately: given subobjects $v_i : V_i \to X$, $i \in I$, with x contained in the image of Fv_i for each i, then x also lies in the image of the finite set $v_i \cap m$, hence $m \subseteq v_i$ by minimality. This proves $m \subseteq \bigcap_{i \in I} v_i$, thus, x lies in the image of $F(\bigcap_{i \in I} v_i$, as required.

Thus, the only fact we need to prove is that the given well-founded coalgebra $a : A \to FA$ is either empty, or (recalling that then $FA = \bar{F}A$) well-founded as an $\bar{F}$-coalgebra. Indeed, given a cartesian subcoalgebra for $\bar{F}$:

$$\begin{tikzcd} B \arrow[r, "b"] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & \bar{F}B \arrow[d, "Fm"] \\ A \arrow[r, "a"'] & \bar{F}A \end{tikzcd}$$

we prove that m is invertible. This is clear if $B \neq \emptyset$ since then the above square is a cartesian subcoalgebra for F. If $B = \emptyset$ then m is the empty map.

From (2) in Remark 3.15 we conclude that the following square

$$\begin{tikzcd}
\emptyset \arrow[r, "p_{F\emptyset}"] \arrow[d, "m=p_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & F\emptyset \arrow[d, "Fp_A=Fm"] \\
A \arrow[r, "a"'] & FA
\end{tikzcd}$$

is also a pullback. Thus p_A is invertible. So $A = \emptyset$. In particular, A is finite. □

Recall from [7, 14.21] that since our category is complete and well-powered, it has large intersections and (strong epi, mono) factorizations of morphisms. We denote by $\mathsf{Sub}(A)$ the complete lattice of all subobjects of A (where intersections give meets). Elements of $\mathsf{Sub}(A)$ are represented by monomorphismsm $m : A' \to A$.

Definition 8.9. Let F be an endofunctor preserving monomorphisms. For every coalgebra $a : A \to FA$ we denote by $\bigcirc$ the 'next-state' operator which is the endofunction on $\mathsf{Sub}(A)$ assigning to every subobject $m : A' \to A$ the pullback $\bigcirc m$ of Fm along a:

$$\begin{tikzcd}
\bigcirc A' \arrow[r, "a(m)"] \arrow[d, "\bigcirc m"'] \arrow[dr, phantom, "\lrcorner", very near start] & FA' \arrow[d, "Fm"] \\
A \arrow[r, "a"'] & FA
\end{tikzcd} \tag{8.2}$$

Since Fm is a monomorphism, so is $\bigcirc m$, and also $a(m)$ is unique.

Finally, observe that $\bigcirc$ is an order-preserving map: if $m \leq n$, then $\bigcirc m \leq \bigcirc n$.

Example 8.10. Let (A, a) be a graph, considered as a coalgebra for the power-set functor $\mathscr{P}$. If A' is a set of vertices of the graph, then $\bigcirc A'$ is the set of nodes all of whose successors belong to A'. This explains the name 'next-state' operator.

Proposition 8.11 (Adámek et al. [12], Corollary 2.19)**.** *A coalgebra* (A, a) *is well-founded iff the only fixed point of* $\bigcirc$ *is* id_A*.*

Proof. (1) Let (A, a) be well-founded. If $m : A' \to A$ is a fixed point of $\bigcirc$ then we have an isomorphism $x : A' \to \bigcirc A'$ with $m = \bigcirc m \cdot x$. Then

$a' = a(m) \cdot x : A' \to FA'$ yields a cartesian subcoalgebra:

$$\begin{tikzcd} A' \arrow[r, "a'"] \arrow[d, "m"'] & FA' \arrow[d, "Fm"] \\ A \arrow[r, "a"'] & FA \end{tikzcd}$$

By well-foundedness, m is invertible and thus represents the same subobject as id_A.

(2) Let $\bigcirc$ have id_A as the unique fixed point. Suppose that $m : (A', a') \to (A, a)$ is a cartesian subcoalgebra. Then the fact that $\bigcirc m$ is a pullback gives a morphism $x : A' \to \bigcirc A'$ such that, $a' = a(m) \cdot x$ and $\bigcirc m \cdot x = m$. It follows that $m \leq \bigcirc m$. Since m is cartesian, x is invertible. Hence m is a fixed point of $\bigcirc$, thus it is invertible. $\square$

Definition 8.12 (Adámek et al. [12])**.** By the *well-founded part* of a coalgebra is meant its greatest well-founded subcoalgebra.

Example 8.13. The well-founded part of a graph is the subcoalgebra on all vertices from which no infinite path starts.

Proposition 8.14. *If F preserves monomorphisms, then for every coalgebra the least fixed point of $\bigcirc$ is the well-founded part.*

Proof. Let (A, a) be a coalgebra. By Example 3.8, the least fixed point of $\bigcirc$ is $\bigvee_{i \in \mathsf{Ord}} m_i$ where the chain $m_i : A_i \to A$ is defined by transfinite recursion: $m_0 = \perp$, $m_{i+1} = \bigcirc m_i$ and for limit ordinals i we have $m_i = \bigvee_{j<i} m_j$. Let $m : B \to A$ be the least fixed point, then by the proof of Proposition 8.11 we know that a coalgebra structure $b : B \to FB$ exists for which m is a homomorphism. This subcoalgebra is well-founded because each cartesian subcoalgebra of (B, b) is a fixed point of $\bigcirc$, hence, is all of B.

Given another well-founded subcoalgebra $m' : (B', b') \to (A, a)$, we prove $m' \leq m$. Let $\bigcirc'$ denote the next-state operator of (B', b'). Its least fixed point is $\bigvee_{i \in \mathsf{Ord}} m'_i$ for a chain $m'_i : B_i \to B'$ analogous to $(m_i)_{i \in \mathsf{Ord}}$ above. We prove that $m'_i \leq m_i$ holds for every ordinal i by transfinite induction. Only the isolated steps need to be worked out: suppose $m'_i \leq m_i$, i.e. $m'_i = m_i \cdot d_i$ for some $d_i : B'_i \to A_i$. The desired morphism $d_{i+1} : B'_{i+1} \to A_{i+1}$ is obtained by the universal property of the inside pullback in the following diagram

$$
\begin{tikzcd}
B'_{i+1} \arrow[rrr, "b'(m'_i)"] \arrow[ddd, "m'_{i+1}"'] \arrow[dr, dotted, "d_{i+1}"] & & & FB'_i \arrow[dl, "Fd_i"'] \arrow[ddd, "Fm'_i"] \\
& A_{i+1} \arrow[r, "a(m_i)"] \arrow[d, "m_{i+1}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & FA_i \arrow[d, "Fm_i"] & \\
& A \arrow[r, "a"'] & FA & \\
B' \arrow[ur, "m'"] \arrow[rrr, "b'"'] & & & FB' \arrow[ul, "Fm'"']
\end{tikzcd}
$$

Since (B', b') is well-founded, the least fixed point of $\bigcirc'$ is all of B' which proves $m'_{i+1} \le m_{i+1}$ as desired. □

For set functors the assumption of preserving monomorphisms can be dropped:

Proposition 8.15. *Let F be a set functor. Every coalgebra has a well-founded part.*

Proof. We apply Remark 3.15. Let (A, a) be a coalgebra, we can assume that $A \neq \emptyset$. Then as a coalgebra for $\bar{F}$ it has by Proposition 8.14 a well-founded part $m : (B, b) \to (A, a)$. If this subcoalgebra is well-founded also for F, we are done. Thus suppose that (B, b) is not well-founded as an F-coalgebra. Then we prove that the F-well-founded part of (A, a) is empty.

To see this observe that Remark 3.15 implies that every nonempty well-founded F-coalgebra (C, c) is also well-founded as an $\bar{F}$-coalgebra. Otherwise it would have a cartesian proper subcoalgebra w.r.t. $\bar{F}$, but that subcoalgebra is necessarily empty (since (C, c) is well-founded as an F-coalgebra). And Remark 3.15(2) implies that the empty F-coalgebra is then also a cartesian subcoalgebra of (C, c), a contradiction.

Now the fact that (B, b) is not a well-founded F-coalgebra implies that the empty F-coalgebra is its cartesian subcoalgebra (since (B, b) is well-founded as an $\bar{F}$-coalgebra). We shall now prove that no nonempty F-subcoalgebra $m' : (B', b') \to (A, a)$ is well-founded. Indeed, if this were a well-founded F-coalgebra, then it would be contained in (B, b) (being well-founded w.r.t. $\bar{F}$). But then clearly the empty coalgebra is cartesian as an F-subcoalgebra of (B', b'), a contradiction. □

Theorem 8.16. *Let F be a set functor. The well-founded part of each fixed point of F is an initial algebra.*

Proof. Let $a : A \to FA$ be a fixed point of F. Let $m : B \to A$ be the well-founded part of A (see Proposition 8.15). Since B is well-founded, there is a coalgebra structure b so that the square below is a pullback.

$$\begin{tikzcd} B \arrow[r, "b"] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & FB \arrow[d, "Fm"] \\ A \arrow[r, "a"'] & FA \end{tikzcd}$$

Since a is an isomorphism, so is b.

By Theorem 3.13, F has an initial algebra, say $(I, i : FI \to I)$. By initiality, we have a morphism of coalgebras

$$\begin{tikzcd} I \arrow[r, "i^{-1}"] \arrow[d, "n"'] \arrow[dr, phantom, "\lrcorner", very near start] & FI \arrow[d, "Fn"] \\ B \arrow[r, "b"'] & FB \end{tikzcd}$$

Moreover, n is a split monomorphism: by Theorem 8.4 we have a coalgebra morphism $e : (B, b) \to (I, i^{-1})$. Then $e \cdot n = \mathsf{id}_I$ because $e \cdot n$ is an algebra endomorphism of (I, i). Since both horizontal arrows are invertible, this square, too, is a pullback. From the well-foundedness of (B, b), we see that n is an isomorphism. Thus, the algebra (B, b^{-1}) is initial, being isomorphic to the initial algebra via n. □

Corollary 8.17. *For every set functor F, the only well-founded fixed point is the initial algebra.*

Remark 8.18. We stated Theorem 8.16 for set functors. But it holds more generally. Suppose that the base category $\mathscr{C}$ is finitely complete and cocomplete, and that it has a constructive class $\mathscr{M}$ of monomorphisms for which $\mathscr{C}$ is $\mathscr{M}$-wellpowered. Then the statement holds for all endofunctors which preserve $\mathscr{M}$-monomorphisms.

In the remainder of this section F denotes a set functor preserving intersections. We will recall from [12] the construction of the terminal coalgebra using well-pointed coalgebras (see Definition 8.20) and that of the initial algebra using well-founded, well-pointed coalgebras.

Notation 8.19. A *pointed coalgebra* is a coalgebra (A, a) equipped with a chosen *initial state* $x \in A$. We denote by $\mathsf{Coalg}_p\, F$ the category of pointed F-coalgebras and coalgebra morphisms preserving initial states.

Definition 8.20 (Adámek et al. [12],Definition 3.15)**.** A *well-pointed* coalgebra is a pointed coalgebra (A, a, x) with no proper subcoalgebra and no proper quotient. More precisely, every injective morphism of $\mathsf{Coalg}_p\, F$ with codomain (A, a, x) is invertible, and so is every surjective one with domain (A, a, x).

Example 8.21. For the functor $F = \{0, 1\} \times (-)^S$ a pointed coalgebra is precisely a deterministic automaton. It is reachable iff it has not proper subcoalgebra and observable iff it has no proper quotient. Thus, well-pointed coalgebras are precisely the minimal deterministic automata.

Construction 8.22. Given a coalgebra (A, a) and a state $x \in A$, since F preserves intersections there is the least subcoalgebra $(A_0, a_0) \hookrightarrow (A, a)$ containing x. Now form the smallest quotient (i.e. the wide pushout of all quotients in $\mathsf{Coalg}\, F$) $e : (A_0, a_0) \to (A_1, a_1)$ in $\mathsf{Coalg}\, F$. The coalgebra $(A_1, a_1, e(x))$ is, as proved in [12], well-pointed. We denote it by

$$a^+(x) = (A_1, a_1, e(x)).$$

Note that this coalgebra is unique up to isomorphism.

Notation 8.23. Let T be a choice class of all well-pointed coalgebras. That is, every well-pointed coalgebra is isomorphic in $\mathsf{Coalg}_p\, F$ with precisely one member of T.

For every coalgebra (A, a), Construction 8.22 defines a function $a^+ : A \to T$.

Example 8.24. For $F = \{0, 1\} \times (-)^S$ we have the set T representing all minimal deterministic automata; it is canonically isomorphic to the set $\mathscr{P} S^*$ of all formal languages on the alphabet S. Indeed, every language is accepted by an essentially unique minimal automaton. And $a^+(x)$ is the minimization of a given automaton (A, a, x).

Theorem 8.25 (Adámek et al. [12], Theorem 3.24)**.** *A set functor F preserving intersections has a terminal coalgebra iff T is a set. Moreover, T carries the following structure $\tau : T \to FT$ of the terminal coalgebra assigning to every (A, a, x) in T the element of FT represented by the function*

$$1 \xrightarrow{x} A \xrightarrow{a} FA \xrightarrow{Fa^+} FT.$$

Example 8.26. For $F = \{0,1\} \times (-)^S$ the usual description $\nu F = \mathscr{P} S^*$ corresponds to our description of νF consisting of all minimal automata because those two coalgebras are isomorphic.

Remark 8.27. Even in the case where T is a proper class, (T, τ) above is still a terminal coalgebra. However, it is not a terminal coalgebra for F, but for the extension F' of F to the category of classes (given by $F'X = \bigcup FY$, where Y ranges over all subsets of the class X).

Theorem 8.28 (Adámek et al. [12], Theorem 3.40)**.** *Let F be a set functor preserving intersections. The initial F-algebra is the algebra consisting of all well-founded, well-pointed coalgebras. More precisely, let $T_0 \subseteq T$ be the collection of all well-founded coalgebras in T. Then μF exists iff T_0 is a set, and then the subcoalgebra T_0 of (T, τ) (considered as an algebra) is initial.*

Example 8.29. (1) For polynomial functors H_Σ, we saw in Theorem 2.5 that νH_Σ is formed by all Σ-trees. This is isomorphic to the set of all well-pointed coalgebras for H_Σ because every Σ-tree is the tree expansion of a well-pointed coalgebras, unique up to isomorphism.

The well-founded, well-pointed coalgebras yield as tree expansions precisely the well-founded Σ-trees.

(2) Since $F = \{0,1\} \times (-)^S$ fulfils $F\emptyset = \emptyset$, no pointed coalgebras is well-founded. Indeed, $\mu F = \emptyset$.

9. Classes and Non-Well-Founded Sets

In this short section we show that the slogan

> "every set functor has an initial algebra and a terminal coalgebra"

can be made into a true statement. But the resulting algebra or coalgebra need not be small. A beautiful example is presented by the power-set functor $\mathscr{P}$: its initial algebra is the class of all sets (in the usual well-founded set theory). And its terminal coalgebra is the class of all non-well-founded sets, as developed by Aczel in his book [2]. In the theory of non-well-founded sets the Foundation Axiom:

$$\text{there exists no inifinite sequence of sets } \cdots x_2 \in x_1 \in x_0$$

is replaced with the following Anti-Foundation Axioms:

every graph has a decoration.

A *decoration* of a given graph is a function d from the set of all vertices of the graph to the class of all sets such that for every vertex x we have $d(x) = \{d(y) \mid y \text{ a neighbor of } x\}$. For example, the decoration of the following graph

is the non-well-founded set $\{\{\{\cdots\}\}\}$.

Aczel proved that in non-well-founded set theory every set-based endofunctor on the category of classes has an initial algebra and a terminal coalgebra. He worked only with functors preserving weak pullbacks, but in subsequent work, Aczel and Mendler proved the same result without that assumption, and without working with non-well-founded sets [4]. Later, Adámek, Milius and Velebil [14] proved that, assuming the Axiom of Choice for classes, all endofunctors on the category of classes are set-based. Let us now explain these concepts and results.

In the following we work in a set theory which (as usual) distinguishes sets and classes (so that e.g. all sets form a class).

Notation 9.1. We denote by

$$\mathsf{Class}$$

the category of classes and functions between them.

The following definition is closely related to λ-accessibility, see Example 6.8(1).

Definition 9.2 (Aczel [2])**.** An endofunctor F on Class is called *set-based* if for every element $x \in FX$, where X is any class, there exists a *small subobject* $m : Y \hookrightarrow X$, i.e. where Y is a set, such that x lies in the image of Fm.

Example 9.3. The power-set functor $\mathscr{P}$ extends to a set-based functor $\bar{\mathscr{P}}$ on Class defined on a class X by

$$\bar{\mathscr{P}}X = \{Y \subseteq X \mid Y \text{ is a set}\}.$$

More generally, every set functor F extends to a set-based functor $\bar{F}$ on Class: for every class X let FX be the filtered colimit of $Fm[FY]$ for all small subobjects $m : Y \hookrightarrow X$.

Theorem 9.4 (Aczel and Mendler [4], Theorem 2.1)**.** *Every set-based endofunctor of* Class *has a terminal coalgebra.*

A simple proof was presented by Adámek et al. [14, Proposition 3.18]. Let us first extend the concept of a signature $\Sigma = (\Sigma_n)_{n\in\mathsf{Card}}$ by letting arities n range over all cardinal numbers and by allowing each Σ_n to be large (i.e. a proper class). We obtain an endofunctor H_Σ on Class completely analogous to polynomial endofunctors on Set: $H_\Sigma X = \coprod_{n\in\mathsf{Card}} \Sigma_x \times X^n$. Its terminal coalgebra T_Σ can be described as the coalgebra T_Σ of all Σ-trees, the proof is the same as that of Theorem 2.5. Every set-based functor F is a quotient of some H_Σ: indeed, put $\Sigma_n = Fn$ for every cardinal n and use the Yoneda lemma to get a natural transformation $\varepsilon : H_\Sigma \to F$ with surjective components.

The terminal H_Σ-coalgebra $\tau_\Sigma : T_\Sigma \to H_\Sigma T_\Sigma$ yields an F-coalgebra by composing τ_Σ with ε_{T_Σ}. An equivalence relation $\sim$ on T_Σ is called a *congruence* if $T_\Sigma/{\sim}$ carries the structure of an F-coalgebra τ for which the quotient function $T_\Sigma \to T_\Sigma/{\sim}$ is a coalgebra morphism. The wide pushout $\approx$ of all congruences is the largest congruence, and one then proves that the corresponding F-coalgebra $(T_\Sigma/{\approx}, \tau)$ is terminal. To see this, let $\beta : B \to FB$ be a coalgebra. Turn it into an H_Σ-coalgebra $(B, u\cdot\beta)$ by choosing a splitting of u of ε_B (i.e, $\varepsilon_B \cdot u = \mathsf{id}_{FB}$) and consider the unique coalgebra morphism $h : (B, u\cdot\beta) \to (T_\Sigma, \tau_\Sigma)$ for H_Σ. Compose h with the quotient function $e : T_\Sigma \to T_\Sigma/{\approx}$ to obtain a coalgebra morphism. Using that e is a coalgebra morphism and that ε is natural we see that the diagram below commutes:

$$\begin{tikzcd}
B \arrow[r, "\beta"] \arrow[rrr, bend left, "\beta"] \arrow[d, "h"'] & FB \arrow[r, "u"] & H_\Sigma B \arrow[r, "\varepsilon_B"] \arrow[d, "H_\Sigma h"] & FB \arrow[d, "Fh"] \\
T_\Sigma \arrow[rr, "\tau_\Sigma"] \arrow[d, "e"'] & & H_\Sigma T_\Sigma \arrow[r, "\varepsilon_{T_\Sigma}"] & FT_\Sigma \arrow[d, "Fe"] \\
T_\Sigma/{\approx} \arrow[rrr, "\tau"'] & & & F(T_\Sigma/{\approx})
\end{tikzcd}$$

Finally, uniqueness of the coalgebra morphism is easily seen: given two coalgebra morphisms $f_1, f_2 : (B, \beta) \to (T_\Sigma/{\approx}, \tau)$, their coequalizer $c : T_\Sigma/{\approx} \to T_\Sigma{\approx'}$ yields a congruence $c\cdot e : T_\Sigma \to T_\Sigma/{\approx'}$. Since e is the largest congruence, it follows that c must be an isomorphism, whence $f_1 = f_2$.

Corollary 9.5. *Every set-based endofunctor F of* Class *has an initial algebra. It is the transfinite colimit of the chain $F^i\emptyset$ ($i \in \mathsf{Ord}$).*

Indeed, since F has a fixed point, namely the terminal coalgebra νF, this corollary is proved precisely in the same way as Theorem 3.13.

In the following theorem we assume that our set theory is such that all proper classes have the same cardinality. This is true if we assume that the Axiom of Choice holds for classes.

Theorem 9.6 (Adámek, Milius and Velebil [14], Theorem 2.2)**.** *Assuming the Axiom of Choice for classes, every endofunctor on* Class *is set-based.*

Proof. The Axiom of Choice for classes implies that all classes are isomorphic. Let F be an endofunctor of Class. We can assume that F preserves finite intersections: one can use Remark 3.15 noting that Trnková's proof [68] is easily seen to work in Class as well as in Set. Given a class X there exists a collection of subclasses $X_i \subseteq X$, $i \in I$, such that for $i \neq j$ the intersection $X_i \cup X_j$ is always a set, and I strictly larger than any class; this was proved by Sierpinski [62] and Tarski [65]. We can assume without loss of generality that X_i is a proper class for each i (since by discarding all small members, the collection retains the above properties). Thus, there exists isomorphisms $w_i : X \to X_i$, $i \in I$.

For every element $x \in FX$ we are to prove that there exists a subset $m : Y \hookrightarrow X$ with x contained in the image of Fm. Let $v_i : X_i \hookrightarrow X$ be the inclusion map. The elements $y_i = F(v_i \cdot w_i)(x)$ of X cannot be pairwise distinct because X is a class and I is strictly larger than X. Given $i \neq j$ with $y_i = y_j$, since F preserves the intersection $v_i \cap v_j$, we conclude that $Y = X_i \cap X_j$ has the desired property. □

Corollary 9.7. *Assuming the Axiom of Choice for classes, every endofunctor on* Class *has an initial algebra and a terminal coalgebra.*

Example 9.8. (1) The initial algebra for the power-set functor is carried by the class S of all sets. More precisely, this is $\mu\bar{\mathscr{P}}$ for the extension $\bar{\mathscr{P}}$ of Example 9.3. This follows from Example 3.10. Observe that $\bar{\mathscr{P}}(S) = S$ and the coalgebra structure is identity.

(2) The terminal coalgebra for $\bar{\mathscr{P}}$ was described by Turi and Rutten [70]; it is carried by the class of all non-well-founded sets.

10. Completely Iterative Algebras

We have seen in Theorem 8.4 that an initial algebra is characterized by the universal property of being the terminal well-founded coalgebra. We now turn to a similar characterization of terminal coalgebras. This characterization arises in connection with an important property of terminal coalgebras (considered as algebras): they are *completely iterative*, i.e. they admit unique solutions of certain recursive equations. The idea of such algebras stems from work in general algebra by Evelyn Nelson [55] und Jerzy Tiuryn [67]. Nelson introduced iterative algebras for a signature as algebras with unique solutions of finite recursive systems of equations. Dropping the finiteness assumption one arrives at the notion of a completely iterative algebra, introduced and studied in [47].

Assumption 10.1. Throughout the rest of this section we assume that $\mathscr{C}$ is a category with binary coproducts.

Definition 10.2 (Milius [47])**.** Let F be an endofunctor. By a *flat equation morphism* in an object A we mean a morphism $e : X \to FX + A$. A *solution* of e in an algebra $a : FA \to A$ is a morphism $e^\dagger : X \to A$ such that the square below commutes:

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & A \\ FX + A \arrow[r, "Fe^\dagger + A"'] & FA + A \arrow[u, "{[a, A]}"'] \end{tikzcd} \tag{10.1}$$

Finally, an F-algebra (A, a) is called *completely iterative* (or, *cia* for short) provided that every flat equation morphism in A has a unique solution.

Remark 10.3. The terminology "equation morphism" and "solution" stems from the fact that for a polynomial set functor H_Σ, flat equations morphisms can be interpreted as systems of mutually recursive equations. In fact, a flat equation morphism in a Σ-algebra A is given by a set $X = \{x_i \mid i \in I\}$ of variables and a system

$$x_i \approx t_i, \qquad i \in I \tag{10.2}$$

of formal equations, where t_i is either a flat term $\sigma(\vec{x})$ or an element of A. A solution $e^\dagger : X \to A$ then assigns to every variable an element of A, turning the formal equations $\approx$ into actual identities in A. For example, consider the polynomial functor $FX = 1 + X \times X$ associated to the signature Σ with one constant symbol c and one binary operation symbol $*$. Then the recursive equation system

$$x_1 \approx x_2 * x_1 \qquad x_2 \approx c$$

corresponds to a flat equation morphism and its solution in the algebra $A = \nu F$ of all binary trees is given by $e^\dagger$ with

$$e^\dagger(x_2) = c \qquad \text{and} \qquad e^\dagger(x_1) = (c * (c * (c * \cdots))).$$

Proposition 10.4 (Milius [47], Theorem 2.8)**.** *The terminal coalgebra νF (considered as algebra) is a cia.*

Proof. Let $e : X \to FX + \nu F$ be a flat equation morphism. From e we form the following F-coalgebra

$$\overline{e} = (X + \nu F \xrightarrow{[e,\mathsf{inr}]} FX + \nu F \xrightarrow{FX+t} FX + F(\nu F) \xrightarrow{\mathsf{can}} F(X + \nu F)),$$

where $\mathsf{can} = [F\mathsf{inl}, F\mathsf{inr}]$. Let $h : X + \nu F \to \nu F$ be the corresponding unique coalgebra homomorphism and define

$$e^\dagger = (X \xrightarrow{\mathsf{inl}} X + \nu F \xrightarrow{h} \nu F).$$

One readily shows that $h \cdot \mathsf{inr}$ is a coalgebra homomorphism from νF to itself, whence $h \cdot \mathsf{inr} = \mathsf{id}$. Now it is not difficult to prove that $e^\dagger$ is a solution of e iff $[e^\dagger, \mathsf{id}]$ is a coalgebra homomorphism from $(X + \nu F, \overline{e})$ to the terminal coalgebra $(\nu F, t)$. Since the latter exists uniquely, so does the former. For further details details see [47, Example 2.5]. □

Examples 10.5. (1) The collection of all (finite and infinite) binary trees is a cia for $FX = 1 + X \times X$. The collection of all finitely branching strongly extensional trees is a cia for $\mathscr{P}_\mathsf{f}$, see Example 3.23(2).

(2) We denote by $\mathscr{B}$ the bag functor; this functor assigns to a set X the set of finite multisets on X, i.e. the free commutative monoid on X. It terminal coalgebra can be described as the coalgebra of all unordered finitely branching trees, hence this is a cia for $\mathscr{B}$.

(3) The algebra of addition on the extended natural numbers

$$\widetilde{\mathbb{N}} = \{ 1, 2, 3, \dots \} \cup \{ \infty \}$$

is a cia for the functor $FX = X \times X$, see [16].

(4) As shown in [47] a unary algebra $a : A \to A$ (here $F = \mathsf{Id}$ on Set) is a cia if and only if

    (a) there exists a unique fixed point $x_0 \in A$ of a and

    (b) if an infinite sequence $y_0, y_1, y_2, \dots$ fulfils $a(y_{n+1}) = y_n$ $(n < \omega)$ then $y_n = x_0$ for all n.

(5) Classical algebras are seldom cias. Consider one binary operation: $FX = X \times X$. A group is a cia iff its unique element is the unit 1, since the recursive equation $x \approx x \cdot 1$ has a unique solution. A lattice is a cia iff it has a unique element; consider the equation $x \approx x \vee x$.

In the category CMS of complete metric spaces cias are abundant; the following is essentially a consequence of Banach's fixed point theorem:

Proposition 10.6 (Adámek, Milius, and Velebil [15], Lemma 2.9)**.** *Let $F : \mathsf{CMS} \to \mathsf{CMS}$ be a locally contracting endofunctor (see Definition 5.2). Then every non-empty algebra is completely iterative.*

This result yields further interesting examples of cias, where solutions of recursive equations are fractals. The most basic of these examples is the following:

Example 10.7 (Milius and Moss [51])**.** Let A be the set of closed subsets of the interval $[0, 1]$. Then A is a complete metric space equipped with the *Hausdorff metric*:

$$d(S, T) = \max\{\sup_{x \in S} \inf_{y \in T} d(x, y), \sup_{y \in T} \inf_{x \in S} d(x, y)\}.$$

We also consider the functor $F(X, d) = (X \times X, \frac{1}{3} d_{\mathsf{max}})$ sending the complete metric space (X, d) to $X \times X$ equipped with the maximum metric scaled by $\frac{1}{3}$; this functor is clearly locally contracting. Then A is an algebra for F with structure a given by

$$a(S, T) = \frac{1}{3} S \cup \left(\frac{1}{3} T + \frac{2}{3} \right)$$

with the obvious interpretation of addition and multiplication on the closed subsets $S, T \subseteq [0, 1]$. Now let X be the one point space and $e : X \to FX + A$ be given by $e(*) = (*, *)$ where $*$ denotes the element of X. Then $e^{\dagger}(*)$ is the famous Cantor dust.

Remark 10.8. One of the reasons why cias are interesting is that one can uniquely solve much more general recursive equation than the above flat ones in them. For example, [3, 47] contain a solution theorem which, when instantiated for a polynomial set functor, states that mutually recursive systems (10.2), where the right-hand sides t_i are arbitrary non-trivial Σ-trees have unique solutions. In [51], cias are shown to admit unique solutions of mutually recursive function definitions, so called *recursive program schemes*, and in [52] it is demonstrated how Turi and Plotkin's abstract operational rules [57] give rise to new cia structures on the terminal coalgebra. The latter yield theorems that instantiate to several known unique solution theorems in the literature (e.g. Milner's solution theorem for CCS [54]) and yield a modular framework for the specification of operations by abstract operational rules; modularity here means that unique solution theorems are preserved when adding operations specified by abstract GSOS rules to a process calculus.

Cias constitute a full subcategory of the category of all F-algebras. That the choice of all homomorphisms of F-algebras is appropriate for cias follows from the fact, established in [47, Proposition 2.3], that algebra morphisms preserve solutions of flat equations in the obvious sense.

We now turn to constructions of cias in our general category $\mathscr{C}$, which will then lead to the proof that initial cias and terminal coalgebras are the same.

Lemma 10.9. *If (A, a) is a cia, then so is (FA, Fa).*

Proof. Let $e : X \to FX + FA$ be a flat equation morphism in FA. Form the equation morphism

$$\overline{e} = (X \xrightarrow{e} FX + FA \xrightarrow{FX+a} FX + A)$$

and define

$$e^\dagger = (X \xrightarrow{e} FX + FA \xrightarrow{[\overline{e}^\dagger, FA]} FA).$$

It is not difficult to check that $e^\dagger$ is a unique solution of e in FA. For the details see [47, Proposition 2.6]. □

Proposition 10.10. *The category of cias is closed under all limits that exist in $\mathscr{C}$.*

The proof is analogous to the proof of a similar result [16, Proposition 2.20] concerning the weaker notion of iterative algebras, and so we omit it.

Corollary 10.11. *Let $\mathscr{C}$ be a complete category. Then in the terminal-coalgebra chain (Definition 3.20) all algebras $v_{i+1,i} : F(F^i 1) = F^{i+1}1 \to F^i 1$, $i \in \mathsf{Ord}$, are cias.*

Proof. This is easy to see by transfinite induction. In fact, $v_{1,0} : F1 \to 1$ is trivially a cia. For successor steps use Lemma 10.9. Finally, for the limit step apply Proposition 10.10. □

Theorem 10.12 (Milius [47], Theorem 2.8)**.** *The initial cia is precisely the same as the terminal coalgebra, i.e. (I, i) is an initial cia iff (I, i^{-1}) is a terminal coalgebra.*

Proof. (1) For a coalgebra $c : C \to FC$ and an algebra $a : FA \to A$ consider the following diagram:

$$\begin{tikzcd}
C \arrow[r, "h"] \arrow[d, "c"] \arrow[dd, bend right=60, "e"'] & A \\
FC \arrow[r, "Fh"'] \arrow[d, "\mathsf{inl}"] & FA \arrow[u, "a"'] \arrow[d, "\mathsf{inl}"'] \\
FC + A \arrow[r, "Fh+\mathsf{id}"'] & FA + A \arrow[uu, bend right=60, "{[a,A]}"']
\end{tikzcd} \tag{10.3}$$

Let us define $e = \mathsf{inl} \cdot c$ so that the left-hand part of this diagram commutes. Notice also that the right-hand part and the lower square of the diagram obviously commute. Now the outside of the diagram commutes iff the upper square does; in other words, h is a coalgebra-to-algebra homomorphism from (C, c) to (A, a) iff it is a solution of the flat equation morphism e in the algebra A.

(2) Suppose that $i : FI \to I$ is an initial cia. The algebra (FI, Fi) is a cia by Lemma 10.9, and then an argument similar to Lambek's Lemma shows that i is an isomorphism. So we have the coalgebra $i^{-1} : I \to FI$, and we need to verify that it terminal. Indeed, for every coalgebra (C, c) replace (A, a) in Diagram (10.3) by the algebra (I, i). Then since this algebra is a cia we have a unique coalgebra-to-algebra homomorphism, i.e. a unique coalgebra homomorphism from (C, c) to (I, i^{-1}).

(3) Conversely, suppose that $\tau : \nu F \to F(\nu F)$ is a terminal coalgebra. Then it is a cia by Proposition 10.4, and it remains to verify its initiality. So let (A, a) be any cia and let (C, c) in Diagram (10.3) be $(\nu F, \tau)$. Since A is a cia we have a unique solution of $e = \mathsf{inl} \cdot \tau$ in A, equivalently a unique coalgebra-to-algebra homomorphism from $(\nu F, \tau)$ to (A, a), i.e. a unique F-algebra homomorphism from the cia νF to the cia A. □

Remark 10.13. (1) An analogous result to Theorem 10.12 can be stated for free cias: TX is a free cia on the object X iff it is a terminal coalgebra for the functor $F(-) + X$. Assuming the existence of a free cia TX on every object X of $\mathscr{C}$, it turns out that T is the object assigment of a monad on $\mathscr{C}$. Furthermore this monad is characterized by a universal property: it is the free completely iterative monad on F. These results appear in [47, 3] generalizing work on (free) completely iterative theories by Elgot, Bloom and Tindell [34].

(2) Dually to recursive coalgebras in Remark 8.2 we have *corecursive algebras*. They were introduced by Capretta, Uustalu and Vene [29] and further studied in [6]. A corecursive algebra for F is an F-algebra A such that for every F-coalgebra there exists a unique coalgebra-to-algebra morphism into A. It is easy to see that

$$\text{completely iterative} \implies \text{corecursive}.$$

However, the converse does not hold; see [29] for a counterexample.

(3) A free corecursive algebra on an object X was described in [6]. Assuming that a terminal coalgebra νF and a free algebra ΦX on X exist, the free corecursive algebra is their coproduct $\nu F \oplus \Phi X$ in the category of algebras for F. For example, for a polynomial endofunctor on Set this means that the free corecursive algebra is formed by all Σ-trees that have only finitely many leaves labelled in X (but possibly infinitely many leaves labelled by constant symbols from Σ). *Op. cit.* also gives a construction of free corecursive algebras by a transfinite recursion which reaches a fixed point in the following way: $U_0 = \nu F$, $U_{i+1} = FU_i + X$, and at limit ordinals take colimits.

11. The Rational Fixed Point

We turn our attention to the rational fixed point of a functor F. Like the initial fixed point μF and the terminal one νF, this third fixed point has a characterization as an initial object (as an algebra) and a terminal object (as a coalgebra).

Assumption 11.1. Throughout the rest of this section we assume that $\mathscr{C}$ is a locally finitely presentable category and that F is a finitary endofunctor on $\mathscr{C}$ (see Definition 6.7).

11.1. Locally Finitely Presentable Coalgebras

We first turn to the coalgebraic description of the rational fixed point as the terminal locally finitely presentable coalgebra (see [48]). Let us denote by

$$\mathsf{Coalg}_{\mathsf{f}}\, F$$

the full subcategory of $\mathsf{Coalg}\, F$ formed by all coalgebras on finitely presentable carriers, e.g. finite coalgebras in the case where $\mathscr{C} = \mathsf{Set}$.

Definition 11.2 (Bonsangue, Milius, and Silva [27], Milius [48], Definition 3.7)**.** An F-coalgebra (X, c) is called *locally finitely presentable* (or *lfp*, for short) if it is the colimit of the diagram formed by all coalgebra morphisms from coalgebras in $\mathsf{Coalg}_{\mathsf{f}}\, F$ to (X, c). Equivalently, the canonical forgetful functor

$$\mathsf{Coalg}_{\mathsf{f}}\, F/(X, c) \to \mathscr{C}/X$$

is cofinal.

In lieu of spelling out the meaning of this definition explicitly we recall the following result, which gives a structural characterisation of lfp coalgebras that is often more useful:

Theorem 11.3 (Milius [48], Corollary 3.13)**.** *A coalgebra is lfp iff it is a filtered colimit of a diagram of coalgebras from $\mathsf{Coalg}_{\mathsf{f}}\, F$.*

Examples 11.4. Let us recall from [48, 27] more concrete descriptions of lfp coalgebras in some categories of interest.

(1) If $\mathscr{C} = \mathsf{Set}$ then a coalgebra is lfp iff it is *locally finite*, which means that every element of its carrier is contained in a finite subcoalgebra.

(2) Suppose that $\mathscr{C}$ is a *locally finite variety*, i.e. a category of algebras specified by finitary operations and equations such that every free algebra on a finite set of generators is finite. Examples include Boolean algebras, distributive lattices or join semi-lattices. Then an F-coalgebra is lfp iff every finite subalgebra of its carrier is contained in a finite subcoalgebra.

(3) Let $\mathscr{C}$ be the category of vector spaces over a fixed field. Then a coalgebra (X, c) is lfp if and only if every element of its carrier is contained in a subcoalgebra whose carrier is finite-dimensional, i.e. (X, c) is *locally finite-dimensional*.

From now on we denote the category of lfp F-coalgebras by $\mathsf{Coalg}_{\mathsf{lfp}}\, F$.

Remark 11.5. Let $D : \mathscr{D} \to \mathscr{C}$ be a filtered diagram in a locally finitely presentable category. A cocone $w_i : Di \to C$, $i \in \mathscr{D}$ is a colimit cocone iff for every morphism $r : R \to C$ with R finitely presentable, the following hold:

(a) r factorizes through some w_i, i.e. there exists $i \in D$ and r' such that $r = w_i \cdot r'$, and

(b) given two factorizations $r = w_i \cdot r_t$ $(t = 1, 2)$ there exists a morphism $h : i \to j$ in $\mathscr{D}$ such that the corresponding connecting morphism of the diagram D merges the r_t, i.e. such that $Dh \cdot r_1 = Dh \cdot r_2$.

Indeed, this states precisely that the hom-functors $\mathscr{C}(R, -)$ of finitely presentable objects R collectively create filtered colimits; this follows from [21, 1.26] applied to the category $\mathscr{A}$ of finitely presentable object of $\mathscr{C}$.

Lemma 11.6. *Let $\mathscr{C}$ be a category with finite colimits. Then every diagram in $D : \mathscr{D} \to \mathscr{C}$ can be extended to a filtered diagram with the same colimit (in fact, essentially the same cocones).*

Proof. Let $J : \mathscr{D} \to \hat{\mathscr{D}}$ be the free completion of $\mathscr{D}$ under finite colimits. Recall that this implies that precomposition with J yields an equivalence between the functor category $[\mathscr{D}, \mathscr{C}]$ and the category of all functors in $[\hat{\mathscr{D}}, \mathscr{C}]$ preserving finite colimits. A construction of $\hat{\mathscr{D}}$ can be found e.g. in [42].

Thus, D extends to the functor $\hat{D} : \hat{\mathscr{D}} \to \mathscr{C}$ given by left Kan extension along J, i.e. $\hat{D} = \mathsf{Lan}_J\, D$. Furthermore, cocones on $\hat{D}$ with vertex X, which are natural transformations from $\hat{D}$ to $\Delta_X : \hat{D} \to \mathscr{C}$, the constant functor on X, bijectively correspond to natural transformations from D to $\Delta_X \cdot J$ via the bijection

$$\mathsf{Nat}(\hat{D}, \Delta_X) \cong \mathsf{Nat}(D, \Delta_X \cdot J) \tag{11.1}$$

that maps a cocone $d : \hat{D} \to \Delta_X$ to its restriction $dJ : D \cong \hat{D} \cdot J \to \Delta_X \cdot J$. On the right $\Delta_X \cdot J$ is also the constant functor with domain $\mathscr{D}$. The latter are precisely the cocones on D with vertex X. This implies that D and $\hat{D}$

have the same colimit; indeed, a cocone $c : \hat{D} \to \Delta_C$ is a colimit of $\hat{D}$ iff we have a bijection $\mathscr{C}(C, X) \cong \mathsf{Nat}(\hat{D}, \Delta_X)$ given by precomposition with (the components of) c. Now compose that last bijection with the one in (11.1) to see that C with the restricted cocone cJ is, equivalently, a colimit of D. $\square$

Proposition 11.7. *If (C, c) is an lfp coalgebra, then so is (FC, Fc).*

Proof. It is sufficient to prove the result for coalgebras with C finitely presentable. The general case then follows: given $c : C \to FC$ in $\mathsf{Coalg}_{\mathsf{lfp}}\, F$, expressed as a filtered colimit of coalgebras $c_i : C_i \to FC_i$ $(i \in I)$ with C_i finitely presentable, then $Fc : FC \to FFC$ is a filtered colimit of the coalgebras (FC_i, Fc_i) since F is finitary. Thus (FC, Fc) is lfp since each (FC_i, Fc_i) is lfp, See Theorem 11.3.

Now suppose $c : C \to FC$ is a coalgebra with C finitely presentable. We prove that (FC, Fc) is a colimit of the diagram D in $\mathsf{Coalg}_{\mathsf{f}}\, F$ that we now describe. An object of D is determined by an (arbitrary) morphism $p : P \to FC$ with P finitely presentable. Given p we form the following coalgebra of $\mathsf{Coalg}_{\mathsf{f}}\, F$:

$$\bar{p} = (P + C \xrightarrow{[p,c]} FC \xrightarrow{F\mathsf{inr}} F(P + C)).$$

Let D be the diagram of all these coalgebras, where connecting morphisms from $(P + C, \bar{p})$ to $(Q + C, \bar{q})$ are those coalgebra morphisms h which fulfil $[p, c] = [q, c] \cdot h$:

$$\begin{tikzcd} & P + C \arrow[r, "\bar{p}"] \arrow[dl, "{[p,c]}"'] \arrow[d, "h"] & F(P + C) \arrow[d, "Fh"] \\ FC & Q + C \arrow[r, "\bar{q}"'] \arrow[l, "{[q,c]}"] & F(Q + C) \end{tikzcd} \tag{11.2}$$

By proving that (FC, Fc) is the colimit of D we know that this coalgebra is lfp: while D is itself not filtered, its closure under finite colimits in $\mathsf{Coalg}_{\mathsf{f}}\, F$ is and has the same colimit (see Lemma 11.6). We first prove some preliminary facts:

(a) The unique morphism $u : 0 \to FC$, where 0 is the initial object, yields $\bar{u} = (c : C \to FC)$ as object of D. Moreover, $c : C \to FC$ itself yields the object $(C + C, \bar{c})$ for which the codiagonal $\nabla : C + C \to C$ is a connecting morphism from $(C+C, \bar{c})$ to $(C, \bar{u})$ as shown by the diagram below:

$$
\begin{tikzcd}
& C+C \arrow[dl, "{[c,c]}"'] \arrow[r, "{[c,c]}"] \arrow[dd, "\nabla"'] \arrow[rr, bend left, "\bar{c}"] & FC \arrow[r, "F\mathsf{inr}"] \arrow[ddr, equal] & F(C+C) \arrow[dd, "F\nabla"] \\
FC & & & \\
& C \arrow[ul, "c"] \arrow[r, "c"'] \arrow[rr, bend right, "\bar{u}"'] & FC \arrow[r, equal] & FC
\end{tikzcd}
$$

(b) Every morphism m from $p : P \to FC$ to $q : Q \to FC$ in the slice category $\mathscr{C}_{\mathsf{fp}}/FC$, i.e. $q \cdot m = p$, yields the connecting morphism $m + C$ of D as shown by the commutative diagram below:

$$
\begin{tikzcd}
& P+C \arrow[dl, "{[p,c]}"'] \arrow[r, "{[p,c]}"] \arrow[dd, "m+C"'] \arrow[rr, bend left, "\bar{c}"] & FC \arrow[r, "F\mathsf{inr}"] \arrow[dd, equal] \arrow[ddr, "F\mathsf{inr}"] & F(P+C) \arrow[dd, "F(m+C)"] \\
FC & & & \\
& Q+C \arrow[ul, "{[q,c]}"] \arrow[r, "{[q,c]}"'] \arrow[rr, bend right, "\bar{q}"'] & FC \arrow[r, "F\mathsf{inr}"'] & F(Q+C)
\end{tikzcd}
$$

(c) D has the following cocone in $\mathsf{Coalg}\, F$:

$$[p, c] : (P + C, \bar{p}) \to (FC, Fc),$$

where p ranges over $\mathscr{C}_{\mathsf{fp}}/FC$. Indeed, the fact that each $[p, c]$ is a coalgebra morphism is clear:

$$
\begin{tikzcd}
P+C \arrow[r, "{[p,c]}"] \arrow[d, "{[p,c]}"'] & FC \arrow[r, "F\mathsf{inr}"] \arrow[dr, "Fc"] & F(P+C) \arrow[d, "{F[p,c]}"] \\
FC \arrow[rr, "Fc"'] & & FFC
\end{tikzcd}
$$

Compatibility follows from the left-hand triangle in Diagram (11.2).

(d) We are ready to prove that the above cocone with vertex (FC, Fc) is a colimit of D. In order to prove the universal property we use that the forgetful functor $U : \mathsf{Coalg}\, F \to \mathscr{C}$ creates all colimits (see Remark 6.5).

Hence, it suffices to prove that $[p, c] : P + C \to FC$, for all p in $\mathscr{C}_{\mathsf{fp}}/FC$, form a colimit of UD. So suppose that

$$a_p : P + C \to A \quad (p \text{ in } \mathscr{C}_{\mathsf{fp}}/FC)$$

is a cocone of UD in $\mathscr{C}$. We shall show that there exists a unique $a : FC \to A$ such that $a_p = a \cdot [p, c]$.

First, it is easy to see that morphisms $a_p \cdot \mathsf{inl} : P \to A$ form a cocone of the canonical diagram $\mathscr{C}_{\mathsf{fp}}/FC \to \mathscr{C}$ whose colimit is FC: in fact, given a morphism m as in (b) above, the given cocone fulfils $a_q \cdot (m + C) = a_p$, hence, $a_q \cdot \mathsf{inl} \cdot m = a_p \cdot \mathsf{inl}$. Therefore there exists a unique $a : FC \to A$ such that $a_p \cdot \mathsf{inl} = a \cdot p$ for all p in $\mathscr{C}_{\mathsf{fp}}/FC$. Thus, to conclude $a_p = a \cdot [p, c]$, it only remains to prove that $a_p \cdot \mathsf{inr} = a \cdot c$ holds for all p. The connecting morphism ∇ in part (a) yields

$$a_u \cdot \nabla = a_c : C + C \to A.$$

This implies that

$$a_u = a_c \cdot \mathsf{inl} = a \cdot c$$

by the definition of a. Moreover, the unique morphism $u : 0 \to P$ yields the connecting morphism $u + C = \mathsf{inr} : (C, \bar{u}) \to (P + C, \bar{p})$ using part (b). Thus, we conclude

$$a_p \cdot \mathsf{inr} = a_u = a \cdot c$$

as desired. □

Definition 11.8 (Adámek, Milius, Velebil [16], Section 3)**.** The *rational fixed point*

$$\varrho F \xrightarrow{r} F(\varrho F) = \operatorname{colim}(\mathsf{Coalg_f}\, F \hookrightarrow \mathsf{Coalg}\, F)$$

is the coalgebra obtained as the colimit of all coalgebras on finitely presentable carriers. We show in Theorem 11.10 that its structure r is invertible.

Theorem 11.9. *The the rational fixed point $(\varrho F, r)$ is the terminal lfp coalgebra.*

Indeed, this is a consequence of the fact established in [50, Theorem 2.7] that $\mathsf{Coalg_{lfp}}\, F$ is the Ind-completion of $\mathsf{Coalg_f}\, F$, i.e. the completion of that category under filtered colimits. Hence $\mathsf{Coalg_{lfp}}\, F$ is a locally finitely presentable category, whence complete and has therefore a terminal object. Moreover, it is easy to see that the terminal object is given by the colimit of the inclusion of the category $\mathsf{Coalg_f}\, F$ into its Ind-completion.

Theorem 11.10. *The rational fixed point is a fixed point of F.*

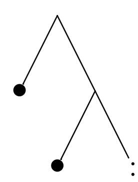
This was proved in [16, Theorem 3.3]. We provide an alternative proof similar to the proof of Lambek's Lemma for terminal coalgebras.

Proof. By Proposition 11.7 we know $(F(\varrho F), Fr)$ is an lfp coalgebra. Therefore we have a coalgebra morphism $s : (F(\varrho F), Fr) \to (\varrho F, r)$. Since $r : (\varrho F, r) \to (F(\varrho F), Fr)$ is clearly coalgebra morphism, we have that $s \cdot r$ is a coalgebra morphism from $(\varrho F, r)$ to itself. Thus, $s \cdot r = \mathsf{id}$ by Theorem 11.9. It follows that

$$r \cdot s = Fs \cdot Fr = F(s \cdot r) = F\mathsf{id} = \mathsf{id}. \qquad \square$$

Examples 11.11. (1) For a polynonial endofunctor H_Σ on Set, an initial algebra is carried by the set of all finite Σ-trees and a terminal coalgebra by the set of all (finite and infinite) Σ-trees (see Theorem 2.5). The rational fixed-point ϱF lies in between; it is carried by the set of all *rational* Σ*-trees*, i.e. Σ-trees having (up to isomorphism) only finitely many subtrees (see Ginali [38]).

(2) As a concrete example consider $FX = 1 + X \times X$ on Set. Its terminal coalgebra consists of all binary trees, see Example 3.6. The rational fixed point is the subcoalgebra formed by all rational binary trees. For example, all finite trees and the following infinite one



are rational, but the tree represented by

$$(t_0 * (t_1 * (t_2 * (\cdots) \cdots))),$$

where t_i is the complete binary tree of depth i, is not rational.

(3) Similarly, all rational finitely branching strongly extensional trees form the rational fixed point of $\mathscr{P}_{\mathsf{f}}$ (cf. Example 2.3(2) and see [16]).

(4) Let $FX = \{0,1\} \times X^\Sigma$ be the functor whose coalgebras are deterministic automata (see Example 2.7). Recall that the terminal coalgebra νF consists of all formal languages. Its subcoalgebra formed by all regular languages is the rational fixed point for F (see [16, 11]).

(5) Let $FX = \mathbb{R} \times X$ with $\nu F = \mathbb{R}^\omega$ the streams of reals, i.e. infinite sequences $\sigma = (\sigma(0), \sigma(1), \sigma(2), \ldots)$ (cf. Example 2.2). Streams have been studied in a coalgebraic setting by Jan Rutten [59, 60]. The convolution product of two streams is given by

$$(\sigma \times \tau)(n) = \sum_{i=0}^{n} \sigma(i) \cdot \tau(n-i)$$

and for streams σ with $\sigma(0) \neq 0$ there exists an inverse σ^{-1}, i. e., $\sigma \times \sigma^{-1} = (1, 0, 0, 0, \ldots)$. A *rational* stream is a stream of the form $\sigma \times \tau^{-1}$, where σ and τ have finitely many non-zero entries and $\tau(0) \neq 0$. When F is considered as an endofunctor on the category of real vector spaces then the rational fixed point for F consists of all rational streams F. However, when we consider F as an endofunctor on Set the rational fixed point for F is given by streams σ that are eventually periodic, i. e., those streams $\sigma = u\overline{v}$, where u and v are finite words on $\mathbb{R}$ (see [48]).

(6) The previous example can be generalized from real streams to formal power-series over a given semiring $\mathbb{S}$. Here one considers the functor $FX = \mathbb{S} \times X^A$ on the category of modules for $\mathbb{S}$. Notice that the F-coalgebras carried by modules $\mathbb{S}^n$ can be identified with *weighted automata*, i.e., non-deterministic automata whose transitions have weights in $\mathbb{S}$ and whose states have outputs in $\mathbb{S}$ (see e.g. [32]). The final coalgebra for F is carried by the set $\mathbb{S}^{A^*}$ of *formal power series* (or *weighted languages*) with the coalgebra structure $\langle o, t\rangle : \mathbb{S}^{A^*} \to k \times (\mathbb{S}^{A^*})^A$ with $o(L) = L(\varepsilon)$ and $t(L)(a) = \lambda w.L(aw)$.

Now whenever $\mathbb{S}$ is a *Noetherian* semiring (i. e., every submodule of a finitely generated module is itself finitely generated) then ϱF is the subcoalgebra of $\nu F = \mathbb{S}^{A^*}$ given by the *rational* formal power series of weighted automata theory (see e. g. [32]). Examples of Noetherian semirings are fields, finite semirings, principal ideal domains (such as the ring of integers) and hence all finitely generated commutative rings by Hilbert's basis theorem. The tropical semiring and also the usual semiring of natural numbers are not Noetherian.

(7) Consider the category of 'sets in context', i. e., the presheaf category $[\mathscr{F}, \mathsf{Set}]$ where $\mathscr{F}$ is the category of finite sets and maps. Objects Γ of $\mathscr{F}$ are regarded as contexts of variables and we think of a presheaf

$X : \mathscr{F} \to \mathsf{Set}$ as assigning to every context Γ a set $X(\Gamma)$ of 'terms in context Γ'. This setting was considered by Fiore, Plotkin and Turi [35] to capture variable binding and in particular λ-terms categorically. Indeed, let $FX = V + X \times X + \delta(X)$, where $V : \mathscr{F} \to \mathsf{Set}$ is the inclusion and $\delta(X)(\Gamma) = X(\Gamma + 1)$. Then μF is the presheaf of λ-terms up to α-equivalence. The terminal coalgebra νF is formed by all (finite and infinite) λ-trees, and the rational fixed point by all rational ones (up to α-equivalence). See [17].

Note that in all the above examples, the rational fixed point occurs as a subcoalgebra of the terminal coalgebra collecting precisely the behaviours of all coalgebra in $\mathsf{Coalg}_{\mathsf{f}}\, F$. This is no coincidence; in fact, for every set functor F, ϱF is always the union of all images of the coalgebra morphisms $(X, c) \to \nu F$, where (X, c) ranges over all finite F-coalgebras (see Proposition 11.12).

This can be generalized to locally finitely presentable categories under a few side conditions, as we now explain. Recall from [21] that a *finitely generated* object is an object X such that its covariant hom-functor $\mathscr{C}(X, -)$ preserves directed unions (i. e., colimits of directed diagrams of monomorphisms). Clearly, every finitely presentable object is finitely generated, but the converse does not hold in general. (For example, in the category of monoids, they differ, cf. Example 6.8(3)). The following proposition follows from [13, Proposition 4.6 and Remark 4.3]; it appears explicitly in [27, Proposition 3.12].

Proposition 11.12. *Suppose that in $\mathscr{C}$ finitely generated objects are finitely presentable, and that F preserves monomorphisms. Then ϱF is the subcoalgebra of νF given by the union of images of all F-coalgebra morphisms $(X, c) \to (\nu F, \tau)$ where (X, c) ranges over $\mathsf{Coalg}_{\mathsf{f}}\, F$.*

All categories of Example 11.11 satisfy the above assumption; see [27, Example 3.13] for more. But there are also categories for which this fails (e.g. groups, monoids or modules for the semiring $\mathbb{S} = (\mathbb{Z}_2)^{\mathbb{N}}$). Moreover, there are instances where the rational fixed point is not a subcoalgebra of νF (see [27, Example 3.15] for a concrete example). We shall come back to this point at the end of the next subsection.

11.2. Iterative Algebras

Recall that the terminal coalgebra νF is, equivalently, an initial completely iterative algebra for F. Similarly, the rational fixed point is the initial *iterative algebra*.

Iterative algebras for a signature were introduced by Nelson [55] (see Tiuryn [67] for a related concept) as an easy approach to iterative theories of Elgot [33]. In [13, 16] this was generalized from Set to arbitrary locally finitely presentable categories. In this subsection we mention the most important results on iterative algebras.

Definition 11.13 (Adámek, Milius, Velebil [16], Definition 2.5)**.** A flat equation morphism $e : X \to FX + A$ is called *finitary* if X is a finitely presentable object of $\mathscr{C}$.

An F-algebra $a : FA \to A$ is called *iterative* if every finitary flat equation morphism has a unique solution in A (cf. Definition 10.2).

Let us remark that iterative algebras constitute a full subcategory of the category of all F-algebras: every homomorphism of F-algebras preserves solutions of flat equation morphisms in the expected sense (see [16, Proposition 2.18]).

Examples 11.14. Every cia is, of course, an iterative algebra. We list two examples of iterative algebras which are not in general cias.

(1) In Set, a unary algebra $a : A \to A$ is iterative if and only if a has a unique fixed point and no other cycles in A. For example, $\mathbb{Z} \cup \{\infty\}$ with the successor operation is iterative but not a cia (cf. Example 10.5(3)).

(2) The algebra of addition on the extended real numbers $\widetilde{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is an iterative algebra for the functor $FX = X \times X$, see [16]. Notice that this is not a completely iterative algebra: the system of equations $x_0 \approx x_1 + 1, x_1 \approx x_2 + 1, \dots$ has more than one solution (e. g. ${x_n}^\dagger = \infty$ or ${x_n}^\dagger = -n$).

Proposition 11.15 (Adámek, Milius, Velebil [16], Theorem 2.20)**.** *The category of iterative algebras is closed under limits and filtered colimits in the category of F-algebras. Thus, limits and filterted colimits are constructed on the level of the base category $\mathscr{C}$.*

Corollary 11.16. *Every object of $\mathscr{C}$ generates a free iterative algebra.*

Indeed, full subcategories of a locally finitely presentable category closed under limits and filtered colimits are reflective, see [21]. It follows that the reflection of μF in the category of iterative F-algebras is the initial iterative F-algebra.

In [16] it was also proved that the colimit in Definition 11.8 yields an initial iterative algebra for F. Thus we have the following result:

Theorem 11.17 (Adámek, Milius, Velebil [16], Theorem 3.3)**.** *The rational fixed point ϱF is an initial iterative algebra.*

Remark 11.18. Similarly as in Definition 11.8 we obtain for every object Y of $\mathscr{C}$ a *free* iterative algebra RY on Y by forming the colimit of all coalgebras for $F(-) + Y$ on finitely presentable carriers, i.e. a colimit of the inclusion functor

$$\mathsf{Coalg_f}(F(-) + Y) \hookrightarrow \mathsf{Coalg}(F(-) + Y).$$

This yields a monad R on $\mathscr{C}$, which is characterized as the free iterative monad on the endofunctor F. This generalizes and extends classical work on iterative theories by Elgot [33] and on iterative algebras for a signature by Nelson [55] and Tiuryn [67].

For more details on the topic of iterative algebras and (free) iterative monads we refer the reader to [16].

The Locally Finite Fixed Point. To conclude this section let us come back to Proposition 11.12 and the discussion following it, where we saw that the rational fixed point need not be a subcoalgebra of νF. However, in all applications, the regular behaviour of systems is defined by taking the image of all the behaviours of finite(ly presented) systems in the semantic domain, i.e. the terminal coalgebra. Hence, in applications one wants the rational fixed point to be subcoalgebra of νF. Unfortunately, the condition in Proposition 11.12 that finitely generated objects be finitely presentable is, if true at all, usually non-trivial to verify in cases where it is unknown.

Technically, the a priori choice of taking finitely presentable objects as the right abstraction of 'finite set' is not completely canonical. One may ask what happens if one chooses finitely generated objects instead. This has recently been investigated in [53]. In fact, one can rework much of the theory we have seen by systematically replacing finitely presentable by finitely generated objects. One assumes that F is a finitary endofunctor preserving monomorphisms. Then, for example, taking the colimit of the diagram of all F-coalgebras on finitely generated carriers yields a coalgebra ϑF, which is a fixed point of F called the *locally finite fixed point*. Furthermore, the locally finite fixed point is characterized by a universal property both as an algebra and a coalgebra: on the one hand it is the terminal locally finitely generated coalgebra, and on the other hand it is the initial fg-iterative algebra (for details see *op. cit.*).

An advantage of ϑF is that, unlike the rational fixed point, it is always a subcoalgebra of νF. Furthermore, under additional assumptions on $\mathscr{C}$ (which hold e.g. for all varieties $\mathscr{C}$) there is a strong connection to the rational fixed point: ϑF is then the image of ϱF in the terminal coalgebra. This means that in all concrete examples of rational fixed points which are of widespread interest, we in fact have $\vartheta F \cong \varrho F$, so these examples might as well be studied as locally finite fixed points.

In addition, we also obtain several examples of finite behaviour domains that could so far not be obtained as instances of the rational fixed point, e.g. (real-time deterministic resp. non-deterministic) context-free languages, constructively $\mathbb{S}$-algebraic formal power-series (and any other instance of the generalized powerset construction by Silva et al. [63]) and the monad of Courcelle's algebraic trees (see [30]). Our description of algebraic trees as a locally finite fixed point yields the first characterization of those trees by a universal property solving an open problem from [18].

Unified Fixed Points. In recent work, Urbat [71] gives a uniform account of several results on fixed points that are based on coalgebras. His theory is parametric in a class $\mathbb{I}$ of diagram schemes and a class $\mathscr{M}$ of morphisms coming from a factorization system $(\mathscr{E}, \mathscr{M})$ on the base category $\mathscr{C}$, where $\mathscr{E}$ is some class of epimorphisms. For an endofunctor $F : \mathscr{C} \to \mathscr{C}$ one considers those coalgebras that have an $(\mathbb{I}, \mathscr{M})$-presentable carrier, i.e. a carrier X such that $\mathscr{C}(X, -)$ preserves colimits in $\mathscr{C}$ that have a diagram scheme in $\mathbb{I}$ and whose colimit injections lie in $\mathscr{M}$.

Under suitable conditions on $\mathbb{I}$, $\mathscr{M}$ and a given endofunctor $F : \mathscr{C} \to \mathscr{C}$, Urbat proves that the colimit T_F of all coalgebras with an $(\mathbb{I}, \mathscr{M})$-presentable carrier is a fixed point of F and that this is characterized by a universal property both as a coalgebra and as an algebra (the proofs of these characterizations use that all diagrams in $\mathbb{I}$ are filtered). As instances one obtains νF, ϱF and ϑF as shown in the following table:

$\mathbb{I}$	$(\mathscr{E}, \mathscr{M})$	T_F
categories with a terminal object	(iso, all)	νF
small filtered categories	(iso, all)	ϱF
small filtered categories	(strong epis, monos)	ϑF

A fourth fixed point φF is obtained in the case where $\mathscr{C}$ is a variety, $\mathbb{I}$ the class of all small sifted diagrams and $(\mathscr{E}, \mathscr{M}) = (\mathrm{iso}, \mathrm{all})$. However, in this case $\mathbb{I}$ does not consist of filtered diagrams, and hence, a characterization of this fixed point by a universal property remains open. Milius [49] studies conditions under which this new fixed point φF coincides with ϱF and ϑF.

Well-pointed Coalgebras. Finally, let us point out that, like the terminal coalgebra and the initial algebra, the rational fixed point of a set functor also has a characterization by well-pointed coalgebras (cf. Theorems 8.25 and 8.28). Recall from those theorems the coalgebra (T, τ) of all representatives of well-pointed F-coalgebras up to isomorphism.

Theorem 11.19 (Adámek et al. [12], Theorem 3.55)**.** *If F is a finitary set functor, its rational fixed point is the coalgebra of all finite well-pointed coalgebras. More precisely, the collection T_f of all finite coalgebras in T forms a (small) subcoalgebra of (T, τ) which is ϱF.*

Example 11.20. For $F = \{0, 1\} \times (-)^S$ we get that ϱF consists of all finite minimal deterministic automata. This coalgebra is isomorphic to that formed by all regular languages on S.

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